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Soft Separation Axioms and Functions with Soft Closed Graphs

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Abstract. Several notions on soft topology are studied and their basic properties are investigated by using the concept of soft open sets and soft closure operators which are derived from the basics of soft set theory established by Molodtsov [7]. In this paper we introduce some soft separation axioms called Soft R_0 and soft R_1 in soft topological spaces which are defined over an initial universe with a fixed set of parameters. Many characterizations and properties of these spaces are found. Necessary and sufficient conditions for a soft topological space to be a soft R_i for $i = 0, 1$ space are also presented. Furthermore, the concept of functions with soft closed graph and soft cluster sets are defined. Many results on theses two concepts are proved also it is proved that a function has a soft closed graph if and only if its soft cluster set is degenerate.

1. Introduction

The study of soft sets and their properties was initiated by Molodtsov [7] in 1999. After his introduction of soft set theory as a common mathematical application in dealing with the vagueness of not well defined objects, Several researchers applying on formal modeling, reasoning and computing. Shabir, M. and Naz, M. [9] in 2011 described a soft topological space and they introduced so many basic notations and gave their properties in detail. Mathematicians gave in several papers different and, many interesting topological concepts such as, soft connectedness defined by Zorlutuna, I. and Cakir, H. [13] in 2015. In the same year Wang p. and He, J. [11], introduced the concept of soft compactness. Separation axioms, have been extended in soft topological spaces and many types of soft separation axioms are introduced. Husain S, and Ahmed B. [4] in 2015 introduced separation axioms by using distinct point in the universal set while in 2018 Bayramov S. and Aras C. G. [2] defined some separation axioms by using distinct soft points.
The aim of this paper, is to introduce and discuss in detail a study of two soft separation axioms which we call them soft R_0 and R_1 which are defined over an initial universe with a fixed set of parameters. Also we shed light on the notion of a function with a soft closed graph and the soft cluster set of a function, by using notations of soft open sets and soft points defined in [2].

Throughout the present paper, X will be a nonempty initial universal set and E will be a set of parameters and A be a non-empty subset of E. A pair (F, A) is called a soft set over X, where F is a mapping $F : A \to P(X)$. The collection of soft sets (F, A) over a universal set X with the parameter set A is denoted by $SP(X)_A$.

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2. Preliminaries

In this section, we give some definitions and results on soft set theory which will be used in the sequel. First, we recall the following notions more details about properties of such notions can be found in [1], [2], [3], [4] and [6].

Definition 2.1. *For two soft sets (F, A) and (G, B) over a common universe X, we say that (F, A) is a soft subset of (G, B), if*

1. *$A \subseteq B$ and*
2. *for all $e \in A$, $F(e) \subseteq G(e)$*

We write $(F, A) \sqsubseteq (G, B)$.

Definition 2.2. *The complement of a soft set (F, A) is denoted by $(F, A)^c$ or $\tilde{X} \setminus (F, A)$ and is defined by $(F, A)^c = (F^c, A)$ where $F^c : A \to P(X)$ is a mapping given by $F^c(e) = X \setminus F(e)$, for all $e \in A$.*

Definition 2.3. *A soft set (F, A) over X is said to be empty soft set denoted by $\tilde{\phi}$ if for all $e \in A$, $F(e) = \phi$ and (F, A) over X is said to be absolute soft set denoted by $\tilde{A}$ if for all $e \in A$, $F(e) = X$.*

Definition 2.4. *The union of two soft sets of (F, A) and (G, B) over the common universe X is the soft set $(H, C) = (F, A) \sqcup (G, B)$, where $C = A \cup B$ and for all $e \in C$,*

$$H(e) = \begin{cases} F(e) & : \ if \quad e \in A - B \\ G(e) & : \ if \quad e \in B - A \\ F(e) \cup G(e) & : \ if \quad e \in A \cap B \end{cases}$$

In particular, $(F, A) \sqcup (G, A) = F(e) \cup G(e)$ for all $e \in A$.

Definition 2.5. *The intersection (H, C) of two soft sets (F, A) and (G, B) over a common universe X, denoted $(F, A) \sqcap (G, B)$, is defined as $C = A \cap B$, and $H(e) = F(e) \cap G(e)$ for all $e \in C$.*

In particular, $(F, A) \sqcap (G, A) = F(e) \cap G(e)$ for all $e \in A$.

Definition 2.6. *Let $x \in X$, then (x, E) denotes the soft set over X for which $x(e) = \{x\}$, for all $e \in E$.*
Let (F, E) be a soft set over X and $x \in X$. We say that $x \in (F, E)$ read as x belongs to the soft set (F, E) whenever $x \in F(e)$ for all $e \in E$.

Definition 2.7. *The soft set (F, E) is called a soft point, denoted by (x_e, E) or x_e, if for the element $e \in E$, $F(e) = \{x\}$ and $F(e) = \phi$ for all $e \in E \setminus \{e\}$.*
We say that $x_e \in (G, E)$ if $x \in G(e)$.

Two soft points x_e and $y_{e'}$ are distinct if either $x \neq y$ or $e \neq e'$.
It is clear that $x_e \in (x, E)$ always.

Definition 2.8. *[9] Let τ be a collection of soft sets over a universe X with a fixed set A of parameters. Then $\tau \subseteq SP(X)_A$ is called a soft topology if,*

1. *$\tilde{\phi}$ and $\tilde{X}$ belongs to τ.*
2. *The union of any number of soft sets in τ belongs to τ.*
3. *The intersection of any two soft sets in τ belongs to τ.*

The triplet (X, τ, A) is called a soft topological space over X. The members of $\tilde{\tau}$ are called soft open sets in $\tilde{X}$ and complements of them are called soft closed sets in $\tilde{X}$ and they are denoted by $SO(\tilde{X})$ and $SC(\tilde{X})$ respectively. Logical operation on soft set are denoted by usual set theoretical operations with symbol $(\sim)$ above. Soft interior and soft closure are denoted by $\tilde{s}int$ and $\tilde{s}cl$ respectively.

Definition 2.9. *[9] Let (X, τ, A) be a soft topological space and let (G, A) be a soft set. Then*

1. *The soft closure of (G, A) is the soft set*
   $\tilde{s}cl(G, A) = \sqcap\{(K, B) \in SC(\tilde{X}) : (G, A) \sqsubseteq (K, B)\}$
2. *The soft interior of (G, A) is the soft set*
   $\tilde{s}int(G, A) = \sqcup\{(H, B) \in SO(\tilde{X}) : (H, B) \sqsubseteq (G, A)\}$.

Definition 2.10. *[3] Let (X, τ, A) be a soft topological space, (G, A) be a soft set over $\tilde{X}$ and $x_e \in \tilde{X}$. Then (G, A) is said to be a soft neighborhood of x_e if there exists a soft open set (H, A) such that $x_e \in (H, A) \sqsubseteq ((G, A)$.*

Definition 2.11. *[5] Let $SP(X)_A$ and $SP(Y)_B$ be families of soft sets. Let $u : X \to Y$ and $p : A \to B$ be mappings. Then a mapping $f_{pu} : SP(X)_A \to SP(Y)_B$ is defined as:*

1. *Let (F, A) be a soft set in $SP(X)_A$. The image of (F, A) under f_{pu}, written as $f_{pu}(F, A) = (f_{pu}(F), p(A))$, is a soft set in $SP(Y)_B$ such that*

$$f_{pu}(F)(e') = \begin{cases} \bigcup_{e \in p^{-1}(e') \cap A} u(F(e)) & : \ if \quad p^{-1}(e') \cap A \neq \phi \\ \phi & : \ if \quad p^{-1}(e') \cap A = \phi \end{cases}$$

   for all $e' \in B$.

2. *Let (G, B) be a soft set in $SP(Y)_B$. Then the inverse image of (G, B) under f_{pu}, written as $f_{pu}^{-1}(G, B) = (f_{pu}^{-1}(G), p^{-1}(B))$, is a soft set in $SP(X)_A$ such that*

$$f_{pu}^{-1}(G)(e) = \begin{cases} u^{-1}(G(p(e))) & : \ if \quad p(e) \in B \\ \phi & : \ otherwise \end{cases}$$

   for all $e \in A$.

The soft function f_{pu} is called surjective if p and u are surjective and it is called injective if p and u are injective.

Definition 2.12. *[13] Let (X, τ, A) and (Y, μ, B) be two soft topological spaces. A soft mapping $f_{pu} : \tilde{X} \to \tilde{Y}$ is called soft continuous if $f_{pu}^{-1}((G, B)) \in \tau$ for all $(G, B) \in \mu$.*

Definition 2.13. *[12] A soft filter $\mathcal{F}$ converges to a soft point $x_e \in \tilde{X}$ in a soft topological space (X, τ, A), if every soft neighborhood of the soft point x_e belongs to the soft filter $\mathcal{F}$. It is denoted by $\mathcal{F} \to x_e$.*

Definition 2.14. *[11] Let $\mathcal{F}$ be a soft filter in a soft topological space (X, τ, A), a soft point x_e is called soft accumulation point of $\mathcal{F}$, if $x_e \in \tilde{s}cl(G, A)$ for every $(G, A) \in \mathcal{F}$.*

Theorem 2.15. *[11] A soft filter $\mathcal{F}$ converges to a soft point x_e , then x_e is the soft accumulation point of $\mathcal{F}$, if $\mathcal{F}$ is a maximal soft filter and x_e is a soft accumulation point of $\mathcal{F}$, then the soft filter $\mathcal{F}$ converges to the soft point x_e.*

Proposition 2.16. *[8] Let $f_{pu} : SP(X)_A \to SP(Y)_B$ be a soft function. If $\mathcal{F}$ is a soft ultra filter in $\tilde{X}$, then $f_{pu}(\mathcal{F})$ is a soft ultra filter in $\tilde{Y}$.*

Definition 2.17. *[2] A soft topological space (X, τ, A) is said to be:*

1. *Soft T_0, if for each pair of distinct soft points x_e , $y_{e'} \in SP(X)_A$, there exist soft open sets (F, A) and (G, A) such that either $x_e \in (F, A)$ and $y_{e'} \notin (F, A)$ or $y_{e'} \in (G, A)$ and $x_e \notin (G, A)$.*
2. *Soft T_1, if for each pair of distinct soft points x_e , $y_{e'} \in SP(X)_A$, there exist two soft open sets (F, A) and (G, A) such that $x_e \in (F, A)$ but $y_{e'} \notin (F, A)$ and $y_{e'} \in (G, A)$ but $x_e \notin (G, A)$.*
3. *Soft T_2, if for each pair of distinct soft points x_e , $y_{e'} \in SP(X)_A$, there exist two disjoint soft open sets (F, A) and (G, A) containing x_e and $y_{e'}$ respectively.*

Proposition 2.18. *[2]*

1. *Every soft T_2-space $\Rightarrow$ soft T_1-space $\Rightarrow$ soft T_0-space.*
2. *A soft topological space (X, τ, A) is soft T_1 if and only if each soft point is soft closed.*

3. Soft R_0 and soft R_1 Spaces

In this section we introduce new types of soft separation axioms over the universal set X and a fixed set of parameters by using the soft points defined in [2] called soft R_i spaces for $i = 0, 1$. We obtain several properties and characterizations of these spaces.

Definition 3.1. *A soft topological space (X, τ, A) is called soft R_0 if for every soft open set (F, A), $\tilde{s}cl(\{x_e\}) \sqsubseteq (F, A)$ whenever $x_e \in (F, A)$.*

Definition 3.2. *A soft topological space (X, τ, A) is called soft R_1 if for $x_e, y_{e'} \in \tilde{X}$ with $\tilde{s}cl(\{x_e\}) \neq \tilde{s}cl(\{y_{e'}\})$, there exist disjoint soft open sets (F, A) and (G, A) such that $\tilde{s}cl(\{x_e\}) \sqsubseteq (F, A)$ and $\tilde{s}cl(\{y_{e'}\}) \sqsubseteq (G, A)$.*

Definition 3.3. *Let (X, τ, A) be a soft topological space, then the soft kernel of the soft set (F, A) is defined to be the intersection of all soft open sets containing (F, A) and it is denoted by $\tilde{s}ker(F, A)$ that is $\tilde{s}ker(F, A) = \sqcap\{(G, A) \in SO(\tilde{X}) : (F, A) \sqsubseteq (G, A)\}$.*

Lemma 3.4. *For any two soft points x_e and $y_{e'}$ in a soft topological space (X, τ, A), we have $y_{e'} \in \tilde{s}ker(\{x_e\})$ if and only if $x_e \in \tilde{s}cl(\{y_{e'}\})$.*

Proof. Suppose thaty $y_{e'} \notin \tilde{s}ker(\{x_e\})$, then there exists a soft open sets (F, A) containing x_e such that $y_{e'} \notin (F, A)$. Therefore, we have $x_e \notin \tilde{s}cl(\{y_{e'}\})$. The proof of the converse part is similar. □

In the following theorem, we give characterizations of soft R_0 spaces.

Theorem 3.5. *Let (X, τ, A) be a soft topological space, then the following properties are equivalent:*

1. *(X, τ, A) is R_0,*
2. *For any $(K, A) \in SC(\tilde{X})$ and $x_e \notin (K, A)$, there exists $(F, A) \in SC(\tilde{X})$ such that $(K, A) \sqsubseteq (F, A)$ and $x_e \notin (F, A)$,*
3. *For any $(K, A) \in SC(\tilde{X})$ and $x_e \notin (K, A)$, implies that $(K, A) \sqcap \tilde{s}cl(\{x_e\}) = \tilde{\phi}$,*
4. *For any distinct soft points $x_e, y_{e'} \in \tilde{X}$, either $\tilde{s}cl(\{x_e\}) = \tilde{s}cl(\{y_{e'}\})$ or $\tilde{s}cl(\{x_e\}) \sqcap \tilde{s}cl(\{y_{e'}\}) = \tilde{\phi}$.*

Proof. (1)$\Rightarrow$ (2). Let $(K, A) \in SC(\tilde{X})$ and $x_e \notin (K, A)$. Then by (1), $\tilde{s}cl(\{x_e\}) \sqsubseteq \tilde{X} \setminus (K, A)$. Let $(F, A) = \tilde{X} \setminus (K, A)$, then $(F, A) \in SO(\tilde{X})$, $(K, A) \sqsubseteq (F, A)$ and $x_e \notin (F, A)$.

(2)$\Rightarrow$ (3). Let $(K, A) \in SC(\tilde{X})$ and $x_e \notin (K, A)$. Then there exists $(F, A) \in SO(\tilde{X})$ such that $(K, A) \sqsubseteq (F, A)$ and $x_e \notin (F, A)$. Hence, by (2), $(F, A) \sqcap \tilde{s}cl(\{x_e\}) = \tilde{\phi}$, this implies that $(K, A) \sqcap \tilde{s}cl(\{x_e\}) = \tilde{\phi}$.

(3)$\Rightarrow$ (4). Let x_e and $y_{e'}$ be two distinct soft points of $\tilde{X}$. Suppose that $\tilde{s}cl(\{x_e\}) \neq \tilde{s}cl(\{y_{e'}\})$, then there exists a soft point z_c such that $z_c \in \tilde{s}cl(\{x_e\})$ and $z_c \notin \tilde{s}cl(\{y_{e'}\})$ [or $z_c \in \tilde{s}cl(\{y_{e'}\})$ and $z_c \notin \tilde{s}cl(\{x_e\})$] and there exists $(F, A) \in SO(\tilde{X})$ such that $y_{e'} \notin (F, A)$ and $z_c \in (F, A)$, so $x_e \in (F, A)$. Therefore, we get $x_e \notin \tilde{s}cl(\{y_{e'}\})$, then by (3), we obtain $\tilde{s}cl(\{x_e\}) \sqcap \tilde{s}cl(\{y_{e'}\}) = \tilde{\phi}$.

(4)⇒ (1). Let $(F,A) \in SO(\tilde{X})$ and $x_e \in (F,A)$, for each $y_{e'} \notin (F,A)$. Hence, $x_e \neq y_{e'}$ and $x_e \notin \tilde{s}cl(\{y_{e'}\})$, this shows that $\tilde{s}cl(\{x_e\}) \neq \tilde{s}cl(\{y_{e'}\})$. By (4), we have $\tilde{s}cl(\{x_e\}) \sqcap \tilde{s}cl(\{y_{e'}\}) = \tilde{\phi}$, for each $y_{e'} \in \tilde{X} \setminus (F,A)$. On the other hand, since $(F,A) \in SO(\tilde{X})$ and $y_{e'} \in \tilde{X} \setminus (F,A)$, we have $\tilde{s}cl(\{y_{e'}\}) \sqsubseteq \tilde{X} \setminus (F,A)$. Hence $\tilde{X} \setminus (F,A) = \sqcup \tilde{s}cl(\{y_{e'}\})$ where $y_{e'} \in \tilde{X} \setminus (F,A)$. Therefore we obtain that $\tilde{X} \setminus (F,A) \sqcap \tilde{s}cl(\{y_{e'}\}) = \tilde{\phi}$ and $\tilde{s}cl(\{x_e\}) \sqsubseteq (F,A)$. This shows that (X,τ,A) is R_0. □

Lemma 3.6. *Let x_e and $y_{e'}$ be any distinct soft points in a soft topological space (X,τ,A). Then $\tilde{s}ker(\{x_e\}) \neq \tilde{s}ker(\{y_{e'}\})$ if and only if $\tilde{s}cl(\{x_e\}) \neq \tilde{s}cl(\{y_{e'}\})$.*

Proof. Suppose that $\tilde{s}ker(\{x_e\}) \neq \tilde{s}ker(\{y_{e'}\})$. Then there exists a soft point $z_c \in \tilde{X}$ such that $z_c \in \tilde{s}ker(\{x_e\})$ and $z_c \notin \tilde{s}ker(\{y_{e'}\})$. Since $z_c \in \tilde{s}ker(\{x_e\})$ so $\{x_e\} \sqcap \tilde{s}cl(\{z_c\}) \neq \tilde{\phi}$. This implies that $x_e \in \tilde{s}cl(\{z_c\})$ and since $z_c \notin \tilde{s}cl(\{y_{e'}\})$ we have $\{y_{e'}\} \sqcap \tilde{s}cl(\{z_c\}) = \tilde{\phi}$. Since $x_e \in \tilde{s}cl(\{z_c\})$, so $\tilde{s}cl(\{x_e\}) \sqsubseteq \tilde{s}cl(\{z_c\})$ and hence $\{y_{e'}\} \sqcap \tilde{s}cl(\{x_e\}) = \tilde{\phi}$. Therefore, $\tilde{s}cl(\{x_e\}) \neq \tilde{s}cl(\{y_{e'}\})$.
Conversely, Suppose that $\tilde{s}cl(\{x_e\}) \neq \tilde{s}cl(\{y_{e'}\})$. Then there exists a soft point $z_c \in \tilde{X}$ such that $z_c \in \tilde{s}cl(\{x_e\})$ and $z_c \notin \tilde{s}cl(\{y_{e'}\})$. Hence, there exists a soft open set (F,A) containing z_c (and hence x_e) but not $y_{e'}$, that is $y_{e'} \notin \tilde{s}ker(\{x_e\})$. Therefore, $\tilde{s}ker(\{x_e\}) \neq \tilde{s}ker(\{y_{e'}\})$. □

Theorem 3.7. *A soft topological space (X,τ,A) is soft R_0 if and only if for any two distinct soft points x_e , $y_{e'} \in \tilde{X}$, $\tilde{s}ker(\{x_e\}) \neq \tilde{s}ker(\{y_{e'}\})$ implies $\tilde{s}ker(\{x_e\}) \sqcap \tilde{s}ker(\{y_{e'}\}) = \tilde{\phi}$.*

Proof. Necessity, suppose that (X,τ,A) is soft R_0. Thus by Lemma 3.6, for any distinct soft points x_e , $y_{e'} \in \tilde{X}$, if $\tilde{s}ker(\{x_e\}) \neq \tilde{s}ker(\{y_{e'}\})$, then $\tilde{s}cl(\{x_e\}) \neq \tilde{s}cl(\{y_{e'}\})$. Assume that $z_c \in \tilde{s}ker(\{x_e\}) \sqcap \tilde{s}ker(\{y_{e'}\})$. Since $z_c \in \tilde{s}ker(\{x_e\})$ and by Lemma 3.4, it follows that $x_e \in \tilde{s}cl(\{z_c\})$. Since $x_e \in \tilde{s}cl(\{x_e\})$, by Theorem 3.5 $\tilde{s}cl(\{x_e\}) = \tilde{s}cl(\{y_{e'}\})$. Similarly we have $\tilde{s}cl(\{x_e\}) = \tilde{s}cl(\{z_c\}) = \tilde{s}cl(\{y_{e'}\})$, which is a contradiction. Therefore $\tilde{s}ker(\{x_e\}) \sqcap \tilde{s}ker(\{y_{e'}\}) = \tilde{\phi}$.

Sufficiency, let (X,τ,A) be a soft topological space such that for any distinct soft points x_e , $y_{e'} \in \tilde{X}$, $\tilde{s}ker(\{x_e\}) \neq \tilde{s}ker(\{y_{e'}\})$ implies that $\tilde{s}ker(\{x_e\}) \sqcap \tilde{s}ker(\{y_{e'}\}) = \tilde{\phi}$. If $\tilde{s}cl(\{x_e\}) \neq \tilde{s}cl(\{y_{e'}\})$, then by Lemma 3.6, $\tilde{s}ker(\{x_e\}) \neq \tilde{s}ker(\{y_{e'}\})$. Therefore $\tilde{s}ker(\{x_e\}) \sqcap \tilde{s}ker(\{y_{e'}\}) = \tilde{\phi}$. Now if $z_c \in \tilde{s}cl(\{x_e\}) \sqcap \tilde{s}cl(\{y_{e'}\})$, then $z_c \in \tilde{s}cl(\{x_e\})$ implies that $x_e \in \tilde{s}ker(\{z_c\})$ so, $\tilde{s}ker(\{x_e\}) \sqcap \tilde{s}ker(\{z_c\}) \neq \tilde{\phi}$ and by the same way we obtain that $\tilde{s}ker(\{y_{e'}\}) \sqcap \tilde{s}ker(\{z_c\}) \neq \tilde{\phi}$. By hypothesis, we have , $\tilde{s}ker(\{x_e\}) = \tilde{s}ker(\{z_c\}) = \tilde{s}ker(\{y_{e'}\})$, which is contradiction so, $\tilde{s}cl(\{x_e\}) \sqcap \tilde{s}cl(\{y_{e'}\}) = \tilde{\phi}$. Therefore, by Theorem 3.5, (X,τ,A) is soft R_0. □

Theorem 3.8. *Let (X,τ,A) be a soft topological space. Then the following statements are equivalent:*

1. *(X,τ,A) is soft R_0,*
2. *For any non-empty soft set (F,A) , $(G,A) \in SO(\tilde{X})$ with $(F,A) \sqcap (G,A) \neq \tilde{\phi}$, there exists $(K,A) \in SC(\tilde{X})$ such that $(F,A) \sqcap (K,A) \neq \tilde{\phi}$, and $(K,A) \sqsubseteq (G,A)$.*
3. *For any $(G,A) \in SO(\tilde{X})$, $(G,A) = \sqcup\{(K,A) \in SC(\tilde{X}) : (K,A) \sqsubseteq (G,A)\}$,*
4. *For any $(K,A) \in SC(\tilde{X})$, $(K,A) = \sqcap\{(G,A) \in SO(\tilde{X}) : (K,A) \sqsubseteq (G,A)\}$,*
5. *For any $x_e \in \tilde{X}$, $\tilde{s}cl(\{x_e\}) \sqsubseteq \tilde{s}ker(\{x_e\})$.*

Proof. (1)⇒ (2). Let (F,A) be a non empty subset of $\tilde{X}$ and $(G,A) \in SO(\tilde{X})$ such that $(F,A) \sqcap (G,A) \neq \tilde{\phi}$. Let $x_e \in (F,A) \sqcap (G,A)$. Since $x_e \in (G,A) \in SO(\tilde{X})$, so by (1), we have $\tilde{s}cl(\{x_e\}) \sqsubseteq (G,A)$. Set $(K,A) = \tilde{s}cl(\{x_e\})$, then $(K,A) \in SC(\tilde{X})$ such that $(K,A) \sqsubseteq (G,A)$ and $(F,A) \sqcap (K,A) \neq \tilde{\phi}$.

(2)⇒ (3). Let $(G,A) \in SO(\tilde{X})$. Then $\sqcup\{(K,A) \in SC(\tilde{X}) : (K,A) \sqsubseteq (G,A)\} \sqsubseteq (G,A)$. Now let x_e be any soft point of (G,A). By (2), there exists $(K,A) \in SC(\tilde{X})$, such that $x_e \in (K,A)$ and $(K,A) \sqsubseteq (G,A)$. Therefore, we have $x_e \in (K,A) \sqsubseteq \sqcup\{(K,A) \in SC(\tilde{X}) : (K,A) \sqsubseteq (G,A)\}$. Hence $(G,A) = \sqcup\{(K,A) \in SC(\tilde{X}) : (K,A) \sqsubseteq (G,A)\}$.

(3)⇒ (4). Obvious.

(4)⇒ (5). Let x_e be any soft point of $\tilde{X}$ and $y_{e'} \notin \tilde{s}ker(\{x_e\})$. So there exists $(H,A) \in SO(\tilde{X})$ such that $x_e \in (H,A)$ and $y_{e'} \notin (H,A)$. Hence $\tilde{s}cl(\{y_{e'}\}) \sqcap (H,A) = \tilde{\phi}$. By(4), we have $[\sqcap\{(G,A) \in SO(\tilde{X}) : \tilde{s}cl(\{y_{e'}\}) \sqsubseteq$

$(G, A)\}] \sqcap (H, A) = \tilde{\phi}$, so $x_e \notin (G, A)$ and $\tilde{s}cl(\{y_{e'}\}) \sqsubseteq (G, A)$. Therefore, $\tilde{s}cl(\{x_e\}) \sqcap (G, A) = \tilde{\phi}$ and hence $y_{e'} \notin \tilde{s}cl(\{x_e\})$. Consequently we obtain that $\tilde{s}cl(\{x_e\}) \sqsubseteq \tilde{s}ker(\{x_e\})$.

(5)⇒ (1). Let $(G, A) \in SO(\tilde{X})$ and $x_e \in (G, A)$, let $y_{e'} \in \tilde{s}ker(\{x_e\})$. Then $x_e \in \tilde{s}cl(\{y_{e'}\})$ and $y_{e'} \in (G, A)$ this implies that $\tilde{s}ker(\{x_e\}) \sqsubseteq (G, A)$. Therefore, we obtain $\tilde{s}cl(\{x_e\}) \sqsubseteq \tilde{s}ker(\{x_e\}) \sqsubseteq (G, A)$. This shows that (X, τ, A) is soft R_0. □

Theorem 3.9. *A soft topological space (X, τ, A) is soft R_0 if and only if $\tilde{s}cl(\{x_e\}) = \tilde{s}ker(\{x_e\})$, for all $x_e \in \tilde{X}$.*

Proof. Suppose that (X, τ, A) is a soft R_0 space. By Theorem 3.8(5), $\tilde{s}cl(\{x_e\}) \sqsubseteq \tilde{s}ker(\{x_e\})$ for all $x_e \in \tilde{X}$. Let $y_{e'} \in \tilde{s}ker(\{x_e\})$, then $x_e \in \tilde{s}cl(\{y_{e'}\})$ and by Theorem 3.8, $\tilde{s}cl(\{x_e\}) = \tilde{s}cl(\{y_{e'}\})$. Therefore, $y_{e'} \in \tilde{s}cl(\{x_e\})$ and hence $\tilde{s}ker(\{x_e\}) \sqsubseteq \tilde{s}cl(\{x_e\})$. This shows that $\tilde{s}cl(\{x_e\}) = \tilde{s}ker(\{x_e\})$, for all $x_e \in \tilde{X}$.
The converse part follows from Theorem 3.8. □

Theorem 3.10. *For a soft topological space (X, τ, A), the following statement are equivalent :*

1. *(X, τ, A) is a soft R_0 space,*
2. *$x_e \in \tilde{s}cl(\{y_{e'}\})$ if and only if $y_{e'} \in \tilde{s}cl(\{x_e\})$ for any two distinct soft points x_e , $y_{e'} \in \tilde{X}$.*

Proof. (1) ⇒ (2). Assume that (X, τ, A) is a soft R_0 space. Let $x_e \in \tilde{s}cl(\{y_{e'}\})$ and (H, A) be any soft open set containing $y_{e'}$, so by definition, $\tilde{s}cl(\{y_{e'}\}) \sqsubseteq (H, A)$, hence $x_e \in (H, A)$. Therefore, every soft open set containing $y_{e'}$ contains x_e, so $y_{e'} \in \tilde{s}cl(\{x_e\})$.

(2) ⇒ (1). Let (G, A) be any soft open set containing x_e, if $y_{e'} \notin (G, A)$, then $x_e \notin \tilde{s}cl(\{y_{e'}\})$ and By (2), we have $y_{e'} \notin \tilde{s}cl(\{x_e\})$. This implies that $\tilde{s}cl(\{x_e\}) \sqsubseteq (G, A)$, hence (X, τ, A) is a soft R_0 space. □

The following corollary follows from Theorem 3.9 and Theorem 3.10.

Corollary 3.11. *A soft topological space (X, τ, A) is a soft R_0 space if and only if $\tilde{s}ker(\{x_e\}) \neq \tilde{s}ker(\{y_{e'}\})$ for all distinct soft points x_e , $y_{e'} \in \tilde{X}$.*

Lemma 3.12. *Let (X, τ, A) be a soft topological space and $(F, A) \sqsubseteq \tilde{X}$. Then $\tilde{s}ker(F, A) = \{x_e \in \tilde{X} : \tilde{s}cl(\{x_e\}) \sqcap (F, A) \neq \tilde{\phi}\}$.*

Proof. Let $x_e \in \tilde{s}ker(F, A)$ and $\tilde{s}cl(\{x_e\}) \sqcap (F, A) = \tilde{\phi}$. So we have $x_e \notin \tilde{X} \setminus \tilde{s}cl(\{x_e\})$ which is a soft open set containing (F, A). This is impossible, because $x_e \in \tilde{s}ker(F, A)$. Therefore, $\tilde{s}cl(\{x_e\}) \sqcap (F, A) \neq \tilde{\phi}$. On the other hand, if $\tilde{s}cl(\{x_e\}) \sqcap (F, A) \neq \tilde{\phi}$ and $x_e \notin \tilde{s}ker(F, A)$. Then there exists a soft open set (H, A) containing (F, A) and $x_e \notin (H, A)$. Let $y_{e'} \in \tilde{s}cl(\{x_e\}) \sqcap (F, A)$, so (H, A) is a soft neighborhood of $y_{e'}$ in which $x_e \notin (H, A)$, which is a contradiction , so $x_e \in \tilde{s}ker(F, A)$. □

Theorem 3.13. *For a soft topological space (X, τ, A), The following statement are equivalent:*

1. *(X, τ, A) is a soft R_0,*
2. *If (K, A) is soft closed, then $(K, A) = \tilde{s}ker(K, A)$,*
3. *If (K, A) is soft closed and $x_e \in (K, A)$, then $\tilde{s}ker(\{x_e\}) \sqsubseteq (K, A)$,*
4. *If $x_e \in \tilde{X}$, then $\tilde{s}ker(\{x_e\}) \sqsubseteq \tilde{s}cl(\{x_e\})$.*

Proof. (1)⇒ (2). Let (K, A) be a soft closed set and $x_e \notin (K, A)$. Then $\tilde{X} \setminus (K, A)$ is soft open set containing x_e. Since $\tilde{X}$ is a soft R_0 space, so $\tilde{s}cl(\{x_e\}) \sqsubseteq \tilde{X} \setminus (K, A)$, thus $\tilde{s}cl(\{x_e\}) \sqcap (K, A) = \tilde{\phi}$, by Lemma 3.12, $x_e \notin \tilde{s}ker(K, A)$. Therefore, $\tilde{s}ker(\{x_e\}) \sqsubseteq (K, A)$, hence $(K, A) = \tilde{s}ker(K, A)$.

(2)⇒ (3). In general $(F, A) \sqsubseteq (G, A)$ implies that $\tilde{s}ker(F, A) \sqsubseteq \tilde{s}ker(G, A)$. Therefore, from (2), it follows that $\tilde{s}ker(\{x_e\}) \sqsubseteq (K, A)$.

(3)⇒ (4). Since $x_e \in \tilde{s}cl(\{x_e\})$ and $\tilde{s}cl(\{x_e\})$ is soft closed, so by (3), we get that $\tilde{s}ker(\{x_e\}) \sqsubseteq \tilde{s}cl(\{x_e\})$.

(4)⇒ (1). Let $x_e \in \tilde{s}cl(\{y_{e'}\})$, then by Lemma 3.4, $y_{e'} \in \tilde{s}ker(\{x_e\})$. Since $x_e \in \tilde{s}cl(\{x_e\})$ and $\tilde{s}cl(\{x_e\})$ is soft closed, by (4) we obtain $y_{e'} \in \tilde{s}ker(\{x_e\}) \sqsubseteq \tilde{s}cl(\{x_e\})$. Therefore, $x_e \in \tilde{s}cl(\{y_{e'}\})$ implies that $y_{e'} \in \tilde{s}cl(\{x_e\})$. Similarly, if $y_{e'} \in \tilde{s}cl(\{x_e\})$, we get $x_e \in \tilde{s}cl(\{y_{e'}\})$, so by Theorem 3.10, (X, τ, A) is a soft R_0. □

Proposition 3.14. *If a soft topological space (X, τ, A) is a soft R_1 space, then it is soft R_0.*

Proof. Suppose that (X, τ, A) is soft R_1. Let (H, A) be any soft open set containing a soft point x_e. Then for each $y_{e'} \in \tilde{X} \setminus (H, A)$, $\tilde{s}cl(\{x_e\}) \neq \tilde{s}cl(\{y_{e'}\})$. Since (X, τ, A) is soft R_1, there exist two disjoint soft open sets (K, A) and (G, A) such that $\tilde{s}cl(\{x_e\}) \sqsubseteq (K, A)$ and $\tilde{s}cl(\{y_{e'}\}) \sqsubseteq (G, A)$. Let $(F, A) = \sqcup\{(G, A) : y_{e'} \in \tilde{X} \setminus (H, A)\}$, then $\tilde{X} \setminus (H, A) \sqsubseteq (F, A)$, $x_e \notin (F, A)$ and (F, A) is a soft open set. Therefore, $\tilde{s}ker(\{x_e\}) \sqsubseteq \tilde{X} \setminus (F, A) \sqsubseteq (H, A)$. Hence, (X, τ, A) is soft R_0 space. □

Proposition 3.15. *If a soft topological space (X, τ, A) is a soft T_1 space, then it is soft R_0.*

Proof. The proof is obvious since in a soft T_1 space, every soft point is soft closed. □

The following examples shows that the converses of Proposition 3.14 and Proposition 3.15 are not true in general.

Example 3.16. *Let X be any infinite set, $A = \{e_1, e_2\}$ and τ a topology consists of $\tilde{\phi}$, $\tilde{X}$ and all soft sets (F, A), where (F, A) is defined as: $F(e_1) = G$ where G is a subset of X and $X \setminus G$ is finite and $F(e_2) = \phi$.*
Then (X, τ, A) is a soft topological space over X. It can be easily shown that this space is soft R_0 and not soft R_1.

Example 3.17. *Let $X = \{x_1, x_2\}$, $A = \{e_1, e_2\}$ and*
$\tau = \{\tilde{\phi}, \tilde{X}, (F_1, A), (F_2, A)\}$, where
$(F_1, A) = \{(e_1, \{x_2\}), (e_2, \{x_1\})\}$,
$(F_2, A) = \{(e_1, \{x_1\}), (e_2, \{x_2\})\}$. Then (X, τ, A) is a soft topological space over X. This space is both soft R_0 and soft R_1 but it is not soft T_1.

Proposition 3.18. *A soft topological space (X, τ, A) is soft T_1 if and only if it is both soft T_0 and soft R_0.*

Proof. Necessity, follows from Proposition 3.15 and the fact that every soft T_1 is soft T_0.
Sufficiency, assume that (X, τ, A) is both soft T_0 and soft R_0 space. Let x_e,$y_{e'} \in \tilde{X}$ be any pair of distinct soft points. Since (X, τ, A) is both soft T_0, there exists a soft open set (H, A) which contains one of the points but not the other. Suppose that $x_e \in (H, A)$ and $y_{e'} \notin (H, A)$. Since $\tilde{X}$ is soft R_0, then $\tilde{s}cl(\{x_e\}) \sqsubseteq (H, A)$. As $y_{e'} \notin (H, A)$ implies $y_{e'} \notin \tilde{s}cl(\{x_e\})$. Hence $y_{e'} \in (G, A) = \tilde{X} \setminus \tilde{s}cl(\{x_e\})$ and it is clear that $x_e \notin (G, A)$, this implies that there exist soft open sets (G, A) and (H, A) containing x_e and $y_{e'}$ respectively such that $x_e \notin (G, A)$ and $y_{e'} \notin (H, A)$. Therefore, (X, τ, A) is soft T_1. □

Theorem 3.19. *A soft topological space (X, τ, A) is soft R_0 if and only if for every soft closed set (K, A) and $x_e \notin (K, A)$, there exists a soft open set (G, A) such that $x_e \notin (G, A)$ and $(K, A) \sqsubseteq (G, A)$.*

Proof. Let (X, τ, A) be soft R_0, $x_e \in \tilde{X}$ and (K, A) be soft closed subset such that $x_e \notin (K, A)$. Then $\tilde{X} \setminus (K, A)$ is a soft open set containing x_e. Hence, $\tilde{s}cl(\{x_e\}) \sqsubseteq \tilde{X} \setminus (K, A)$ and then $(K, A) \sqsubseteq \tilde{X} \setminus \tilde{s}cl(\{x_e\})$. Now let $(G, A) = \tilde{X} \setminus \tilde{s}cl(\{x_e\})$, then (G, A) is a soft open set does not contains x_e and $(K, A) \sqsubseteq (G, A)$.
Conversely: Let $x_e \in (G, A)$ where (G, A) is a soft open set in $\tilde{X}$. Then $\tilde{X} \setminus (G, A)$ is a soft closed set and $x_e \notin (G, A)$ implies by hypothesis, that there exists a soft open set (H, A) such that $x_e \notin (H, A)$ and $\tilde{X} \setminus (G, A) \sqsubseteq (H, A)$. Now $\tilde{X} \setminus (H, A) \sqsubseteq (G, A)$ and $x_e \in \tilde{X} \setminus (H, A)$, but $\tilde{X} \setminus (H, A)$ is soft closed, hence $\tilde{s}cl(\{x_e\}) \sqsubseteq \tilde{X} \setminus (H, A) \sqsubseteq (G, A)$ this implies that (X, τ, A) is soft R_0. □

Theorem 3.20. *A soft topological space (X, τ, A) is soft T_2 if and only if it is both soft T_0 and soft R_1.*

Proof. Let $\tilde{X}$ be soft T_2, then from Proposition 2.18(1), it is soft T_0 and to show $\tilde{X}$ is a soft R_1 space, let x_e , $y_{e'} \in \tilde{X}$ such that $\tilde{s}cl(\{x_e\}) \neq \tilde{s}cl(\{y_{e'}\})$ and since $\tilde{X}$ is soft T_1 space, so by Proposition 2.18(2), every singleton set in $\tilde{X}$ is soft closed, that is $\tilde{s}cl(\{x_e\}) = \{x_e\}$ and $\tilde{s}cl(\{y_{e'}\}) = \{y_{e'}\}$ and since $\tilde{X}$ is a soft T_2 space so there exist two disjoint soft open sets (G, A) and (H, A) such that $\{x_e\} = \tilde{s}cl(\{x_e\}) \sqsubseteq (G, A)$ and $\{y_{e'}\} = \tilde{s}cl(\{y_{e'}\}) \sqsubseteq (H, A)$. Thus $\tilde{X}$ is soft R_1 space.
Conversely, let $\tilde{X}$ be soft T_0 and soft R_1 and x_e , $y_{e'} \in \tilde{X}$ such that $x_e \neq y_{e'}$. Since $\tilde{X}$ is soft T_0, so by Definition 2.17, there exists a soft open set (G, A) containing one of the points but not the other. Suppose that $x_e \in (G, A)$ and $y_{e'} \notin (G, A)$ implies that $(G, A) \sqcap \{y_{e'}\} = \tilde{\phi}$, and then $x_e \notin \tilde{s}cl(\{y_{e'}\})$ this implies that $\tilde{s}cl(\{x_e\}) \neq \tilde{s}cl(\{y_{e'}\})$ and since $\tilde{X}$ is soft R_1, so there exist two disjoint soft open sets (G, A) and (H, A) such that $\tilde{s}cl(\{x_e\}) \sqsubseteq (G, A)$ and $\tilde{s}cl(\{y_{e'}\}) \sqsubseteq (H, A)$ implies that $x_e \in (G, A)$ and $y_{e'} \in (H, A)$. Thus (X, τ, A) is soft T_2. □

4. Functions with soft closed graphs

In this section we introduce the concepts of the soft closed graph and the soft cluster set of a function $f_{pu} : (X, \tau, A) \to (Y, \mu, B)$ and give several related properties. Also, we proved that a function has a soft closed graph if and only if its soft cluster set at a fixed point degenerate.

Definition 4.1. *The graph of a function $f_{pu} : (X, \tau, A) \to (Y, \mu, B)$ is denoted by $G(f_{pu})$ and it is soft closed in $\tilde{X} \times \tilde{Y}$, if for each $(x_e, y_{e'}) \in G(f_{pu})$, there exist two soft open sets (U, A) and (V, B) containing x_e and $y_{e'}$ respectively such that $(U, A) \times (V, B) \sqcap G(f_{pu}) = \tilde{\phi}$.*

The following lemma follows from Definition 4.1.

Lemma 4.2. *The function $f_{pu} : (X, \tau, A) \to (Y, \mu, B)$ has a soft closed graph if and only if for each $x_e \in \tilde{X}$ and $y_{e'} \in \tilde{Y}$ such that $f_{pu}(x_e) \neq y_{e'}$, there exist two soft open sets (U, A) and (V, B) containing x_e and $y_{e'}$ respectively such that $f_{pu}((U, A) \sqcap (V, B)) = \tilde{\phi}$.*

Proposition 4.3. *If $f_{pu} : (X, \tau, A) \to (Y, \mu, B)$ is an injective function with soft closed graph, then $\tilde{X}$ is a soft T_1 space.*

Proof. Let $(x_1)_{e_1}$ and $(x_2)_{e_2}$ be two distinct points in $\tilde{X}$. Since f_{pu} is injective, so $f_{pu}((x_1)_{e_1}) \neq f_{pu}((x_2)_{e_2})$. Let $f_{pu}((x_1)_{e_1}) = (y_1)_{e'_1}$ thus $f_{pu}((x_2)_{e_2}) = (y_2)_{e'_2}$, by Lemma 4.2, there exist two soft open sets (U, A) and (V, B) containing $(x_2)_{e_2}$ and $(y_1)_{e'_1}$ respectively, such that $f_{pu}((U, A)) \sqcap (V, B) = \tilde{\phi}$, then $(U, A) \sqcap f_{pu}^{-1}(V, B) = \tilde{\phi}$. We get $f_{pu}((x_1)_{e_1}) = (y_1)_{e'_1} \in (V, B)$, then $(x_1)_{e_1} \in f_{pu}^{-1}(V, B)$ implies that, $(x_1)_{e_1} \notin (U, A)$. Again consider $f_{pu}((x_2)_{e_2}) = (y_2)_{e'_2}$ implies that $f_{pu}((x_1)_{e_1}) = (y_2)_{e'_2}$. Since the graph of f_{pu} is soft closed, so there exist soft open sets (U_1, A) containing $(x_1)_{e_1}$ and (V_1, A) containing $(y_2)_{e'_2}$ such that $f_{pu}((U_1, A) \sqcap (V_1, A)) = \tilde{\phi}$, so $(U_1, A) \sqcap f_{pu}^{-1}(V_1, A) = \tilde{\phi}$, we obtain $f_{pu}((x_2)_{e_2}) = (y_2)_{e'_2} \in (V_1, A)$, so $(x_2)_{e_2} \in f_{pu}^{-1}(V_1, A)$ and hence $(x_2)_{e_2} \notin (U_1, A)$. Therefore, $\tilde{X}$ is a soft T_1 space. □

Proposition 4.4. *If $f_{pu} : (X, \tau, A) \to (Y, \mu, B)$ is a surjective function with soft closed graph, then $\tilde{Y}$ is a soft T_1 space.*

Proof. Let $(y_1)_{e'_1}$ and $(y_2)_{e'_2}$ be two distinct points in $\tilde{Y}$. Since f_{pu} is surjective, so there exists a point $(x_1)_{e_1} \in \tilde{X}$, with $f_{pu}((x_1)_{e_1}) = (y_1)_{e'_1}$ then $f_{pu}((x_1)_{e_1}) \neq (y_2)_{e'_2}$. Therefore, $((x_1)_{e_1}, (y_2)_{e'_2}) \notin G(f_{pu})$, since the graph of f_{pu} is soft closed, by Lemma 4.2, there exist two soft open sets (U_1, A) containing $(x_1)_{e_1}$ and (V_2, A) containing $(y_2)_{e'_2}$ such that $f_{pu}((U_1, A) \sqcap (V_2, A)) = \tilde{\phi}$, We obtain $(y_2)_{e'_2} \in (V_2, A)$, and $(x_1)_{e_1} \in (U_1, A)$ implies that $f_{pu}((x_1)_{e_1}) \in f_{pu}(U_1, A)$, so $(y_1)_{e'_1} \notin (V_2, A)$. Again from the surjectivity of f_{pu} there exists $(x_2)_{e_2} \in \tilde{X}$ with $f_{pu}((x_2)_{e_2}) = (y_2)_{e'_2}$, then $f_{pu}((x_2)_{e_2}) \neq (y_1)_{e'_1}$, thus $((x_2)_{e_2}, (y_1)_{e'_1}) \notin G(f_{pu})$ and the graph of f_{pu} is soft closed, there exist two soft open sets (U_2, A) and (V_1, A) containing $(x_2)_{e_2}$ and $(y_1)_{e'_1}$ respectively, such that $f_{pu}((U_2, A)) \sqcap (V_1, A) = \tilde{\phi}$. We get $(x_2)_{e_2} \in (U_2, A)$ implies that $(y_2)_{e'_2} = f_{pu}((x_2)_{e_2}) \in f_{pu}((U_2, A))$, so $(y_2)_{e'_2} \notin (V_1, A)$. It follows that $\tilde{Y}$ is soft T_1. □

The following corollary follows from Proposition 4.3 and Proposition 4.4.

Corollary 4.5. *If $f_{pu} : (X, \tau, A) \to (Y, \mu, B)$ is a bijective function with soft closed graph, then both $\tilde{X}$ and $\tilde{Y}$ are soft T_1 spaces.*

Proposition 4.6. *If $f_{pu} : (X, \tau, A) \to (Y, \mu, B)$ is is soft continuous and $\tilde{Y}$ is a soft T_2 space, then $G(f_{pu})$ is soft closed.*

Proof. Let $(x_e, y_{e'}) \notin G(f_{pu})$. Then $f_{pu}(x_e) \neq y_{e'}$ and since $\tilde{Y}$ is a soft T_2 space, there exist soft open sets (U, A) and (V, B) such that $f_{pu}(x_e) \in (U, A)$, $y_{e'} \in (V, B)$ and $((U, A) \sqcap (V, B)) = \tilde{\phi}$. Since f_{pu} is soft continuous, so there exists a soft open set (G, A) containing x_e such that $f_{pu}(G, A) \sqsubseteq (U, A)$. Hence, we have $f_{pu}((G, A)) \sqcap (V, B) = \tilde{\phi}$. Therefore, by Lemma 4.2, $G(f_{pu})$ is soft closed. □

Definition 4.7. *Let $f_{pu} : (X, \tau, A) \to (Y, \mu, B)$ be any soft function. The soft cluster set of f_{pu} at x_e is denoted by $\tilde{s}C(f_{pu}, x_e)$ is the set of all points $y_{e'} \in \tilde{Y}$ such that whenever there exists a filter base $\mathcal{F}$ soft converges to the point x_e, the filter base $< f_{pu}(\mathcal{F}) >$ soft converges to the point $y_{e'}$.*

The following theorem is a characterization of soft cluster set of a function f_{pu}.

Theorem 4.8. *Let $f_{pu} : (X, \tau, A) \to (Y, \mu, B)$ be any function and $x_e \in \tilde{X}$. Then the following statements are equivalent:*

1. $y_{e'} \in \tilde{s}C(f_{pu}, x_e)$,
2. $y_{e'} \in \sqcap\{\tilde{s}cl(f_{pu}((U, A)) : \forall \quad (U, A) \in \tilde{s}N(x_e)\}$,
3. $f_{pu}(\tilde{s}N(x_e))$ *is soft accumulates to* $y_{e'}$,
4. $f^{-1}_{pu}(\tilde{s}N(y_{e'}))$ *is soft accumulates to* x_e,
5. $x_e \in \sqcap\{\tilde{s}cl(f^{-1}_{pu}((V, B)) : \forall \quad (V, B) \in \tilde{s}N(y_{e'})\}$,

Proof. (1)⇒ (2). Let $y_{e'} \in \tilde{s}C(f_{pu}, x_e)$, so there exists a filter base $\mathcal{F}$ soft converges to the point x_e and $< f_{pu}(\mathcal{F}) >$ soft converges to the point $y_{e'}$. Suppose that (U, A) is any soft open set containing x_e, since $\mathcal{F}$ soft converges to x_e, so $(U, A) \in \mathcal{F}$ and $< f_{pu}(\mathcal{F}) >$ soft converges to the point $y_{e'}$. Therefore, $y_{e'} \in \tilde{s}cl(f_{pu}(x_e))$ implies that $y_{e'} \in \tilde{s}cl(f_{pu}((U, A))$ for each soft open set (U, A) containing x_e. Hence $y_{e'} \in \sqcap\{\tilde{s}cl(f_{pu}((U, A)) : \forall \quad (U, A) \in \tilde{s}N(x_e)\}$.

(2)⇒ (3). Let $y_{e'} \in \sqcap\{\tilde{s}cl(f_{pu}((U, A)) : \forall \quad (U, A) \in \tilde{s}N(x_e)$, so $y_{e'} \in \tilde{s}cl(f_{pu}((U, A))$ for each soft open set (U, A) containing x_e. Then $f_{pu}((U, A)) \sqcap (V, B) \neq \tilde{\phi}$ for each soft open sets (U, A) containing x_e and (V, B) containing $y_{e'}$ implies that $f_{pu}(\tilde{s}N(x_e) \sqcap (V, B) \neq \tilde{\phi}$ for every soft open set (V, B) containing $y_{e'}$ and hence $f_{pu}(\tilde{s}N(x_e))$ is soft accumulates to $y_{e'}$.

(3)⇒ (4). Let $f_{pu}(\tilde{s}N(x_e))$ is soft accumulates to $y_{e'}$, which implies that $f_{pu}(\tilde{s}N(x_e) \sqcap (V, B) \neq \tilde{\phi}$ for each $(V, B) \in \tilde{s}N(y_{e'})$, thus $(U, A) \sqcap f^{-1}_{pu}(\tilde{s}N(y_{e'})) \neq \tilde{\phi}$ for every soft open set (U, A) in $\tilde{X}$ containing x_e it follows that $f^{-1}_{pu}(\tilde{s}N(y_{e'}))$ is soft accumulates to x_e.

(4)⇒ (5). Assume that $f^{-1}_{pu}(\tilde{s}N(y_{e'}))$ is soft accumulates to x_e, so $(U, A) \sqcap f^{-1}_{pu}(\tilde{s}N(y_{e'})) \neq \tilde{\phi}$, for every soft open set (U, A) containing x_e. It follows that $(U, A) \sqcap f^{-1}_{pu}((V, B)) \neq \tilde{\phi}$ for every soft open set (U, A) in $\tilde{X}$ containing x_e and (V, B) in $\tilde{Y}$ containing $y_{e'}$. Hence, $x_e \in \tilde{s}cl(f^{-1}_{pu}((V, B)))$ for every soft open set (V, B) containing $y_{e'}$. This shows that $x_e \in \sqcap\{\tilde{s}cl(f^{-1}_{pu}((V, B)) : \forall \quad (V, B) \in \tilde{s}N(y_{e'})\}$.

(5)⇒ (1). Since $SO(\tilde{X}, x_e)$ is a filter base which is soft converges to the point x_e, then $SO(\tilde{X}, x_e)$ is contained in an ultra filter $\mathcal{F}$ on $\tilde{X}$ which is also soft converges to x_e, so there exists $(F, A) \in \mathcal{F}$ such that $(F, A) \sqsubseteq (U, A)$ for every soft open set (U, A) in $\tilde{X}$ containing x_e, so $(U, A) \in \mathcal{F}$. By (5), $x_e \in \tilde{s}cl(f^{-1}_{pu}((V, B))$ for every soft open set (V, B) containing $y_{e'}$. So $(U, A) \sqcap f^{-1}_{pu}((V, B)) \neq \tilde{\phi}$, implies that $f_{pu}((U, A)) \sqcap (V, B) \neq \tilde{\phi}$ for every soft open set (U, A) containing x_e and (V, B) containing $y_{e'}$. Hence, by Proposition 2.16, $f_{pu}(\mathcal{F})$ is an ultra filter base which is soft accumulates to the soft point $y_{e'}$,so by Theorem 2.15, $f_{pu}(\mathcal{F})$ is soft convergent to $y_{e'}$. □

By using the concept of soft cluster set of a function $f_{pu} : (X, \tau, A) \to (Y, \mu, B)$ we obtain some properties and characterizations of the soft graph of the function f_{pu}. We start by the following result which is a relation between a function with soft closed graph and soft cluster set of the function. First we introduce the concept of degenerate soft cluster set.

Definition 4.9. *Let (X, τ, A) be a soft topological space, the degenerate soft cluster set of f_{pu} is a soft cluster set which contains exactly one element.*

Theorem 4.10. *Let $f_{pu} : (X, \tau, A) \to (Y, \mu, B)$ be any function, the graph of f_{pu} is soft closed if and only if the soft cluster set of f_{pu} at x_e is degenerate.*

Proof. Let $y_{e'}$ be any point in $\tilde{Y}$ different from $f_{pu}(x_e)$. By Lemma 4.2, there exist $(U, A) \in SO(\tilde{X}, x_e)$ and $(V, B) \in SO(\tilde{Y}, y_{e'})$ such that $f_{pu}((U, A)) \sqcap (V, B) \neq \tilde{\phi}$. This implies that $y_{e'} \notin \tilde{s}cl(f_{pu}((U, A))$ and by Theorem 4.8(2), $y_{e'} \in \tilde{s}C(f_{pu}, x_e)$. Hence $\tilde{s}C(f_{pu}, x_e) = \{f_{pu}(x_e)\}$.
Conversely, suppose $G(f_{pu})$ is not soft closed. This implies that there exists $(x_e, y_{e'}) \notin G(f_{pu})$ such that $f_{pu}((U, A)) \sqcap (V, B) \neq \tilde{\phi}$ for every soft open set (U, A) in $\tilde{X}$ containing x_e and (V, B) in $\tilde{Y}$ containing $y_{e'}$, then $y_{e'} \in \tilde{s}cl(f_{pu}((U, A))$ for each soft open set (U, A) containing x_e by Theorem 4.8(2), $y_{e'} \in \tilde{s}C(f_{pu}, x_e)$ which contradicts the fact that $\tilde{s}C(f_{pu}, x_e) = \{f_{pu}(x_e)\}$. Therefore, $G(f_{pu})$ is soft closed. □

The following results follows from Theorem 4.10 and the definition of soft cluster set of a function.

Corollary 4.11. *The function $f_{pu} : (X, \tau, A) \to (Y, \mu, B)$ has a soft closed graph if and only if there exists a filter base $\mathcal{F}$ soft converges to a point x_e, the filter base $f_{pu}(\mathcal{F})$ soft converges to a point $y_{e'}$ and $y_{e'} = f_{pu}(x_e)$..*

Corollary 4.12. *Let $f_{pu} : (X, \tau, A) \to (Y, \mu, B)$ be any function. The graph of f_{pu} is soft closed if and only if*

$$f_{pu}(x_e) \in \sqcap\{\tilde{s}cl(f_{pu}((U, A)) : \forall \quad (U, A) \in \tilde{s}N(x_e)\}$$

.

5. Conclusion

In the last two decades the soft set theory, new definitions, examples, new classes of soft sets, and properties for mappings between different classes of soft sets are introduced and studied. After then, the theory of soft topological spaces is investigated. This paper continues the study of the theory of soft topological spaces. In section 3, we present the notion of soft R_i spaces for $i = 0, 1$, we get several characterizations and properties of these two spaces. In section 4, we obtain nice results concerning functions with soft closed graphs and its relations with the notion of soft convergence and cluster set of a function.

References

[1] M. Akdag, and A. Ozkan, On Soft Preopen Sets and Soft Pre Separation Axioms, Gazi University Journal of Science, 27(4)(2014) 1077–1083.
[2] S. Bayramov and and C. G. Aras, A new approach to separability and compactness in soft topological spaces, (TWMS) Journal of Pure and Applied Mathematics 9(1) (2018) 82–93.
[3] S. Hussain and B Ahmad, Some properties of soft topological spaces, Computers and Mathematics with Applications 62(2011) 4058-4067.
[4] S. Hussain and B. Ahmad , Soft separation axioms in soft topological spaces, Hacettepe Journal of Mathematics and Statistics 44(3) (2015) 559–568.
[5] A. Kharal and B. Ahmad, Mappings on soft classes, New Mathematics and Natural Computation 7(3) (2011) 471–481. https://doi.org/10.1142/S1793005711002025.
[6] P. K. Maji, R. Biswas and R. Roy, Soft set theory, Computers and Mathematics with Applications 45(2003) 555–562.
[7] D. Molodtsov, Soft set theory-first results, Computers and Mathematics with Applications 37 (1999) 19–31.
[8] R. Sahin and A. Kucuk, Soft filters and their convergence properties, Annals of Fuzzy Mathematics and Informatics 6(3) (2013) 529–543.

[9] M. Shabir and M. Naz, On soft topological spaces, Computers and Mathematics with Applications 61 (2011) 1786–1799. https://doi.org/10.1016/j.camwa.2011.02.006.

[10] M. E. Al-Shafei, M. Abo-Elhamayel and T. M. Al-Shami, Partial soft separation axioms and soft compact spaces, Filomat 32:13(2018) 4755–4771.
https://doi.org/10.2298/FIL1813755E

[11] P. Wang and J. He, Characterization of soft compact spaces based on soft filter, Journal of Theoretical and Applied Information Technology 79(3)(2015) 431–436.

[12] S. Yuksel, N. Tozlu and Z. G. Ergul, Soft filter, Mathematical Sciences (2014) 8:119.
https://link.springer.com/article/10.1007/s40096-014-0119-4

[13] I. Zorlutuna, M. Akdag, W. K. Min and S. Atmaca, Remarks on soft topological spaces, Annals of Fuzzy Mathematics and Informatics 3(2) (2012) 171–185.

[14] I. Zorlutuna and H. Cakr, On Continuity of Soft Mappings, Applied Mathematics and Information Sciences 9(1) (2015) 403–409.