Linear Algebra

Definition

Let S be a non-empty set. Any function $f : S \times S \to S$ is called a binary operation on S or equivalently, an operation $(*)$ on S is called a binary operation on S (or S is closed under the operation $(*)$), if $a * b \in S$, for all $a, b \in S$.

Examples

The operations $(+)$, $(.)$ and $(-)$ all are binary operations on $Z, Q, \Re$ and C, where Z is the set of all integers, Q is the set of all rational numbers, $\Re$ is the set of all real numbers and C is the set of all complex numbers.

Definition

A group is a pair $(G,*)$, where G is a non-empty set and $(*)$ is a binary operation on G satisfying the following conditions:

1. $(*)$ is associative on G, that is, $a*(b*c)=(a*b)*c$, for all $a,b,c \in G$.

2. There exists an element $e \in G$,(called identity element of G) such that $a*e=a=e*a$, for all $a \in G$.

3. For each $a \in G$, there exists an element $b \in G$ such that $a*b=e=b*a$, (b is called the inverse of a in G and we denote this by $a^{-1}=b$). If the above three conditions satisfied, then we say $(G,*)$ is a group (simply G is a group) and if in addition to the above conditions, the following holds:

4. $a*b=b*a$, for all $a,b \in G$, then the group G is called an abelian group (or a commutative group).

Examples: (Z,+,.), (Q,+,.), (R,+,.) and (C,+,.) are groups.

The Field

- Let *,o are two binary operations on S, then (S,*,o) is called a field if satisfy the following conditions:

1. (S,*)is an abelian group.

2. (S-{e},o) is an abelian group, where e is an identity element of a binary operation *.
3. Distributive Law: ao (b*c)=(aob)*(aoc) for all a,b & c belong to S.

Example:

(R,+,.) & (C,+,.) are fields

Vectors

Vector : Is **a quantity that has both magnitude and direction but not position**.

Examples of such quantities are velocity and acceleration.

In their modern form, vectors appeared late in the 19th century when Josiah Willard Gibbs and Oliver Heaviside (of the United States and Britain, respectively) independently developed vector analysis to express the new laws of electromagnetism discovered by the Scottish physicist James Clerk Maxwell. Since that time, vectors have become essential in physics, mechanics, electrical engineering, and other sciences to describe forces mathematically.

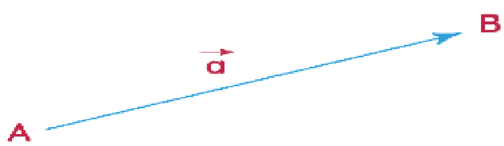
Or;

Vectors are geometrical entities that have magnitude and direction. A vector can be represented by a line with an arrow pointing towards its direction and its length represents the magnitude of the vector. Therefore, vectors are represented by arrows, they have initial points and terminal points.

A vector is a Latin word that means carrier. Vectors carry a point A to point B. The length of the line between the two points A and B is called the **magnitude of the vector** and the direction of the displacement of point A to point B is called the **direction of the vector** AB. Vectors are also called Euclidean vectors or Spatial vectors.

Representation of Vectors

Vectors are usually represented in bold lowercase such as **a** or using an arrow over the letter as $\vec{a}$. Vectors can also be denoted by their initial and terminal points with an arrow above them, for example, vector AB can be denoted as $\overrightarrow{AB}$. The standard form of representation of a vector is $\vec{A} = a\hat{i} + b\hat{j} + c\hat{k}$. Here, a,b,c are real numbers and $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors along the x-axis, y-axis, and z-axis respectively.



Definition: A vector

- **Is an arrow starting at A and finishing at B denoted by AB**
- **(1) Properties of Addition:**
- $\forall$ ***v, w, u*** $\in$ ***V***

a) $\boldsymbol{v + w = w + v}$ **(Commutativity)**

b) $\boldsymbol{(v + w) + u = v + (w + u)}$ **(Associativity)**

c) $\boldsymbol{v + 0 = v = 0 + v}$ **(Zero element)**

- **(2) Properties of scalar multiplication :**
- $\forall$ ***v, w*** $\in$ ***V*** **and *a, b*** $\in$ **R,[R is the real number field (R,+,.)]**

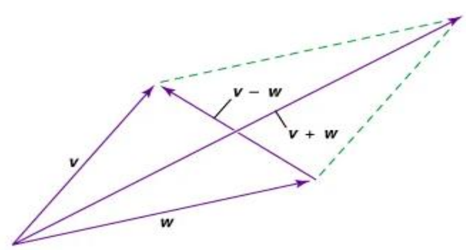
a) **$(a + b) v = a v + b v$** **(Distributivity)**

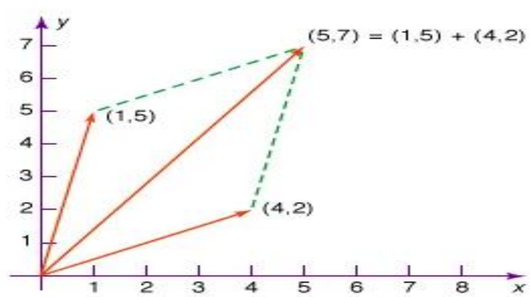
b) **$a (v + w) = a v + a w$**

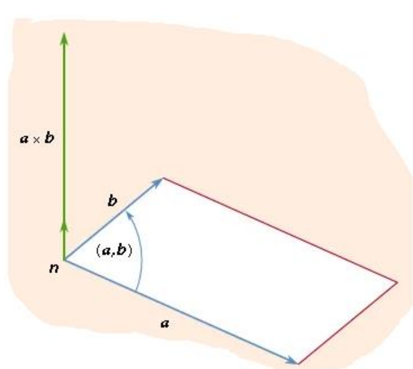
c) **$(a b) v = a (b v) = a b v$** **(Associativity)**

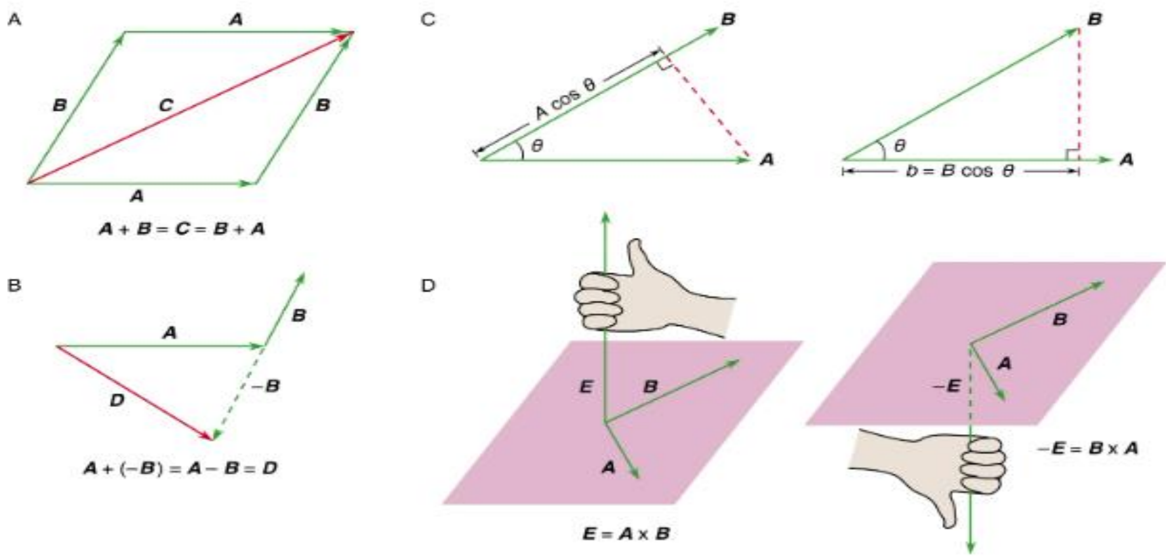
d) **$1 v = v$, $0P=0$**

e) **$P + (-1)P =0$**









vector mathematics

Figure 1: (A) The vector sum ***C*** = ***A*** + ***B*** = ***B*** + ***A***. (B) The vector difference ***A*** + (−***B***) = ***A*** − ***B*** = ***D***. (C, left) *A* cos θ is the component of ***A*** along ***B*** and (right) *B* cos θ is the component of ***B*** along ***A***. (D, left) The right-hand rule used to find the direction of ***E*** = ***A*** × ***B*** and (right) the right-hand rule used to find the direction of −***E*** = ***B*** × ***A***.

Types of Vectors

The vectors are termed as different types based on their magnitude, direction, and their relationship with other vectors. Let us explore a few types of vectors and their properties:

1- **Zero Vectors**

Vectors that have 0 magnitude are called zero vectors, denoted by $\vec{0}$= (0,0,0).

2-Unit Vectors

Vectors that have magnitude equals to 1 are called unit vectors, denoted by $\vec{a}$.

3- **Parallel Vectors**

Two or more vectors are said to be parallel vectors if they have the same direction but not necessarily the same magnitude.

4- **Orthogonal Vectors**

Two or more vectors in space are said to be orthogonal if the angle between them is 90 degrees. In other words, the dot product of orthogonal vectors is always 0.

a·b = |a|·|b|cos90° = 0.

Length and Dot Product of Vectors:

The individual components of the two vectors to be multiplied are multiplied and the result is added to get the dot product of two vectors.

DEFINITION The ***dot product*** or ***inner product*** of $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$ is the number $\boldsymbol{v} \cdot \boldsymbol{w}$:

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2. \tag{1}$$

Example 1 The vectors $\boldsymbol{v} = (4, 2)$ and $\boldsymbol{w} = (-1, 2)$ have a *zero* dot product:

Dot product is zero
Perpendicular vectors

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 + 4 = 0.$$

Lengths and Unit Vectors

An important case is the dot product of a vector *with itself*. In this case v equals w. When the vector is $v = (1, 2, 3)$, the dot product with itself is $v \cdot v = \|v\|^2 = 14$:

Dot product $v \cdot v$
Length squared

$$\|v\|^2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 4 + 9 = \mathbf{14}.$$

DEFINITION The ***length*** $\|v\|$ of a vector v is the square root of $v \cdot v$:

Length = norm(v) **length** $= \|v\| = \sqrt{v \cdot v}$.

DEFINITION ***A unit vector u is a vector whose length equals one.*** Then $u \cdot u = 1$.

An example in four dimensions is $u = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Then $u \cdot u$ is $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$. We divided $v = (1, 1, 1, 1)$ by its length $\|v\| = 2$ to get this unit vector.

Example 4 The standard unit vectors along the x and y axes are written i and j. In the xy plane, the unit vector that makes an angle "theta" with the x axis is $(\cos\theta, \sin\theta)$:

$$\text{Unit vectors} \quad i = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad j = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

Unit vector $u = v/\|v\|$ **is a unit vector in the same direction as v.**

Right angles *The dot product is $v \cdot w = 0$ when v is perpendicular to w.*

COSINE FORMULA If v and w are nonzero vectors then $\dfrac{v \cdot w}{\|v\|\,\|w\|} = \cos\theta.$

The resultant of a dot product of two vectors is a scalar value, that is, it has no direction.

SCHWARZ INEQUALITY $|v \cdot w| \leq \|v\|\,\|w\|$

TRIANGLE INEQUALITY $\|v + w\| \leq \|v\| + \|w\|$

Example 5 Find $\cos\theta$ for $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and check both inequalities.

Solution The dot product is $v \cdot w = 4$. Both v and w have length $\sqrt{5}$. The cosine is $4/5$.

$$\cos\theta = \frac{v \cdot w}{\|v\|\,\|w\|} = \frac{4}{\sqrt{5}\sqrt{5}} = \frac{4}{5}.$$

Proporties of Dot product

1- u.v=v.u
2- (cu).v = u.(cv) = c(u.v)
3- u.(v+w) = u.v+u.w
4- 0.u=0
5- u.u $=|u|^2$

Theorem: Two vectors u and v are said to be orthogonal or perpendicular if the angle between them is 90.

Or u and v are orthogonal $\Leftrightarrow$ u.v=0

Cross Product of vectors:

The vector components are represented in a matrix and a determinant of the matrix represents the result of the cross product of the vectors.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$\vec{A} \times \vec{B} = (b1c2 - c1b2, a1c2 - c1a2, a1b2 - b1a2)$$

Another way to determine the cross product of two vectors A and B is to determine the product of the magnitudes of the two vectors and the sine of the angle between them.

$$\vec{A} \times \vec{B} = |A||B| \sin\theta \, \hat{n}$$

Example 1: Find the angle between the two vectors $2\hat{i} + \hat{j} - 3\hat{k}$ and $3\hat{i} - \hat{j} + \hat{k}$?

Solution:

Given two vectors **a** = $2\hat{i} + \hat{j} - 3\hat{k}$ and **b** = $3\hat{i} - \hat{j} + \hat{k}$

We need to determine the angle between the vectors **a** and **b** using the formula cosθ = **a.b** / |**a**||**b**|

a·b = $(2\hat{i} + \hat{j} - 3\hat{k}) \cdot (3\hat{i} - \hat{j} + \hat{k})$

= (2 × 3) + (1 × -1) + (-3 × 1)

= 6 - 1 - 3

= 2

$|\mathbf{a}| = \sqrt{(2^2 + 1^2 + (-3)^2)}$

$= \sqrt{(4 + 1 + 9)}$

$= \sqrt{14}$

$|\mathbf{b}| = \sqrt{(3^2 + (-1)^2 + (1)^2)}$

$= \sqrt{(9 + 1 + 1)}$

$= \sqrt{11}$

$\cos\theta = 2 / (\sqrt{14} \times \sqrt{11})$

$\cos\theta = 2 / 12.409$

$\cos\theta = 0.161$

$\theta = \cos^{-1}(0.161)$

$\theta = 80.73°$

Answer: The angle between the two vectors is 80.73°.

Example 2: Find the sum of two vectors a = $4\hat{i} + 2\hat{j} - 5\hat{k}$ and b = $3\hat{i} - 2\hat{j} + \hat{k}$?

Solution:

Given two vectors **a** = $4\hat{i} + 2\hat{j} - 5\hat{k}$ and **b** = $3\hat{i} - 2\hat{j} + \hat{k}$

a + **b** = $(4\hat{i} + 2\hat{j} - 5\hat{k}) + (3\hat{i} - 2\hat{j} + \hat{k})$

= $(4 + 3)\hat{i} + (2 - 2)\hat{j} + (-5 + 1)\hat{k}$

= $7\hat{i} + 0\hat{j} - 4\hat{k}$

= $7\hat{i} - 4\hat{k}$

Therefore, the sum of two vectors is $7\hat{i} - 4\hat{k}$

Answer: $7\hat{i} - 4\hat{k}$

Example 3: Find the cross product of two vectors **a** = $4\hat{i} + 2\hat{j} - 5\hat{k}$ and **b** = $3\hat{i} - 2\hat{j} + \hat{k}$ and verify it using cross product calculator?

Solution:

Given two vectors **a** = $4\hat{i} + 2\hat{j} - 5\hat{k}$ and **b** = $3\hat{i} - 2\hat{j} + \hat{k}$

Comparing these to the vector notations we have.

a = $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and **b** = $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Applying cross product formula,

a × **b** = $\hat{i}(a_2b_3 - a_3b_2) - \hat{j}(a_1b_3 - a_3b_1) + \hat{k}(a_1b_2 - a_2b_1)$

$= \hat{i}((2 \times 1) - (-5) \times (-2)) - \hat{j}((4 \times 1 - (-5) \times (3)) + \hat{k}((4) \times (-2) - (2 \times 3))$

$= \hat{i}(2 - 10) - \hat{j}(4 + 15) + \hat{k}(-8 - 6)$

$= -8\hat{i} - 19\hat{j} - 14\hat{k}$

Therefore, the cross product of two vectors is $-8\hat{i} - 19\hat{j} - 14\hat{k}$

Answer: $-8\hat{i} - 19\hat{j} - 14\hat{k}$

Definition 1.1: (Real) Vector Space (V, +,.)

A vector space (over F) consists of a set V along with 2 operations '+' and ' . ' s.t.

(1) **For the vector addition + :**

$\forall$ $v, w, u \in V$

a) $v + w \in V$ **(Closure)**

b) $v + w = w + v$ **(Commutativity)**

c) $(v + w) + u = v + (w + u)$ **(Associativity)**

d) $\exists\, 0 \in V$ s.t. $v + 0 = v$ **(Zero element)**

e) $\exists\, -v \in V$ s.t. $v - v = 0$ **(Inverse)**

(2) For the scalar multiplication :

$\forall$ $v, w \in V$ and $a, b \in$ R, **[R is the real number field (R,+,$\times$)]**

a) $a\, v \in V$ **(Closure)**

b) $(a + b)\, v = a\, v + b\, v$ **(Distributivity)**

c) $a\, (v + w) = a\, v + a\, w$

d) $(a\, b)\, v = a\, (b\, v) = a\, b\, v$ **(Associativity)**

e) $1\, v = v$

Example-1:

- **If n is any positive integer , then $< F^n,+,.,F >$ is a vector space by defining addition and scalar multiplication as:**

$$(a_1,a_2,...,a_n)+(b_1,b_2,...,b_n)=(a_1+b_1,a_2+b_2,...,a_n+b_n)$$
$$\lambda(a_1,a_2,...,a_n)=(\lambda a_1,\lambda a_2,...,\lambda a_n)$$

- **In particular we have seen two dimensional and tridimensional spaces** $(R^2,+,.,R)$ & $(R^3+,.,R)$

Example-2:

- $(M_{n\times n}^{(F)},+,.,F)$ is a vector space under addition and scalar multiplication of matrices.

- Example 3: $F[x]=\{a_0+a_1x+a_2x^2+...+a_nx^n;a_i\in F,n\in N\}$

denotes the set of polynomials with coefficient in a field $(F,+,.)$ then $< F[x],+,.,F >$ is a vector space under polynomial addition and scalar multiplication of polynomial.

Example-3:

- IS R^2 with addition and scalar multiplication defined as

$$(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)$$
$$\lambda(x_1,y_1)=(\lambda x_1,y_1)$$

vector space or not? why?

Example-4:

- IS R^+ with addition and scalar multiplication defined as

$$x + y = xy$$

$$rx = x^r \qquad \forall\ x, y \in R \ \& \ r \in R$$

vector space or not? why?

Homework

1. IS R^2 with addition and scalar multiplication defined as

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$\lambda(x_1, y_1) = (\lambda x_1, 0)$$

vector space or not? why?

2. IS R^3 with addition and scalar multiplication defined as

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1, y_1, z_1)$$

$$\lambda(x_1, y_1, z_1) = (\lambda x_1, \lambda y_1, \lambda z_1)$$

vector space or not? why?

Lemma

In any vector space V,

1. $0_F V = 0$

2. $(-1)v + v = 0$.

3. $a\,0 = 0$.

4. If aV=0 then a=0(F) or V=0

5. (-a) V= a(-V) = - (aV)

$\forall\, v \in V$ **and** $a \in$ **F.**

Proof:

1. $\mathbf{0} = \mathbf{v} - \mathbf{v} = (1+0)\mathbf{v} - \mathbf{v} = \mathbf{v} + 0\mathbf{v} - \mathbf{v} = 0\mathbf{v}$
2. $(-1)\mathbf{v} + \mathbf{v} = (-1+1)\mathbf{v} = 0\mathbf{v} = \mathbf{0}$ By 1
3. $a\,\mathbf{0} = a(0\,\mathbf{v}) = (a\,0)\,\mathbf{v} = 0\,\mathbf{v} = \mathbf{0}$

By 1 By 1

Vector Subspaces

Definition : **Subspaces**

For any vector space, a subspace is a subset that is itself a vector space, under the inherited operations.

DEFINITION A ***subspace*** of a vector space is a set of vectors (including $\mathbf{0}$) that satisfies two requirements: ***If v and w are vectors in the subspace and c is any scalar, then***

(i) $v + w$ is in the subspace

(ii) cv is in the subspace.

First fact: ***Every subspace contains the zero vector***. The plane in $\mathbf{R}^3$ has to go through $(0, 0, 0)$. We mention this separately, for extra emphasis, but it follows directly from rule (**ii**). Choose $c = 0$, and the rule requires $0v$ to be in the subspace.

Example :

- **{ 0 } is a trivial subspace of $\mathbb{R}^n$.**
- **$\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.**

Both are improper subspaces.

All other subspaces are proper.

Example : Subspace is only defined inherited operations.

({1},♣ ; R) is a vector space if we define 1♣1 = 1 and a♦1=1 $\forall a \in F$.

However, neither ({1},♣ ; R) nor ({1},+ ; R) is a subspace of the vector space (R,+ ; R).

Lemma :

Let S be a non-empty subset of a vector space (V, + ,., R).the inherited operations, the following statements are equivalent:

1. **S is a subspace of V.**
2. **S is closed under all linear combinations of pairs of vectors.**
3. **S is closed under arbitrary linear combinations.**

Remark:

Vector space = Collection of linear combinations of vectors.

A subspace containing v and w must contain all linear combinations $cv + dw$.

Example 3 Inside the vector space **M** of all 2 by 2 matrices, here are two subspaces:

(**U**) All upper triangular matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ (**D**) All diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$.

Add any two matrices in **U**, and the sum is in **U**. Add diagonal matrices, and the sum is diagonal. In this case **D** is also a subspace of **U**! Of course the zero matrix is in these subspaces, when a, b, and d all equal zero.

Example

$$S=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\middle|\, x-2y+z=0\right\}$$ is a subspace of R^3.

Example 2/ Matrix Subspace.

$$L=\left\{\begin{pmatrix}a&0\\b&c\end{pmatrix}\middle|\, a+b+c=0\right\}$$ is a subspace of the space of 2×2 matrices.

Home work

- Is $M=\{(x,y);y=2x\}$ **vector subspace of** R^2 **over field R?**

- Is $M=\{\begin{pmatrix}0&a\\b&0\end{pmatrix};a,b\in R\}$ **vector subspace of** $M_{2\times2}^{(R)}$ **over field R?**

- **Is** $M=\{a+bx+cx^2;\ a+2b-c=1\}$ **vector subspace of** $P_2^{(R)}$ **over field R?**

- Is $M=\{(z_1,z_2,z_3);|z_3|=4\}$ **vector subspace of** R^3 **over field C ?**

Algebra of Subspaces

- **The intersection of two vector subspace is also a vector subspace.**
- **The union of two vector subspace may not be a vector subspace.**
- **For example:**
- **Let** $M_1=\{(x,y);x+2y=0\}$
  $M_2=\{(x,y);5x+y=0\}$ **are two vector subspaces**

  But $M_1\cup M_2$ **is not a vector subspace.**

Theorem: $M_1 \cup M_2$ is a vector subspace if and only if $M_1 \sqsubset M_2$ or $M_2 \sqsubset M_1$

Find the intersection of each of the following:

$$1.\ Let\ \ M_1=\{(x,y,z);2x-y+3z=0\}$$
$$and\ \ M_2=\{(x,y,z);x+y-z=0\}$$

$$2.\ Let\ M_1=\{a+bx+cx^2;a+2b-c=0\}$$
$$and\ \ M_2=\{a+bx+cx^2;b=0,a+3c=0\}$$

The sum of Two vector subspace

- The sum of two vector subspace is also a vector subspace.

$$M + N = \{A + B; A \in M, B \in N\}$$

- Example: Let

1- $M = \{(x,0); x \in R\}, \ \& \ N = \{(0, y); y \in R\}$

are vectors on R^2 over field R , find M+N.

2) Let $M = \{(x, y, z); x = 0\}$ & $N = \{(x, y, z); y + z = 0\}$

$Find \ M + N \ \& \ M \cap N.$

Home work

- Find $M + N$ & $M \cap N$ for each of the following:

1) $M = \{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}; a + c = b\}$ & $N = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}; a = 3c\}$

2) $M = \{(2x, x); x \in R\}$ & $N = \{(y, y); y \in R\}$

Direct sum:

$$V = M \oplus N \quad iff \ V = M + N \ and \ M \cap N = \{0\}$$

Which of the following are direct sum:

$$M = \{(x, y, z); x + y + z = 0\}, \ \ N = \{(0,0, z); z \in R\}$$
$$L = \{(x, y, z); x = z\}$$

Linear combination of vectors

- A vector X is defined to be a linear combination of vectors $x_1, x_2, \ldots, x_k$ if its can be written as:

$$X = c_1x_1 + c_2x_2 + \ldots + c_kx_k$$

Where $c_1, c_2, \ldots, c_k$ are scalars.

Examples

- Show that c=(3,-5,-2) is a linear combination of a set s={(1,5,0),(2,0,-1)} but D=(-2,20,7) is not a linear combination of them.

Which of the following vectors are linear combinations of

$$\{1+x, x-x^2+x^4, 5+x^3\}$$

1) x^2
2) 7
3) $-5+x^3$

- **Example 1**

$S = \{\underset{\mathbf{v}_1}{(1, 3, 1)}, \underset{\mathbf{v}_2}{(0, 1, 2)}, \underset{\mathbf{v}_3}{(1, 0, -5)}\}$,

$\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ because

$$\mathbf{v}_1 = 3\mathbf{v}_2 + \mathbf{v}_3 = 3(0, 1, 2) + (1, 0, -5) = (1, 3, 1)$$

Definition 2.13: Span

Let $S = \{ s_1 , \dots, s_n \mid s_k \in (V,+,\mathrm{R}) \}$ be a set of n vectors in vector space V.

The span of S is the set of all linear combinations of the vectors in S, i.e.,

$$span\ S = \left\{ \sum_{k=1}^{n} c_k \mathbf{s}_k \;\middle|\; \mathbf{s}_k \in S\ ,\ c_k \in \mathrm{R} \right\}$$

Also: span S is the smallest vector space containing all members of s.

Example 2.:

$$span\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} = \mathbf{R}^2$$

Proof:

The problem is to showing that for all $x, y \in \mathrm{R}$, $\exists$ unique $a,b \in \mathrm{R}$ s.t.

$$\begin{pmatrix} x \\ y \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

i.e., $\begin{aligned} a+b&=x \\ a-b&=y \end{aligned}$ has a unique solution for arbitrary x & y.

Since $a = \frac{1}{2}(x+y)$ $\quad b = \frac{1}{2}(x-y)$ $\quad \forall\ x, y \in \mathbf{R}$ $\quad$ QED

Example 3: In vector space R^3 over field R,

if T={(0,3,-3)}, S={(1,0,0),(0,2,0)} then find the following:

$A-[S],[T]\ span\ \ S,T\ respective\,ly.$

$B-\ Is\ the\ vector\ \ (5,-3,0) \in [S]?$

$C-[S] \cap [T].$

1) $M + N = [M \cup N]$
2) $[M] + [N] = [M \cup N]$
3) $[S \cap T] \subset [S] \cap [T]$
4) *If* $S \subset T$ *then* $[S] \subset [T]$

Exercises:

1. Consider the set $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| x + y + z = 1 \right\}$

under these operations.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} \qquad r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx - r + 1 \\ ry \\ rz \end{pmatrix}$$

(a) Show that it is not a subspace of R^3.
(b) Show that it is a vector space.
(To save time, you need only prove axioms (d) & (j), and closure under all linear combinations of 2 vectors.)
(c) Show that any subspace of R^3 must pass throw the origin, and so
any subspace of R^3 must involve zero in its description.
Does the converse hold?
Does any subset of R^3 that contains the origin become a subspace when given the inherited operations?

2. Because 'span of' is an operation on sets we naturally consider how it interacts with the usual set operations. Let $[S] \equiv \text{Span } S$.
(a) If $S \subseteq T$ are subsets of a vector space, is $[S] \subseteq [T]$?
Always? Sometimes? Never?
(b) If S, T are subsets of a vector space, is $[S \cup T] = [S] \cup [T]$?
(c) If S, T are subsets of a vector space, is $[S \cap T] = [S] \cap [T]$?
(d) Is the span of the complement equal to the complement of the span?

LINEAR INDEPENDENT SETS; BASES

- An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V is said to be **linearly independent** if the vector equation
$$c_1 v_1 + c_2 v_2 + \ldots + c_p v_p = 0 \qquad \text{----(1)}$$
has *only* the trivial solution, $c_1 = 0, \ldots, c_p = 0$.
- The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if (1) has a nontrivial solution, *i.e.*, if there are some weights, $c_1, \ldots, c_p$, *not all zero*, such that (1) holds.
- In such a case, (1) is called a **linear dependence relation** among $\mathbf{v}_1, \ldots, \mathbf{v}_p$

Example

Determine whether the following set of vectors is linearly dependent or linearly independent

$S = \{ \mathbf{v}_1 = (1, 2, 3), \mathbf{v}_2 = (0, 1, 2), \mathbf{v}_3 = (-2, 0, 1)\}$

Solution: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$

$\Rightarrow c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1) = (0, 0, 0)$

$\Rightarrow (c_1-2c_3, 2c_1+c_2, 3c_1+2c_2+c_3) = (0, 0, 0)$

$\Rightarrow c_1 = c_2 = c_3 = 0$

$\Rightarrow$Therefore, S is **linearly independent**.

- **Example: Show that**

The set *S* = {(1, 0), (0, 1), (–2, 5)} is linearly dependent

Linear independence:

properties

- **Theorem 1:**
  A set of vectors is linearly dependent if and only if one of the vectors is a linear combination of the others.

- **Theorem 2:** Any set of vectors containing the zero vector is linearly dependent.
- **Theorem 3:**
  If a set of vectors is linearly independent, then any subset of these vectors is also linearly independent.
- **Theorem 4:**
  If a set of vectors is linearly dependent, then any larger set, containing this set, is also linearly dependent.

Basis in vector space

- A basis of V is a linearly independent set of vectors in V which spans V.
- Example: F^n the standard basis

$$\epsilon_1 = (1, 0, \ldots, 0), \epsilon_2 = (0, 1, \ldots, 0), \ldots, \epsilon_n = (0, 0, \ldots, 1)$$

- V is finite dimensional if there is a finite basis.

- Is the set $\{-1, 1-x, -2x^2\}$ basis of $P_2(R)$?

Theorem

- **If $S = \{x_1, x_2, \ldots, x_n\}$ is a span for a vector space and $T = \{y_1, y_2, \ldots, y_t\}$ is a linearly independent set of vectors in V, then $t \leq n$**

- **Corollary**
  If $S = \{x_1, x_2, \ldots, x_n\}$ and $T = \{y_1, y_2, \ldots, y_n\}$ are bases for a vector space, then n=m

Homework

- Is a set $\{1, x-2, (x-2)(x+1)\}$ a basis of $P_2(R)$?
- Is a set $\{(1,0), (i,0), (0,1), (0,i)\}$ a basis of C^2 *over field* C. ?
- Is a set $\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$ a basis

of $M_{2\times 2}(R)$?

Find a basis of vector subspace

1) $M = \{(x, y, z); 2x - y + z = 0\}$ *in* R^3 *over field* R.

2) $M = \{a + bx + cx^2 + dx^3; a + b = c - 2d = 0\}$; *in* $P_3(R)$.

Dimension of V is the number of elements of a basis.

Example

$\dim(R^n) = n$

$\dim(C^n)$ *over field complex number is* n

$\dim(C^n)$ *over field real number is* $2n$.

$\dim(P_n)$ *is* $n+1$.

$\dim(M_{2\times 2}(R))$ *is* 4 $[M_n(R) = n^2]$

Theorem:

- **Let V be an n-dimensional vector space, and let**

  $s = \{x_1, x_2, \dots, x_n\}$ **be a set of n vectors;**
- **1) if S is linearly independent, then it's a basis for V.**
- **2) If S spans, then it's a basis for V.**

Some properties of dimension

- **Dim (M+N)=dim (M)+dim(N)-dim(M $\cap$ N)**
- **If M is a vector subspace of V, then** $\dim M \leq \dim V$
- **If dim V=dim M, then M=V.**

(Home work)

- **Find a basis and dimension of each of the following:**

$$1)\ M = \{(x, y, z); x + y = 0\}\ in\ R^3$$

$$2)\ M = \{P_3(x); d/dx\ (P(x)) = 0\}$$

$$3)\ M = \{P_3(x); P(0) = 0\}$$