Basis and Dimentions

Basis of a Subspace

As we discussed in Section 2.6, a subspace is the same as a span, except we do not have a set of spanning vectors in mind. There are infinitely many choices of spanning sets for a nonzero subspace; to avoid reduncancy, usually it is most convenient to choose a spanning set with the *minimal* number of vectors in it. This is the idea behind the notion of a basis.

Definition. Let V be a subspace of $\mathbf{R}^n$. A ***basis*** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in V such that:

1. $V = \text{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. the set $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

A nonzero subspace has *infinitely many* different bases, but they all contain the same number of vectors.

Definition. Let V be a subspace of $\mathbf{R}^n$. The number of vectors in any basis of V is called the ***dimension*** of V, and is written $\dim V$.

Example-1: Find a basis of R^2.

Solution

We need to find two vectors in $\mathbf{R}^2$ that span $\mathbf{R}^2$ and are linearly independent. One such basis is $\left\{\binom{1}{0}, \binom{0}{1}\right\}$:

1. They span because any vector $\binom{a}{b}$ can be written as a linear combination of $\binom{1}{0}, \binom{0}{1}$:

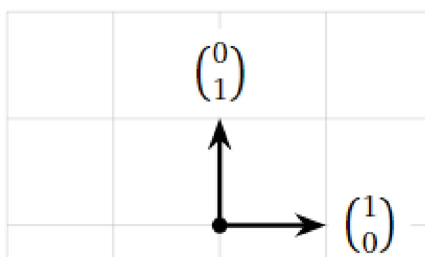
$$\begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

2. They are linearly independent: if

$$x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

then $x = y = 0$.

This shows that the plane $\mathbf{R}^2$ has dimension 2.



Example2: The standard basis of R^n

One shows exactly as in the above example that the standard coordinate vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

form a basis for $\mathbf{R}^n$. This is sometimes known as the ***standard basis***.

In particular, $\mathbf{R}^n$ has dimension n.

The Basis Theorem

Basis Theorem. *Let V be a subspace of dimension m. Then:*

- *Any m linearly independent vectors in V form a basis for V.*
- *Any m vectors that span V form a basis for V.*

In other words, if you *already* know that $\dim V = m$, and if you have a set of m vectors $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ in V, then you only have to check *one* of:

1. $\mathcal{B}$ is linearly independent, *or*
2. $\mathcal{B}$ spans V,

in order for $\mathcal{B}$ to be a basis of V. If you did not already know that $\dim V = m$, then you would have to check *both* properties.

To put it yet another way, suppose we have a set of vectors $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ in a subspace V. Then if any two of the following statements is true, the third must also be true:

1. $\mathcal{B}$ is linearly independent,
2. $\mathcal{B}$ spans V, and
3. $\dim V = m$.

For example, if V is a plane, then any two noncollinear vectors in V form a basis.

Bases as Coordinate Systems

In this section, we interpret a basis of a subspace V as a *coordinate system* on V, and we learn how to write a vector in V in that coordinate system.

> **Fact.** *If $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ is a basis for a subspace V, then any vector x in V can be written as a linear combination*
>
> $$x = c_1v_1 + c_2v_2 + \cdots + c_mv_m$$
>
> *in exactly one way.*

Proof:

Recall that to say $\mathcal{B}$ is a *basis* for V means that $\mathcal{B}$ spans V and $\mathcal{B}$ is linearly independent. Since $\mathcal{B}$ spans V, we can write any x in V as a linear combination of $v_1, v_2, \ldots, v_m$. For uniqueness, suppose that we had two such expressions:

$$x = c_1v_1 + c_2v_2 + \cdots + c_mv_m$$
$$x = c_1'v_1 + c_2'v_2 + \cdots + c_m'v_m.$$

Subtracting the first equation from the second yields

$$0 = x - x = (c_1 - c_1')v_1 + (c_2 - c_2')v_2 + \cdots + (c_m - c_m')v_m.$$

Since $\mathcal{B}$ is linearly independent, the only solution to the above equation is the trivial solution: all the coefficients must be zero. It follows that $c_i - c_i'$ for all i, which proves that $c_1 = c_1'$, $c_2 = c_2', \ldots, c_m = c_m'$.

Example. Consider the standard basis of $\mathbf{R}^3$ from this example

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

According to the above fact, every vector in $\mathbf{R}^3$ can be written as a linear combination of e_1, e_2, e_3, with unique coefficients. For example,

$$v = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 3e_1 + 5e_2 - 2e_3.$$

In this case, *the coordinates of v are exactly the coefficients of e_1, e_2, e_3.*

What exactly are coordinates, anyway? One way to think of coordinates is that they give directions for how to get to a certain point from the origin. In the above example, the linear combination $3e_1 + 5e_2 - 2e_3$ can be thought of as the following list of instructions: start at the origin, travel 3 units north, then travel 5 units east, then 2 units down.

Definition. Let $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ be a basis of a subspace V, and let

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

be a vector in V. The coefficients $c_1, c_2, \dots, c_m$ are the ***coordinates of x with respect to $\mathcal{B}$***. The ***$\mathcal{B}$-coordinate vector of x*** is the vector

$$[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \quad \text{in } \mathbf{R}^m.$$

If we change the basis, then we can still give instructions for how to get to the point $(3,5,-2)$, but the instructions will be different. Say for example we take the basis

$$v_1 = e_1 + e_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We can write $(3,5,-2)$ in this basis as $3v_1 + 2v_2 - 2v_3$. In other words: start at the origin, travel northeast 3 times as far as v_1, then 2 units east, then 2 units down. In this situation, we can say that "3 is the v_1-coordinate of $(3,5,-2)$, 2 is the v_2-coordinate of $(3,5,-2)$, and -2 is the v_3-coordinate of $(3,5,-2)$."

Linear transformaton

Definition. A ***linear transformation*** is a transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ satisfying

$$T(u+v) = T(u) + T(v)$$
$$T(cu) = cT(u)$$

for all vectors u, v in $\mathbf{R}^n$ and all scalars c.

Facts about linear transformations. *Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Then:*

1. *$T(0) = 0$.*
2. *For any vectors $v_1, v_2, \dots, v_k$ in $\mathbf{R}^n$ and scalars $c_1, c_2, \dots, c_k$, we have*

$$T\big(c_1 v_1 + c_2 v_2 + \cdots + c_k v_k\big) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_k T(v_k).$$

Proof:

1. Since $0 = -0$, we have

$$T(0) = T(-0) = -T(0)$$

by the second defining property. The only vector w such that $w = -w$ is the zero vector.

2. Let us suppose for simplicity that $k = 2$. Then

$$\begin{aligned} T(c_1v_1 + c_2v_2) &= T(c_1v_1) + T(c_2v_2) && \text{first property} \\ &= c_1T(v_1) + c_2T(v_2) && \text{second property.} \end{aligned}$$

In engineering, the second fact is called the *superposition principle*; it should remind you of the distributive property. For example, $T(cu + dv) = cT(u) + dT(v)$ for any vectors u, v and any scalars c, d. To restate the first fact:

> A linear transformation necessarily takes the zero vector to the zero vector.

Example: A non linear transformation.

Define $T: \mathrm{R} \to \mathrm{R}$ by $T(x) = x + 1$. Is T a linear transformation?

Solution

We have $T(0) = 0 + 1 = 1$. Since any linear transformation necessarily takes zero to zero by the above important note, we conclude that T is *not* linear (even though its graph is a line).

Note: in this case, it was not necessary to check explicitly that T does not satisfy both defining properties: since $T(0) = 0$ is a consequence of these properties, at least one of them must not be satisfied. (In fact, this T satisfies neither.)

Example: Verify linearity: dilation

Define $T : \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = 1.5x$. Verify that T is linear.

Solution

We have to check the defining properties for *all* vectors u, v and *all* scalars c. In other words, we have to treat u, v, and c as *unknowns*. The only thing we are allowed to use is the definition of T.

$$\begin{aligned} T(u+v) &= 1.5(u+v) = 1.5u + 1.5v = T(u) + T(v) \\ T(cu) &= 1.5(cu) = c(1.5u) = cT(u). \end{aligned}$$

Since T satisfies both defining properties, T is linear.

Example: Verify linearity: Rotation.

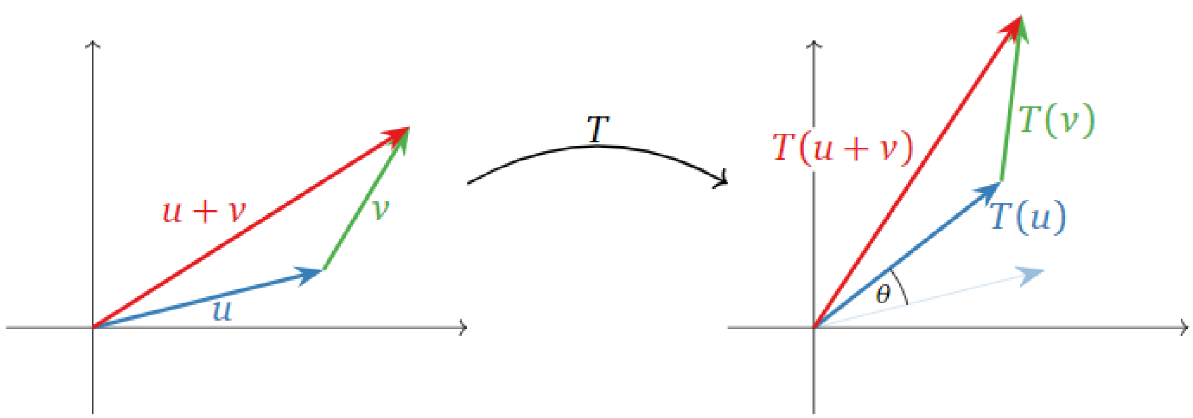
Define $T : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$T(x) = \text{ the vector } x \text{ rotated counterclockwise by the angle } \theta.$$

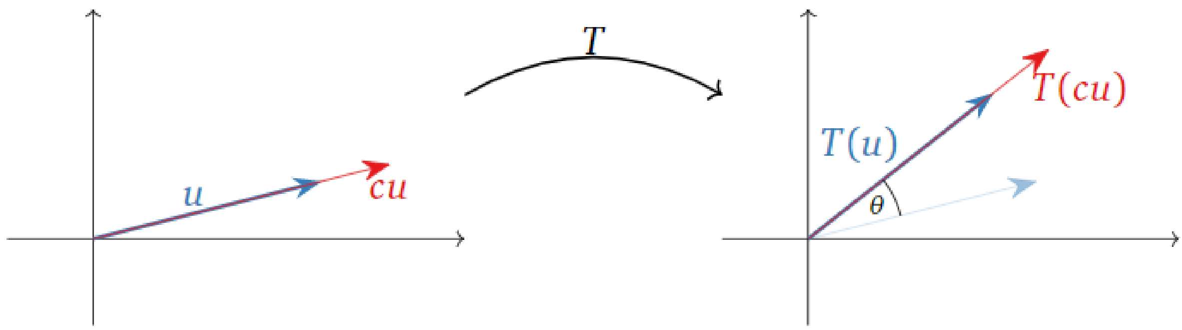
Verify that T is linear.

Solution

Since T is defined geometrically, we give a geometric argument. For the first property, $T(u) + T(v)$ is the sum of the vectors obtained by rotating u and v by θ. On the other side of the equation, $T(u+v)$ is the vector obtained by rotating the sum of the vectors u and v. But it does not matter whether we sum or rotate first, as the following picture shows.



For the second property, $cT(u)$ is the vector obtained by rotating u by the angle θ, then changing its length by a factor of c (reversing direction of $c < 0$. On the other hand, $T(cu)$ first changes the length of c, then rotates. But it does not matter in which order we do these two operations.



This verifies that T is a linear transformation.

Example: Linear transformation defined by formula.

Define $T: \mathbf{R}^2 \to \mathbf{R}^3$ by the formula

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x - y \\ y \\ x \end{pmatrix}.$$

Verify that T is linear.

Solution

We have to check the defining properties for *all* vectors u, v and *all* scalars c. In other words, we have to treat u, v, and c as *unknowns*; the only thing we are allowed to use is the definition of T. Since T is defined in terms of the coordinates of u, v, we need to give those names as well; say $u = \binom{x_1}{y_1}$ and $v = \binom{x_2}{y_2}$. For the first property, we have

$$T\left(\begin{pmatrix}x_1\\y_1\end{pmatrix}+\begin{pmatrix}x_2\\y_2\end{pmatrix}\right)=T\begin{pmatrix}x_1+x_2\\y_1+y_2\end{pmatrix}=\begin{pmatrix}3(x_1+x_2)-(y_1+y_2)\\y_1+y_2\\x_1+x_2\end{pmatrix}$$
$$=\begin{pmatrix}(3x_1-y_1)+(3x_2-y_2)\\y_1+y_2\\x_1+x_2\end{pmatrix}$$
$$=\begin{pmatrix}3x_1-y_1\\y_1\\x_1\end{pmatrix}+\begin{pmatrix}3x_2-y_2\\y_2\\x_2\end{pmatrix}=T\begin{pmatrix}x_1\\y_1\end{pmatrix}+T\begin{pmatrix}x_2\\y_2\end{pmatrix}.$$

For the second property,

$$T\left(c\begin{pmatrix}x_1\\y_1\end{pmatrix}\right)=T\begin{pmatrix}cx_1\\cy_1\end{pmatrix}=\begin{pmatrix}3(cx_1)-(cy_1)\\cy_1\\cx_1\end{pmatrix}$$
$$=\begin{pmatrix}c(3x_1-y_1)\\cy_1\\cx_1\end{pmatrix}=c\begin{pmatrix}3x_1-y_1\\y_1\\x_1\end{pmatrix}=cT\begin{pmatrix}x_1\\y_1\end{pmatrix}.$$

Since T satisfies the defining properties, T is a linear transformation.

Note: we will see in this example below that

$$T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3&-1\\0&1\\1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}.$$

Translatio:

Define $T: \mathbf{R}^3 \to \mathbf{R}^3$ by

$$T(x) = x + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

This kind of transformation is called a ***translation***. As in a previous example, this T is not linear, because $T(0)$ is not the zero vector.

Exercise: Find an example of a transformation that satisfies the first property of linearity but not the second.

Kernel and image of linear transformation

Definition: Let V and W be vector space and let T:V→ W be a linear transformation. Then the image of T denoted by im(T) is defined to be the set { T(v): v ∈ V}. In word, it consists of all vectors in W equal T(v).

The kernel, ker(T), consists of all v∈ V such that T(v)=0, that is,

ker(T) ={v∈ V: T(v) =0}.

Then in fact, both im(T) and ker(T) are subspaces of W and V respectively.

Proposition-1: Let V and W be two vector spaces and T: V→ W be linear transformation. Then ker(T)⊑ V, and im(T) ⊑ W, and both are vector space.

Proof:

First consider ker(T), let v_1, v_2 are vectors in ker(T), and scalers a, b ∈ R, we have to prove $av_1+bv_2 \in \ker(T)$.

$T(av_1+bv_2)= T(av_1)+ T(bv_2) = aT(v_1) +bT(v_2) = a.0 +b.0=0$

Thus ker(T) is a subspace of V.

Next suppose $T(\vec{v}_1), T(\vec{v}_2)$ are two vectors in $\text{im}\,(T)$. Then if a, b are scalars,

$$aT(\vec{v}_2) + bT(\vec{v}_2) = T\,(a\vec{v}_1 + b\vec{v}_2)$$

and this last vector is in $\text{im}\,(T)$ by definition.

Example-1:

Let $T : \mathbb{P}_1 \to \mathbb{R}$ be the linear transformation defined by

$$T(p(x)) = p(1) \text{ for all } p(x) \in \mathbb{P}_1.$$

Find the kernel and image of T.

Solution

We will first find the kernel of T. It consists of all polynomials in $\mathbb{P}_1$ that have 1 for a root.

$$\begin{aligned} \ker(T) &= \{p(x) \in \mathbb{P}_1 \mid p(1) = 0\} \\ &= \{ax + b \mid a, b \in \mathbb{R} \text{ and } a + b = 0\} \\ &= \{ax - a \mid a \in \mathbb{R}\} \end{aligned}$$

Therefore a basis for $\ker(T)$ is

$$\{x - 1\}$$

Notice that this is a subspace of $\mathbb{P}_1$.

Now consider the image. It consists of all numbers which can be obtained by evaluating all polynomials in $\mathbb{P}_1$ at 1.

$$\begin{aligned} \text{im}(T) &= \{p(1) \mid p(x) \in \mathbb{P}_1\} \\ &= \{a + b \mid ax + b \in \mathbb{P}_1\} \\ &= \{a + b \mid a, b \in \mathbb{R}\} \\ &= \mathbb{R} \end{aligned}$$

Therefore a basis for $\text{im}(T)$ is

$$\{1\}$$

Example-2:

Let $T : \mathbb{M}_{22} \mapsto \mathbb{R}^2$ be defined by

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a - b \\ c + d \end{bmatrix}$$

Then T is a linear transformation. Find a basis for $\ker(T)$ and $\text{im}(T)$.

Solution

You can verify that T represents a linear transformation.

Now we want to find a way to describe all matrices A such that $T(A) = \vec{0}$, that is the matrices in $\ker(T)$.

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is such a matrix. Then

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-b \\ c+d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The values of a, b, c, d that make this true are given by solutions to the system

$$\begin{aligned} a-b &= 0 \\ c+d &= 0 \end{aligned}$$

The solution is $a = s, b = s, c = t, d = -t$ where s, t are scalars. We can describe $\ker(T)$ as follows.

$$\ker(T) = \left\{ \begin{bmatrix} s & s \\ t & -t \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$$

It is clear that this set is linearly independent and therefore forms a basis for $\ker(T)$.

We now wish to find a basis for $\operatorname{im}(T)$. We can write the image of T as

$$\operatorname{im}(T) = \left\{ \begin{bmatrix} a-b \\ c+d \end{bmatrix} \right\}$$

Notice that this can be written as

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

However, this is clearly not linearly independent. By removing vectors from the set to create an independent set gives a basis of $\operatorname{im}(\mathrm{T})$.

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Notice that these vectors have the same span as the set above but are now linearly independent.

Theorem-2: Let $T:V\to W$ be a linear transformation where V,W are vector spaces. Suppose the dimension of V is n. Then $n=\dim(\ker(T))+\dim(\mathrm{im}(T))$

Proof: From Proposition -1, $\mathrm{im}(T)$ is a subspaces of W, then there exists a basis for $\mathrm{im}(T),\{T(\vec{v}_1),\cdots,T(\vec{v}_r)\}$.. Similarly, ker(T) is a subspace in V, then there is a basis for $\ker(T)$, $\{\vec{u}_1,\cdots,\vec{u}_s\}$. Then if $\vec{v}\in V$, there exist scalars c_i such that:

$$T(\vec{v})=\sum_{i=1}^{r}c_iT(\vec{v}_i)$$

Hence $T\left(\vec{v}-\sum_{i=1}^{r}c_i\vec{v}_i\right)=0$. It follows that $\vec{v}-\sum_{i=1}^{r}c_i\vec{v}_i$ is in $\ker(T)$.

Hence there are scalars a_i such that

$$\vec{v}-\sum_{i=1}^{r}c_i\vec{v}_i=\sum_{j=1}^{s}a_j\vec{u}_j$$

Hence $\vec{v}=\sum_{i=1}^{r}c_i\vec{v}_i+\sum_{j=1}^{s}a_j\vec{u}_j$. Since $\vec{v}$ is arbitrary, it follows that

$$V=\operatorname{span}\{\vec{u}_1,\cdots,\vec{u}_s,\vec{v}_1,\cdots,\vec{v}_r\}$$

If the vectors $\{\vec{u}_1,\cdots,\vec{u}_s,\vec{v}_1,\cdots,\vec{v}_r\}$are linearly independent, then it will follow that this set is a basis. Suppose then that

$$\sum_{i=1}^{r}c_i\vec{v}_i+\sum_{j=1}^{s}a_j\vec{u}_j=0$$

Apply T to both sides to obtain

$$\sum_{i=1}^{r} c_i T(\vec{v}_i) + \sum_{j=1}^{s} a_j T(\vec{u}_j) = \sum_{i=1}^{r} c_i T(\vec{v}_i) = \vec{0}$$

Since $\{T(\vec{v}_1),\cdots,T(\vec{v}_r)\}$ is linearly independent, it follows that each $c_i=0$. Hence $\sum_{j=1}^{s} a_j \vec{u}_j=0$ and so, since the $\{\vec{u}_1,\cdots,\vec{u}_s\}$ are linearly independent, it follows that each $a_j=0$ also. It follows that $\{\vec{u}_1,\cdots,\vec{u}_s,\vec{v}_1,\cdots,\vec{v}_r\}$ is a basis for V and so

$$n= s+ r= \dim(\ker(T))+ \dim(\mathrm{im}(T))$$

Definition: The rank of linear transformation

Let $T:V\to W$ be a linear transformation and suppose V,W are finite dimensional vector spaces. Then the rank of T denoted as $\mathrm{rank}(T)$ is defined as the dimension of $\mathrm{im}(T)$. The nullity of T is the dimension of $\ker(T)$. Thus the above theorem says that $\mathrm{rank}(T)+\dim(\ker(T))=\dim(V)$.

Definition: Let V, be vector spaces with v_1,v_2 be vectors in V. Then a linear transformation $T:V\mapsto W$ is called **one to one** if whenever $v_1\neq v_2$ it follows that

$$T(v_1)\neq T(v_2)$$

A linear transformation $T:V\mapsto W$ is called **onto** if for all $\vec{w}\in\vec{W}$ there exists $\vec{v}\in V$ such that $T(\vec{v})=\vec{w}$.

Lemma-3: The assertion that a linear transformation T is one to one is equivalent to saying that if $T(\vec{v})=\vec{0}$, then $\vec{v}=0$.

Proof:

Suppose first that T is one to one.

$$T(\vec{0})=T(\vec{0}+\vec{0})=T(\vec{0})+T(\vec{0})$$

and so, adding the additive inverse of $T(\vec{0})$ to both sides, one sees that $T(\vec{0})=\vec{0}$. Therefore, if $T(\vec{v})=\vec{0}$, it must be the case that $\vec{v}=\vec{0}$ because it was just shown that $T(\vec{0})=\vec{0}$.

Now suppose that if $T(\vec{v})=\vec{0}$, then $\vec{v}=0$. If $T(\vec{v})=T(\vec{u})$, then $T(\vec{v})-T(\vec{u})=T(\vec{v}-\vec{u})=\vec{0}$ which shows that $\vec{v}-\vec{u}=0$ or in other words, $\vec{v}=\vec{u}$.

Corollary-4:

Let $T:V\to W$ be a linear map where the dimension of V is n and the dimension of W is m. Then T is one to one if and only if $\ker(T)=\{\vec{0}\}$ and T is onto if and only if $\mathrm{rank}(T)=m$.

Proof: The statement $\ker(T)=\{\vec{0}\}$ is equivalent to saying if $T(\vec{v})=\vec{0}$, it follows that $\vec{v}=\vec{0}$ Thus by Lemma-3, T is one to one.

If T is onto, then $\mathrm{im}(T)=W$ and so $\mathrm{rank}(T)$ which is defined as the dimension of $\mathrm{im}(T)$ is m. If $\mathrm{rank}(T)=m$, then by Theorem , since $\mathrm{im}(T)$ is a subspace of W, it follows that $\mathrm{im}(T)=W$.

Example: Let $S:P_2\to M_{2*2}$ be a linear transformation defined by

$$S(ax^2+bx+c)=\begin{bmatrix} a+b & a+c \\ b-c & b+c \end{bmatrix} \text{ for all } ax^2+bx+c\in\mathbb{P}_2.$$

Prove that S is one to one but not onto.

Solution

Here we will determine that S is one to one, but not onto, using the method provided in Corollary -3.

By definition,

$$\ker(S)=\{ax^2+bx+c\in P_2 \mid a+b=0,\ a+c=0,\ b-c=0,\ b+c=0\}.$$

Suppose $p(x)=ax^2+bx+c\in\ker(S)$. This leads to a homogeneous system of four equations in three variables. Putting the augmented matrix in reduced row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since the unique solution is $a=b=c=0$a=b=c=0, $\ker(S)=\{\vec{0}\}$ker(S)={0→}, and thus SS is one-to-one by Corollary.

Similarly, by Corollary , if S is onto it will have $\operatorname{rank}(S)=\dim(M_{22})=4$The image of S is given by

$$\operatorname{im}(S)=\left\{\begin{bmatrix} a+b & a+c \\ b-c & b+c \end{bmatrix}\right\}=\operatorname{span}\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\right\}$$

These matrices are linearly independent which means this set forms a basis for $\operatorname{im}(S)$. Therefore, the dimension of $\operatorname{im}(S)$, also called rank(S), is equal to 3. It follows that S is not onto.