CHAPTER ONE

Differential Equations (DEs.).

Equation: Equations describe the relations between the dependent and independent variables. An equal sign "=" is required in every equation.

Differential Equations: Equations that involve dependent variables and their *derivatives* with respect to the independent variables are called *differential* equations.

Differential Equation:- Is an equation consist algebraic function or non-algebraic function or both of them which contains derivative. Divide DE. in two types
1- Ordinary DEs. .
2- Partial DEs.

Ordinary Differential Equations: Differential equations that involve only *one independent variable* are called *ordinary* differential equations. Or (Is a relation between independent variable and the dependent variable which contains Ordinary derivatives).

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Partial Differential Equations: Is a relation between independent variables and the dependent variable which contains Partial derivatives.

Partial Differential Equations: Differential equations that involve *two or more independent variables* are called *partial* differential equations.

Examples:

1) $\left(\dfrac{\partial^2 z}{\partial x \partial y}\right) + 6xy\left(\dfrac{\partial^2 z}{\partial x^2}\right) = 2x$ $\quad P$

2) $\dfrac{dy}{dx} + 2xy = e^x$ $\quad O.$

3) $\dfrac{d^2 y}{dx^2} + y\cos x \dfrac{dy}{dx} = \tanh x$ $\quad O.$

4) $\left(\dfrac{\partial f}{\partial x}\right)^2 - 3x\dfrac{\partial f}{\partial x} = \cos x$ $\quad P.$

5) $y''' - 3yy'' - 2x(y')^3 = 7$ $\quad O.$

Order of ODE:-Is an order of the highest derivative in which occurs.

Degree of ODE:-Is the highest power of the highest derivative in which occurs.

Examples:

1) $\left(\frac{\partial^2 z}{\partial x \partial y}\right) + 6xy\left(\frac{\partial^2 z}{\partial x^2}\right) = 2x$ $\quad Or.2 \quad D.1$

2) $\frac{dy}{dx} + 2xy = e^x$ $\quad Or.1 \quad D.1$

3) $\frac{d^2 y}{dx^2} + y\cos x\frac{dy}{dx} = \tanh x$ $\quad Or.2 \quad D.1$

4) $\left(\frac{\partial f}{\partial x}\right)^2 - 3x\frac{\partial f}{\partial x} = \cos x$ $\quad Or.1 \quad D.2$

5) $y''' - 3yy'' - 2x(y')^3 = 7$ $\quad Or.3 \quad D.1$

Note: if the O.D.E containing the roots or rational power then to find the degree of this ODE. can be reduces this roots or rational powers.

Example:1) $y''' + 6\sqrt{(y')^2 + y^2} = 0$

$$\frac{d^3y}{dx^3} + 6\sqrt{\left(\frac{dy}{dx}\right)^2 + y^2} = 0$$

$$\frac{d^3y}{dx^3} = -6\sqrt{\left(\frac{dy}{dx}\right)^2 + y^2}$$ Square both side of the equation

$$\left(\frac{d^3y}{dx^3}\right)^2 = 36\left[\left(\frac{dy}{dx}\right)^2 + y^2\right]$$ **Order 3, degree 2**

2) $\sqrt[3]{(y'')^2} = \sqrt{1+(y')^2}$ (Ordinary DE.) **Or. 2** **D.4**

Linear O.D.E

A.D.E. in any order is said to be linear if satisfies:-

1) The dep.v .is exist and of the first degree.

2) The derivatives $y', y'', y''', \ldots$ exist and each of them of the first degree.

3) The dep.v. and the derivatives not multiply by each other.

Note: If one of these conditions is not satisfied, then the equation considerate non- Linear.

Examples:

1) $y'' + 4xy' + 2y = \tan x$ (L.)

2) $\frac{d^2y}{dx^2} + 3x\left(\frac{dy}{dx}\right)^2 + 5y = x^2$ (Non.L.)

3) $y'' + 4yy' + 2y = \cos x$ (Non.L.)

4) $y'' + 3xy' + 5y^6 = x^2$ (Non.L.)

Quasi-linear D.E

Def: For a non-linear DE., if there are no multiplications among all dependent variables and their derivatives in the highest derivative term, the differential is considered to be quasi-linear.

Example: 1) $(y''')^3 e^x + y'' + y = 0$ $\quad (non-linear\ but\ is\ a\ quasi-linear)$

2) $(y''')e^x + xyy'' + yy' = \sin x$ $\quad (non-linear\ but\ is\ a\ quasi-linear)$

3) $y(y''')e^x + e^x(y'')^3 + (y'x)^5 + y = 7x$ $\quad (non-linear, nonquasi-linear)$

The Solutions of ODE

Solutions : A functional relation between the dependent variable y and the independent variable x that satisfies the given ODE in some interval J is called a solution of the given ODE on J .

Type of solutions:
1) The general solution, denoted by y_G
2) The particular solution, denote1d by y_p.
3) The singular solution, denoted by S.

General Solution: Solutions obtained from integrating the differential equations are called general solutions. The general solution of a nth order ordinary differential equation contains n arbitrary constants resulting from integrating n times.

Example: solve: $\dfrac{dy}{dx} = 2y$, $\dfrac{dy}{y} = 2dx$ $\quad (y \neq 0)$

Lin y=2x+c $\quad$ $y = e^{2x}.e^{c}$, suppose $e^{c}=k$

$y = k\, e^{2x}$ $\quad$ Is the G. solution.

Particular Solution: Particular solutions are the solutions obtained by assigning(giving) specific values to the arbitrary constants in the general solutions.

Example /pervious example

Choose (x, y) = (0, 1) $\Rightarrow k = 1$

$y = e^{2x}$ is the particular solution.

Note: To find the particular solution of O.D.E in the G. solution by giving the value of arbitrary constant as follows.
By giving the value of dependent variable and the value of independent variable (Represent the integral curve).Obtained the value of arbitrary constant and substituted in the G. solution ,we get the particular solution .
(These values of (x, y) is called initial conditions or boundary conditions).

Initial Condition: Constrains that are specified at the initial point, generally time point, are called initial conditions. Problems with specified initial conditions are called initial value problems (IVP).

Example: $y'' + 3y' + 2y = 0$, $y(0) = 1$, $y'(0) = 2$

Boundary Condition: Constrains that are specified at the boundary points, generally space points, are called boundary conditions. Problems with specified boundary conditions are called boundary value problems (BVP).

Boundary value problem:- (B.V.P)

Is an equation in which contains the boundary conditions.

Example: $y'' + y = 0 \quad y(0) = 0, y\left(\frac{\pi}{2}\right) = 2.$,

Singular Solutions: Solutions that cannot be expressed by the general solutions are called singular solutions.

Example: $y' = 2y \Rightarrow \frac{dy}{y} = 2dx$ If y=0 then $\left(\frac{dy}{dx} = 2y\right)$ is undefined, then y =0 is the singular solution.

Elimination the arbitrary constants (Finding the O.D.E if the G. Solution is Exist)

1) Differential the G. solution n-times (the number of arbitrary constant =the order of O.D.E)
2) We obtain (n+1) of equations.
3) we can solve these equations by simultaneous way (or by determinant way).
4) Substituted the value of arbitrary constants in the G. solution, we get the O.D.E.

Example: find the D.E if the G. solution is y=Ax^3+Bx^2+Cx ------ (1)

Solution: y'=3Ax^2+ 2Bx +c ------ (2)

y''=6Ax+2B

y'''=6A

A=$\frac{1}{6}y'''$, B=$\frac{1}{2}y''$-$\frac{1}{2}x\,y'''$, C=$y' + \frac{1}{2}x^2 y'''$ - $x\,y''$

Subset value of A, B and C in the G. solution , we get the O.D.E of third order

Y = $\frac{1}{6}y''' x^3 + (\frac{1}{2}y'' - \frac{1}{2}y''' x)x^2 + (y' + \frac{1}{2}x^2 y''' - x\,y'')\,x$

Example: find the O.D.E if the G. solution is ,(H.W)

1) y=Ae^{-x}+Be^{x}
2) y=C_1sin x +C_2cos x
3) y=Ae^{x}+Bx^2+C x

CHAPTER TWO

Methods for Solving the O.D.E in the first order and first degree

1) Separation variable (separable).
2) Substitution method.
3) Homogenous D.E.
4) Non-homogenous D.E. of linear coefficient.
5) Exact D.E.
6) Non-exact D.E. (Integrating factor).

1) Separable DEs

Separable Function: A function F(x, y) is called separable if can be written of the form.

F(x, y) = g(x).h(y) or $= \dfrac{h(y)}{g(x)}$; $g(x) \neq 0$.

Where g is a function of (x) only and h is a function of (y)only.

Ex1: F(x, y) = x^2y SF

Ex2: F(x, y) = $x^2 \pm$ y Non-SF

(SDE): A D.E M(x, y) dx + N(x, y) dy =0 is called separable if both M and N are S. functions.

Example: solve: sinxcosydx+sinycosxdy=0

Solution: $\dfrac{\sin x}{\cos x}dx + \dfrac{\sin y}{\cos y}dy = 0$

$\int \sin x \dfrac{1}{\cos x}dx + \int \sin y \dfrac{1}{\cos y}dy = 0$

$\ln|\cos x| + \ln|\cos y| = c$

$\ln(\cos x \cos y) = c$ $e^c = c_1$

$\cos x \cos y = c_1$

$\cos y = c_1 \dfrac{1}{\cos x}$

$\cos^{-1}\cos y = \cos^{-1}\left(\dfrac{c_1}{\cos x}\right)$

$y = \cos^{-1}\left(\dfrac{c_1}{\cos x}\right)$ is the g.solution.

Example :(x^2 - y) dx +(x+1) dy =0 Non-S.D.E

Example: solve: $4xdy - ydx = x^2dy$

$$(4x - x^2)dy - ydx = 0 \quad S.D.E \Rightarrow \int \frac{dy}{y} - \int \frac{dx}{(4x - x^2)} = 0$$

$$\ln y - \int \frac{dx}{x(4-x)} = o \Rightarrow \ln y - \int [\frac{A}{x} + \frac{B}{(4-x)}]dx = c$$

Example: solve :

$$x^3y^3dx+\left(x^2+x^2y\right)dy=0$$

$$\{x^3y^3dx+x^2(1+y)dy=0\} \qquad \frac{1}{y^3x^2}$$

$$xdx+\frac{(1+y)}{y^3}dy=0\Rightarrow\frac{x^2}{2}+y^{-3}(1+y)dy=0$$

$$\frac{x^2}{2}+\frac{y^{-2}}{-2}+\frac{y^{-1}}{-1}=C_1\Rightarrow\frac{x^2}{2}-\frac{y^{-2}}{2}-y^{-1}=C_1$$

H.w/ Solve:

1) x^3 dx + $(y+1)^2$ dy = 0

2) x^2 (y+1) dx + y (x-1)dy = 0

2) Substitution method. If the D.E. of the form $y'=(ax+by)----(1)$ Non-Separable D.E., then

Suppose $ax+by=z----(2)$

$$adx+bdy=dz$$

$$dy=\frac{dz-adx}{b}----(3)$$

$$\frac{dy}{dx}=\frac{\frac{dz}{dx}-a}{b}----(4)$$

By subset .equation (2) and (4) in D.E (1) we get.

$$\frac{\frac{dz}{dx}-a}{b}=z\Rightarrow\frac{dz}{dx}=bz+a$$

Is S.D.E can be solved by previous way.

Example: Solve:- $y'=(x+y)^2(is\ not\ SDE)$

Solution: Suppose $x+y=z$ ----- (2)

$$dx+dy=dz\Rightarrow 1+y'=z'\Rightarrow y'=z'-1\text{-----}(3)\Rightarrow z'-1=(z)^2$$

$$z'=(z^2+1)\text{ is SDE}\Rightarrow[\frac{dz}{dx}=(z^2+1)] \quad \frac{1}{1+z^2} \quad \Rightarrow \int\frac{dz}{1+z^2}=\int dx$$

$$\Rightarrow\tan^{-1}(z)=x+c, z=\tan(x+c), y=\tan(x+c)-x\ is\ theg.sol.$$

Example:- Solve: $y'=x^2-8xy+16y^2$ **(H.W.)**

3) Homogenous D.E

Homogenous function: A function F(x, y) is called Homogenous function of n-th degree if satisfy the relation.

$\forall t \in R \Rightarrow F(tx,ty) = t^n f(x,y)$

Example: 1) Test the function

f(x, y) = $x^2 y$, t∈R

F (tx, ty) = $(tx)^2(ty)$

= (t^2x^2) (ty)

= t^3f(x, y) $\Rightarrow$ F is H.F. of 3-rd degree

Example: 2) Test the function

F(x, y) = $\frac{x}{y}\sin(\frac{y}{x}) + \cosh(\frac{y}{x}), \quad t \in R$

$$F(tx,ty) = \frac{tx}{ty}\sin(\frac{ty}{tx}) + \cosh(\frac{ty}{tx}),$$

$F(tx,ty) = t^0 f(x,y)$ $\quad F$ *is H.F of zero* - degree.

Example: 3) Test the function

$F(x,y) = x$

$F(tx,ty) = (tx) = t(x) = t(F(x,y))$

F is H.F. of first degree.

Homogeneous D.E :A D.E. M(x, y) dx+ N(x, y) dy=0 is called H.D.E if both functions M and N are H. functions of the same degree (i.e.)[the H. degree of M=the H. degree of N]

Example: 1) $(x^2 + y^2)dx + x.y\,dy = 0$.

M(x, y) = $(x^2 + y^2)$

M (tx,ty) = $(tx^2 + ty^2) = t^2(x^2 + y^2) = t^2 F(x,y)$ is H.F. of 2-nd degree.

N $(x,y) = (xy)$

N (tx,ty)= $(tx)(ty)$

$= t^2(xy) = t^2 N(x,y)$ $\quad$ N is H.F. of 2-nd degree

DE is H. of 2-nd degree .

Example:- Solve:-

$(x^2 + yx)dx + y^3 dy = 0$. $\quad$ H.W

$(x^2 + y)dx + y^2 dy = 0$. $\quad$ H.W

Note: Every homogeneous function of two variables of n-th degree can be reduced into H.function of one variable of the same degree, by using the relation $\left(V = \frac{y}{x}\right)$.

Solution: Since $F(x,y)$ is H.F of n-th degree, then

$F(tx,ty) = t^n(F(x,y))$ $\quad ----(1)$ t∈R by def. of H.F.

Suppose $t = \frac{1}{x}$ $\quad$ ----- (2)

Sub. equ.(2) in equ.(1) ,we get

$$F(\frac{1}{x}x,\frac{1}{x}y)=\left(\frac{1}{x}\right)^n(F(x,y)) \quad ----(3)$$

Let $V=\frac{y}{x}$ ----- (4)

Sub. equ.(4) in equ.(3) ,we get

$$F(1,v)=\frac{1}{x^n}(F(x,y))$$

$F(x,y)=x^n(F(1,v))$ is a H.F of n-th degree of one variable

Theorem: Every homog.DE of n-th degree can be reduced into separable DE by using the relation $\frac{y}{x}=v$.

Proof :Since M(x, y) dx+ N(x, y) dy=0 -----(1) is HDE of n-th degree.

∴M and N are HF. of n-th degree.

M $(tx,ty)=t^n M(x,y)$

N $(tx,ty)=t^n N(x,y)$ (2)

by note.

$$M(x,y)=x^n(M(1,v)) \quad ----(3)$$

$$N(x,y)=x^n(N(1,v)) \quad ----(4)$$

And must be $N(x,y)\neq 0$

$\frac{y}{x}=v$

$y=xv$ (5)

$dy=xdv+vdx$

Divide equ (3)by (4) ,we get

$$\frac{M(x,y)}{N(x,y)}=\frac{x^n(M(1,v))}{x^n(N(1,v))}=\frac{M(1,v)}{N(1,v)}=g(1,v)\text{-----(6)}$$

Divide equ (1) by $N(x,y)\neq 0$

$$\frac{M(x,y)}{N(x,y)}dx+dy=0\text{-----(7)}$$

Sub. equ.(5),(6)in D.E (7) ,we get the S.D.E.

$g(1,v)dx+(xdv+vdx)=0$ S.D.E.

$[g(1,v)+v]\,dx+xdv=0$

$\frac{dx}{x}+\frac{dv}{v+g(1,v)}=0$,Can be solved by integration immediately, and finally subst the value of v by $(\frac{y}{x})$ we get the general solution.

Example: solve: - $(x^2+y^2)dx+xy\,dy=0 \quad ----(1)$ is HDE of 2-nd degree.

Solution: suppose $\frac{y}{x}=v$

$y=xv$ (2)

$dy=xdv+vdx$

Sub. equ. (2) in equ. (1), we get the S.D.E.

$$(x^2+x^2v^2)dx+x^2v(x\,dv+v\,dx)=0$$

$$[x^2(1+v^2)+x^2v^2]dx+x^3v\,dv=0$$

$$([x^2(1+v^2)+x^2v^2]dx+x^3v\,dv=0) \qquad \frac{1}{x^3(1+v^2)}$$

$$\frac{dx}{x}+\frac{v}{(1+2v^2)}dv=0$$

$$lin\,x+\frac{1}{4}lin(1+2v^2)=c$$

$$4\,lin\,x+lin(1+2v^2)=4c$$

$$lin\,x^4+lin(1+2v^2)=4c$$

$$lin[x^4(1+2v^2)]=4c$$

$x^4(1+2v^2)=c_1$ where $e^{4c}=c_1$

Subst. $v=\frac{y}{x}$

$$x^4(1+2\frac{y^2}{x^2})=c_1$$

$x^4+2y^2x^2=c_1$ *is the G.solu.of DE* (1).

Examples:- Solve the following DEs**. (H.W.)**

1) $2e^{\frac{y}{x}}(1-\frac{y}{x})dx+(1+2e^{\frac{y}{x}})dy=0$.

2) $(2x+3y)dx+(y-x)dy=0$

3) $(3x^2-y^2)dx-xydy=0$

Example: Test the functions **in DE.(1)**

$$(x+y+1)dx+xdy=0---(1)$$

$$M(x,y)=x+y+1$$

$M(tx,ty)=tx+ty+1=t(x,y+\frac{1}{t})\neq t\,M(x,y)$ is non-homog.func.

N(x,y)=x

$N(tx,ty)=t\,x=t\,N(x,y)\;H.F.of\;first$ degree

DE (1) is non-Homog. DE.

4)Non-Homogenous DE With Linear Coefficients:-

Th general form of non-homog DE with linear coefficients is

$$(ax+by+c)dx+(\alpha x+\beta y+\gamma)dy=0------(1)$$

where $a,b,c,\alpha,\beta\,and\,\gamma$ are constants.

To changing the Non-H.D.E. into H.D.E.or S.DE. , there exist two cases:-

Case1:- if $m_1\neq m_2$ (two lines are intersected).

$ax+by+c=0----(2) \Rightarrow m_1=\frac{-a}{b}$

$\alpha x+\beta y+\gamma=0----(3) \Rightarrow m_2=\frac{-\alpha}{\beta}$

$m_1 \neq m_2$

(h, k) the intersection point

$ah+bk+c=0$

$\alpha h+\beta k+\gamma=0$ (4)

Suppose $x=x_1+h$ and $y=y_1+k$

$dx=dx_1$ and $dy=dy_1$ (5)

Subst .equ (5) in D.E (1), we get the H.D.E

$(a(x_1+h)+b(y_1+k)+c)dx_1+(\alpha(x_1+h)+\beta(y_1+k)+\gamma)dy_1=0$

$[(ax_1+by_1)+(bk+ah+c)]dx_1+[(\beta y_1+\alpha x_1)+(\beta k+\alpha h+\gamma)]dy_1=0$

$(ax_1+by_1)dx_1+(\beta y_1+\alpha x_1)dy_1=0----(6)$ is H.D.E.

equ. (6) can be solving by homogenous method by supposes

$v=\frac{y_1}{x_1}$

and $y_1=x_1v$ (7)

$dy_1=x_1dv+vdx_1$

Subst equ. (7) in equ (6) we get the separable D.E., we can solving by integration immediately we get the G.solution .

In which contains two variable x_1, v subst. values of v ,

$x_1=x-h$, and $y_1=y-k$

Example: solve $(2y-x-5)dx+(3x+y+1)dy=0-----(1)$ $(Non-H.D.E)$

Solution:

$2y-x-5=0$

$3x+y+1=0$

$m_1=\frac{-(-1)}{2}=\frac{1}{2}, m_1=\frac{-(-3)}{-1}=-3 \Rightarrow \frac{1}{2} \neq -3 \Rightarrow m_1 \neq m_2 \quad \exists (h,k)=(-1,2)$

2y – x - 5 = 0

$3x+y+1=0$

$-7x-7=0 \Rightarrow (x=-1),$

$(y=2)$

Suppose $x=x_1+h$ and $y=y_1+k$

$x=x_1-1$ and $y=y_1+2$ (A)

$dx=dx_1$ and $dy=dy_1$

Subst equ. (A) in equ (1) we get the H. D. E.

$(2(y_1+2)-(x_1-1)-5)dx_1+(3(x_1-1)+(y_1+2)+1)dy_1=0$

$(2y_1+4-x_1+1-5)dx_1+(3x_1-3+y_1+2+1)dy_1=0$

$(2y_1-x_1)dx_1+(3x+y_1)dy_1=0---(2)$ is HDE

Suppose $v = \frac{y_1}{x_1}$

and $y_1 = x_1 v$ (B)

$dy_1 = x_1 dv + v dx_1$

Subst equ. (B) in equ (2) we get the separable DE

$(2vx_1 - x_1)dx_1 + (3x_1 + vx_1)(x_1 dv + vdx_1) = 0$ Is SDE

$[x_1(v^2 + 5v - 1)dx_1 + x_1^2(3+v)dv = 0]$ $\frac{1}{(v^2+5v-1)}$

$\int \frac{dx}{x} + \int \frac{3+v}{(v^2+5v-1)} dv = 0$

(H.W) solve $(x - 2y - 5)dx - (-3x + 6y + 1)dy = 0$

Case 2: if $m_1 = m_2$ (the two lines are not intersected at the point, but are parallel.

Then suppose $ax + by = z$

$adx + bdy = dz$

$dy = \frac{dz - adx}{b}$ (k)

Subst. equ. (k) in D.E. (1) we get the S.D.E.

$(z + c)dx + (mz + \gamma)\left(\frac{dz - adx}{b}\right) = 0$

Can be solved, by integration immediately finally subst. the value of (z), we get G.solution.
m= is multiple of (z)

Example: solve: (3x-6y+5) dx+ (12x-24y-2) dy =0 -----(1)

Solution: $m_1 = \frac{-3}{-6} = \frac{1}{2}$

$m_2 = \frac{-12}{-24} = \frac{1}{2} \Rightarrow m_1 = m_2$

Then suppose

$3x - 6y = z$

$3dx - 6dy = dz$

$dy = \frac{3dx - dz}{6}$ (2)

Subst. equ. (2) in D.E. (1) we get the S.D.E.

$(z + 5)dx + (4z - 2)(\frac{3dx - dz}{6}) = 0$ is S.D.E.

Example:-Solve the following DEs.(H.W.)

1) (x-y+2) dx+ (-x-y+2) dy =0

2) (6x-8y-5) dy= (3x-4y-2) dx

Note /If the function of the form f (x y) or the DE.of theform y f(x y) dx + x g(x y) dy=0 (1) Suppose yx=v

$y=\frac{v}{x}$

$dy=\frac{xdv-vdx}{x^2}$ (2)

Subst. equ.(2) in the DE.(1) ,we get $\frac{v}{x}f(v)dx+xg(v)(\frac{xdv-vdx}{x^2})$ *is SDE*

Example: solve:- $y(xy+1)dx+x(1+xy+x^2y^2)dy=0$ ------(1)

Solution :

Suppose yx=v

$y=\frac{v}{x}$

$dy=\frac{xdv-vdx}{x^2}$ (2)

Subst. equ.(2) in the D.E.(1)

$\frac{v}{x}(v+1)dx+x(1+v+v^2)(\frac{xdv-vdx}{x^2})=0 \Rightarrow xv(v+1)dx+x(1+v+v^2)(xdv-vdx)=0$

$(xv^2+xv-xv-xv^2-x\,v^3)dx+x^2(1+v+v^2)dv=0$

$\int(-xv^3)dx+x^2(1+v+v^2)dv)=\int 0 \quad S.D.E$

$(\int(-xv^3)dx+x^2(1+v+v^2)dv)=\int 0) \quad \frac{1}{-v^3x^2}\Rightarrow\int\frac{dx}{x}-\int\frac{1+v+v^2}{v^3}dv$

5) Exact D.E

Exact function: A function f(x, y) is called exact function if can be written of the form

$df(x,y)=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0$

Exact DE: A D.E. M(x, y) dx+ N(x, y) dy=0 ----- (1) is called exact D.E. if $\exists\ df(x,y)$ such that

$df(x,y)=M(x,y)dx+N(x,y)dy=0$

$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=M\,dx+N\,dy=0$ (by def. of exact function)

$\frac{\partial f}{\partial x}=M(x,y)$ and $\frac{\partial f}{\partial y}=N(x,y)$

Theorem: The necessary and sufficient conditions of D.E $M(x,y)dx+N(x,y)dy=0$

is Exact D.E. if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ such that M, N, M_x, M_y, N_x, N_y,… are continuous in R.

Proof : part 1 / " The Necessary condition"

Suppose $M(x,y)dx+N(x,y)dy=0$ -----(1) is Exact D.E we must prove $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Since D.E (1) is Exact D.E then $\exists\ df(x,y)=M(x,y)dx+N(x,y)dy=0$ by def. of total function.

$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=M(x,y)dx+N(x,y)dy=0$

$\frac{\partial f}{\partial x} = M(x,y)$ ------(2) and $\frac{\partial f}{\partial y} = N(x,y)$ -------(3)

Differentiation equ. (2) with respect to (y) ,we get $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial M}{\partial y}$

Differentiation equ. (3) with respect to (x) ,we get $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial N}{\partial x}$

Since $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial x\,\partial y}$ (if f is a continuous function in R and all partial derivatives are continuous in R then $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial x\,\partial y}$)

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is proved .

Part 2) (The sufficient condition)

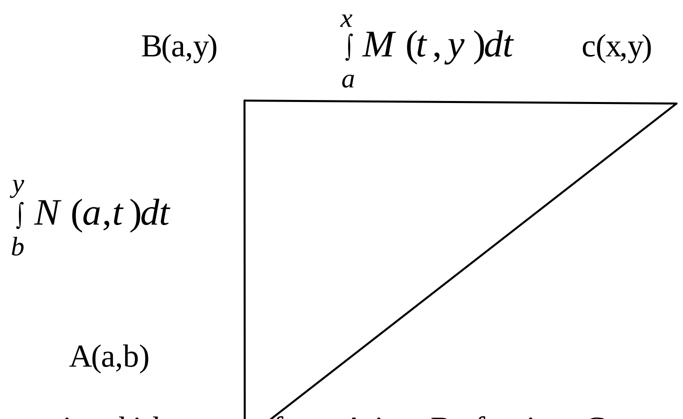
Suppose $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ we must prove $M(x,y)dx + N(x,y)dy = 0$ -----(1) is Exact D.E

To prove D.E (1) is Exact if $\exists\ df(x,y)$ such that $df(x,y)$ =D.E. (1)

Method to find this total function ($df(x,y)$)completed a follows choose arbitrary point A(a,b) in R and by making the integral form A(a,b) in to B(a,y)[such that doesn't any changing in x-axis], and the second integral passes from B(a,y) into c(x,y) [such that doesn't any changing in y-axis]

$$f(x,y) = \int_a^x M(t,y)dt + \int_b^y N(a,t)dt \quad -----(*)$$

A first integral passes horizontally and the second integral is vertical.



Both integrals is linear in which passes from A into B after into C .

The total function is appears in equ (*) remain to prove that $\frac{\partial f}{\partial x} = M(x,y)$ and $\frac{\partial f}{\partial y} = N(x,y)$

Differentiation equ (*) with respect to(x)

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\int_a^x M(t,y)dt + \frac{\partial}{\partial x}\int_b^y N(a,t)dt \quad -----(*)$$

Since the second integral doesn't any changing in x-axis then this integral equal to zero.

$$\frac{\partial f}{\partial x}=M(t,y)\Big]_a^x$$

$$\frac{\partial f}{\partial x}=M(x,y)-M(a,y)$$

$$\frac{\partial f}{\partial x}=M(x,y)$$

Differentiation equ (*) with respect to(y)

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\int_a^x M(t,y)dt+\frac{\partial}{\partial y}\int_b^y N(a,t)dt \quad -----(**)$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ by supposes and subst in equ(**) ,we get

$$\frac{\partial f}{\partial y}=\int_a^x \frac{\partial N}{\partial x}(t,y)dt+\frac{\partial}{\partial y}\int_b^y N(a,t)dt$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial x}\int_a^x N(t,y)dt+\frac{\partial}{\partial y}\int_b^y N(a,t)dt$$

$$\frac{\partial f}{\partial y}=N(x,y)\Big]_a^x+N(a,t)\Big]_b^y$$

$$\frac{\partial f}{\partial y}=N(x,y)-N(a,y)+N(a,y)-N(a,b)$$

$$\frac{\partial f}{\partial y}=N(x,y)$$

$\therefore M(x,y)dx+N(x,y)dy=0$ is exact

Note: To check the problem is exact we can using $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Example: solve: (x+y) dx +(x-y) dy =0 -------- (1)

Solution: $M(x,y)dx+N(x,y)dy=0$

$$\therefore M(x,y)=x+y \Rightarrow \frac{\partial M}{\partial y}=1 \quad \text{and} \quad N(x,y)=x-y \Rightarrow \frac{\partial N}{\partial x}=1$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ than D.E. (1) is exact

$df(x,y)$=(x+y) dx +(x-y) dy =0 s.t

$$\frac{\partial f}{\partial x}=x+y \text{ -----(1)}$$

$$\frac{\partial f}{\partial y}=x-y \text{ -----(2)}$$

Integration both side of equ. (1) with respect to (x) and choosing the arbitrary function of (y) only

$$f(x,y)=\frac{x^2}{2}+yx+h(y) \quad -----(4),\ \frac{\partial f}{\partial y}=x+\frac{\partial}{\partial y}h(y) \text{ -------- (5)}$$

x-y = $x+\frac{\partial}{\partial y}h(y) \Rightarrow$ -y = $\frac{\partial}{\partial y}h(y) \Rightarrow -\int y\,dy = \int h'(y)dy \Rightarrow -\frac{y^2}{2}+c = h(y)$

$\therefore f(x,y) = \frac{x^2}{2} + yx - \frac{y^2}{2} + c$

Example: solve: $2x(ye^{x^2}-1)dx + e^{x^2}dy = 0 ----(1)$

Solution: $\frac{\partial M}{\partial y} = 2xe^{x^2}$, $\frac{\partial N}{\partial x} = 2xe^{x^2}$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ D.E. is exact , $\therefore df(x,y) = 2x(ye^{x^2}-1)dx + e^{x^2}dy = 0$

$\frac{\partial f}{\partial x} = 2x(ye^{x^2}-1)$ ----- (2) and $\frac{\partial f}{\partial y} = e^{x^2}$ -------(3)

$f(x,y) = ye^{x^2} + Q$ (x) where Q is arbitrary function of (x) only

$\frac{\partial f}{\partial x} = y\,2xe^{x^2} + \frac{\partial}{\partial x}Q(x)$ ----- (4) .In both equation (2) and (4) ,we get

$2xye^{x^2} - 2x = y\,2xe^{x^2} + \frac{\partial}{\partial x}Q(x) \Rightarrow Q'(x) = -2x \Rightarrow \int \frac{\partial}{\partial x}Q(x)\,dx = \int -2x\,dx$

$\Rightarrow Q(x) = -x^2 + c \Rightarrow f(x,y) = ye^{x^2} - x^2 + c$ be the G.solution.

H.W.: Solve the following DES:- (1) $y^3 \sin 2x\,dx - \frac{3}{2}y^2 \cos 2x\,dy = 0$.

(2)$(3x^2+3xy^2)dx+(3x^2y - 3y^2+2y)dy = 0$. (3)$(2ye^{2x} + 2x\cos y)dx + (e^{2x} - x^2\sin y)dy = 0$

5) Integrating Factor (IF)

Def.: When multiply both side of Non-Exact DE by a suitable factor in which changed into Exact DE then the suitable factor is called integrating factor and symbols by (IF).

$M(x,y)dx + N(x,y)dy = 0$ ----- (1) is Non-Exact D.E.

$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

I $(M(x,y)dx + N(x,y)dy = 0)$ ----- (2) is Exact D.E.

$\Rightarrow \frac{\partial(IM)}{\partial y} = \frac{\partial(IN)}{\partial x}$

$I\frac{\partial M}{\partial y} + M\frac{\partial I}{\partial y} = I\frac{\partial N}{\partial x} + N\frac{\partial I}{\partial x}$

$I\,M_y + M\,I_y = I\,N_x + N\,I_x$

$I(M_y - N_x) = N\,I_x - M\,I_y$ -----(*)

$M_y - N_x = N\frac{I_x}{I} - M\frac{I_y}{I}$

$M_y - N_x = N(\ln I)_x - M(\ln I)_y$ -----(**)

There exist four cases:

Case 1: if I is a function of (x) only then (I_y =0)

$\therefore [I(M_y - N_x) = N\, I_x]$ $\quad \frac{1}{N\,I}$

$$\frac{(M_y - N_x)}{N} = \frac{I_x}{I}$$

$$\int \frac{(M_y - N_x)}{N} dx = \int \frac{\partial I}{I\,\partial x} dx$$

$$\int \frac{M_y - N_x}{N}) dx = \int \frac{\partial I}{I\,\partial x} dx$$

$$e^{\int (\frac{M_y - N_x}{N}) dx} = e^{lin\, I(x)}$$

$I(x) = e^{\int (\frac{M_y - N_x}{N}) dx}$ is an I.F in which a function of (x) only

Case 2: If I is a function of (y) only, than $(I_x = 0)$ H.W

Example: solve:- $(x^4 + y^4)dx - xy^3 dy = 0$------(1)

Solution: $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ D.E. (1) is Non-Exact.

$I(x) = e^{\int (\frac{M_y - N_x}{N}) dx}$ is an I.F of function in which this function of one variable (x) only

Test $= \frac{M_y - N_x}{N}$

Test $[I(x)] = \frac{4y^3 + y^3}{-xy^3} \Rightarrow = \frac{5y^3}{-xy^3} \Rightarrow = \frac{-5}{x}$

$I(x) = e^{\int (\frac{-5}{x}) dx} \Rightarrow I(x) = e^{-5 lin\, x} \Rightarrow I(x) = e^{lin\, x^5} \Rightarrow I(x) = x^{-5}$ is I.F.

multiply both side of Non-Exact D.E (1) by suitable factor $I(x) = x^{-5}$,we get an exact D.E.

$x^{-5}(x^4 + y^4)dx - x^{-5}xy^3 dy = 0$----- (2)

$(x^{-1} + x^{-5}y^4)dx - x^{-4}y^3 dy = 0$

$\frac{\partial M_1}{\partial y} = 4y^3 x^{-5}$, $\frac{\partial N_1}{\partial x} = 4y^3 x^{-5}$

$\Rightarrow \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

D.E. (2) is Exact D.E. then we can solved by exact method.

Example: solve:- $(2xy^2 - 2y)dx + (3x^2 y - 4x)dy = 0$------ (1)

$\frac{\partial M}{\partial y}$ =4xy-2, $\frac{\partial N}{\partial x}$ =6xy-4 $\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Test $= \frac{N_x - M_y}{M} = \frac{6xy - 4 - 4xy + 2}{2xy^2 - 2y}$

Test $[I(y)] = \frac{2(xy-1)}{2y(xy-1)} = \frac{1}{y}$

$I(y) = e^{\int(\frac{1}{y})dy} \Rightarrow I(y) = e^{\ln y} \Rightarrow I(y) = y$ is I.F. of equ (1)

multiply both side of Non-Exact D.E (1) bya suitable factor $I(y) = y$,we get an exact D.E.

$(2xy^3 - 2y^2)dx + (3x^2y^2 - 4yx)dy = 0$ ------(2)

$\frac{\partial M_1}{\partial y} = 6yx^2 - 4y \quad and \quad \frac{\partial N_1}{\partial x} = 6yx^2 - 4y$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ $\therefore$ D.E. (2) is Exact D.E.

$\therefore \exists\, df(x,y)$ = D.E. (2)

$\frac{\partial f}{\partial x} = 2y^3x - 2y^2 ----(3)\ , \quad \frac{\partial f}{\partial y} = 3y^2x^2 - 4xy\ ----(4)$

.

Case 3: write an I.F in which I is a function of (x+y). H.W

Case 4: if I is a function of $(x^2 + y^2)$,

Then suppose $z = x^2 + y^2 \Rightarrow dz = 2xdx + 2ydy \Rightarrow \frac{\partial z}{\partial x} = 2x \ \& \frac{\partial z}{\partial y} = 2y\ ------(4)$

From equ(**)

$(\ln I)_x = \frac{\partial}{\partial x}(\ln I) = \frac{\partial}{\partial z}(\ln I)\frac{\partial z}{\partial x}$ (by chain rule)

$\Rightarrow (\ln I)_x = 2x\frac{\partial}{\partial z}(\ln I)$ -----(5)

&

$(\ln I)_y = \frac{\partial}{\partial y}(\ln I) = \frac{\partial}{\partial z}(\ln I)\frac{\partial z}{\partial y}$ (by chain rule)

$\Rightarrow (\ln I)_y = 2y\frac{\partial}{\partial z}(\ln I)$ -----(6)

Subst. equ(4) and equ(5) ,(6) in equ (**) ,we obtain

$M_y - N_x = N(\ln I)_x - M(\ln I)_y ---(**)$

$M_y - N_x = N\,2x\frac{\partial}{\partial z}(\ln I) - M\,2y\frac{\partial}{\partial z}(\ln I)$

$M_y - N_x = (2xN - 2yM)\frac{\partial}{\partial z}(\ln I \Rightarrow \frac{M_y - N_x}{(2xN - 2yM)} = \frac{\partial}{\partial z}(\ln I)(x^2 + y^2)$

$\int \frac{M_y - N_x}{(2xN - 2yM)}dz = \int \frac{\partial}{\partial z}(\ln I(z))dz \Rightarrow \ln I(z) = \int \frac{M_y - N_x}{(2xN - 2yM)}dz$

$e^{\ln I(z)} = e^{\int \frac{M_y - N_x}{(2xN - 2yM)}dz} \Rightarrow e^{\ln I(x^2+y^2)} = e^{\int \frac{M_y - N_x}{(2xN - 2yM)}(2xdx + 2ydy)}$ is an integrating factor in which a function of (x^2+y^2) s.t can be changing into exact D.E.

solve:(1) $(x^2 + y^2 - x)dx - ydy = 0$, **(2)** $xdx + (y + 4y(x^2 + y^2))dy = 0$ H.W

Function	Test	I.F
I(x)	$\frac{M_y - N_x}{N}$	$e^{\int (\frac{M_y - N_x}{N}) dx}$
I(y)	$\frac{N_x - M_y}{M}$	$e^{\int (\frac{N_x - M_y}{M}) dy}$
I(x+y)	$\frac{M_y - N_x}{N - M}$	$e^{\int \frac{M_y - N_x}{N - M}(dx + dy)}$
I(x^2+y^2)	$\frac{M_y - N_x}{(2xN - 2yM)}$	$e^{\int \frac{M_y - N_x}{(2xN - 2yM)}(2xdx + 2ydy)}$

Linear First Order Ordinary D.E

The general form of first order and first degree of ordinary DE is $a(x)\frac{dy}{dx} + b(x)y = c(x) ----(1)$;

where a(x) $\neq 0$ and a,b,c are functions of x only.

$$\frac{dy}{dx} + \frac{b(x)}{a(x)}y = \frac{c(x)}{a(x)}$$

$\frac{dy}{dx} + p(x)y = Q(x)$ is the standard form of **LFODE**

$y' + p(x)y = Q(x) ----(2)$ (non-homog. & non-exact)

Suppose $y' + p(x)y = 0 ----(3)$

$\frac{dy}{dx} + p(x)y = 0]$ $\frac{dx}{y}$ SDE

$$\frac{dy}{y} + p(x)dx = 0 \Rightarrow lin\, y + \int p(x)dx = c \Rightarrow lin\, y = c - \int p(x)dx$$

$$lin\, y = e^c e^{-\int p(x)dx}$$

$y = e^c e^{-\int p(x)dx}$ suppose $e^c = c_1$

$$y = c_1 e^{-\int p(x)dx}$$

$[y = c_1 e^{-\int p(x)dx}]$ $e^{\int p(x)dx}$

$$y\, e^{\int p(x)dx} = c_1 \Rightarrow \frac{\partial}{\partial x}(y\, e^{\int p(x)dx}) = \frac{\partial}{\partial x}c_1 \Rightarrow y\, p(x)e^{\int p(x)dx} + e^{\int p(x)dx} y' = 0$$

$e^{\int p(x)dx}[y' + p(x)y] = 0$ exact DE.

$e^{\int p(x)dx}$ is an IF of equ (3)

Since the I.F is a function of (x) only and equ.(2) depended on the independent variable (x) only then this IF. is also an I.F of equ.(2) multiply both side of equ(2) by $e^{\int p(x)dx}$

$[y' + p(x)y = Q(x)]$ $e^{\int p(x)dx}$

$$y' e^{\int p(x)dx} + p(x)y e^{\int p(x)dx} = Q(x)e^{\int p(x)dx} \quad \Rightarrow \frac{d}{dx}(e^{\int p(x)dx} y) = Q(x)e^{\int p(x)dx}$$

$$\int \frac{d}{dx}(e^{\int p(x)dx} y)dx = \int Q(x)e^{\int p(x)dx} dx \Rightarrow y e^{\int p(x)dx} = \int Q(x)e^{\int p(x)dx} dx + c$$

Or $y=\dfrac{\int Q(x)e^{\int p(x)dx}dx+c}{e^{\int p(x)dx}}$, Or $y=e^{-\int p(x)dx}[\int Q(x)e^{\int p(x)dx}dx+c]$

Is the general solution of non-homog. and non-exact D.E(liner first order)

Example: solve: $2xy'+x^2y=3x^3----(1)$

Solution: divide equ (1) by $2x$, we obtain,

$$y'+\frac{x^2}{2x}y=\frac{3x^2}{2}$$

By comparison with $y'+p(x)y=Q(x)$

$$p(x)=\frac{x}{2},Q(x)=\frac{3}{2}x^2$$

$$I.F=e^{\int p(x)dx}=e^{\int\frac{x}{2}dx}=e^{\frac{1}{2}\int xdx}=e^{\frac{x^2}{4}}$$

$$y=e^{-\frac{x^2}{4}}[\int\frac{3}{2}x^2e^{\frac{x^2}{4}}dx+c_1]$$

Example: solve: $y'\sin x=y\cos x+\sin^2x\ ----(1)$

Solution: $y'\sin x-y\cos x=\sin^2 x$

$$y'-\frac{y\cos x}{\sin x}=\sin x\ \Rightarrow y'+p(x)y=Q(x);\ p(x)=-\frac{\cos x}{\sin x}\ \&\ Q(x)=\sin x$$

$$y=(I.F)^{-1}[\int Q(x)e^{\int p(x)dx}dx+c]$$

$$I.F=I.F=e^{\int p(x)dx}=e^{\int-\frac{\cos x}{\sin x}dx}=e^{-\int\frac{1}{\sin x}dx}=e^{-lin|\sin x|}=\frac{1}{\sin x}$$

$$y=\sin x[\int\sin x\frac{1}{\sin x}dx+c]$$

Bernoulli's Equation

Def: A DE $y'+p(x)y=Q(x)y^n----(1)$ is said to be Bernoulli's Equation such that p, and Q are functions of (x) only and n≠0, 1(n∈ R):

If n=0 $\Rightarrow y'+p(x)y=Q(x)$ is (F.O.L.D.E)

If n=1 $\Rightarrow y'+p(x)y=Q(x)y\Rightarrow y'+[p(x)-Q(x)]y=0$ is (S.D.E)

$y'+p(x)y=Q(x)y^n$ ----- (1) (Bernolli's Equation)

To solve DE (1) we will divided it by (y^n) we obtain $y^{-n}y'+p(x)yy^{-n}=Q(x)$

$$y^{-n}y'+p(x)y^{1-n}=Q(x)----(2)$$

Suppose y^{1-n}=u --- (3)

$$(1-n)y^{1-n-1}y'=u'$$

$$y^ny'=\frac{u'}{(1-n)}---(4)$$

Subst both equ (3) &equ (4) in equ (2); we get an FOLDE

$$\frac{u'}{(1-n)}+p(x)u=Q(x)$$

$$[\frac{u'}{(1-n)}+p(x)u=Q(x)] \qquad (1-n)$$

$$u'+(1-n)p(x)u=(1-n)Q(x)----(5) \text{ is FOLDE}$$

equ (5) can be solving by method of FOLDE. we get the value of u .Finally subst. the value of y^{1-n} =u, we get the general solution of Bernoulli's equ.

Example: solve: $(12e^{2x}y^2-y)dx=dy$

Solution: $\frac{dy}{dx}=12e^{2x}y^2-y$

$$\frac{dy}{dx}+y=12e^{2x}y^2 \text{ ----- (1)} \quad \text{Bernoulli's equ.}$$

Divide equ (1) by y^2 we obtain

$$y'y^{-2}+y^{-1}=12e^{2x} ----(2)$$

Let $y^{-1}=u ----(3)$

Differentiation equ (3) with respect to x

$$-y^{-2}y'=u'$$

$$y^{-2}y'=-u'----(4)$$

Subst equ (3) & equ (4)in equ.(2), we obtain a F.O.L.D.E.

$$-u'+u=12e^{2x}$$

$$u'-u=-12e^{2x} \text{ is a FOLDE } p(x)=-1 \ \& \ Q(x)=-12e^{2x}$$

$$u=e^{-\int -1dx}[\int -12e^{2x}e^{\int -1dx}dx+c]$$

$$u=e^{x}[-12\int e^{x}dx+c] \Rightarrow u=e^{x}[-12e^{x}+c] \Rightarrow$$

$$y^{-1}=-12e^{2x}+ce^{x} \Rightarrow \frac{1}{y}=-12e^{2x}+ce^{x}$$

$$y=\frac{1}{-12e^{2x}+ce^{x}} \text{ is the gernarl solution of Birnolli's equ.}$$

Example: solve $\frac{dy}{dx}-\frac{4}{x}y=x\sqrt{y} ---(1)$

Solution: $y'-\frac{4}{x}y=xy^{\frac{1}{2}} \Rightarrow y'y^{-\frac{1}{2}}-\frac{4}{x}y^{\frac{1}{2}}=x ---(2)$

Suppose $-y^{\frac{1}{2}}$ = u --- (3) $\Rightarrow -\frac{1}{2}y^{-\frac{1}{2}}y'=u' \Rightarrow (y^{-\frac{1}{2}}y'=-2u'$ --- (4) , Subst (3) & (4) in equ(2) we obtain FOLDE

$$[-2u'+\frac{4}{x}u=x] \qquad (\frac{-1}{2})$$

$$u'-\frac{2}{x}u=-\frac{x}{2} \text{ is a FOLDE}$$

$$I.F=e^{\int p(x)dx}=e^{-2\int \frac{1}{x}dx}=e^{-2\ln x}=e^{\ln x^{-2}}=x^{-2}$$

$u = x^2[\int \frac{-x}{2} x^{-2} dx + c]$

H.W: solve the following equations.

1) $2y' - xy = x$

2) $\frac{dx}{dy} + \frac{3}{y} x = 2y$

3) $(x+1)y' - y = e^x (x+1)^2$

Riccatti's Equation

The general form of Riccatti's equation is: $y' + p(x)y + Q(x)y^2 = f(x) ----(1)$; where p , Q & f are functions of (x) only ,if Q(x)=0 then equ.(1) becomes F.O.L.D.E or F(x)=0 then equ.(1) becomes is bernoullis equation , and one of the particular solutions of Riccatti's equ. is known ,say $y_p = y_1$ is a particular solution of Riccatti's equ. We can find the general solution by supposing

$y_G = y_p + v(x) --- (2)$

$y_G = y_1 + v(x)$

$y' = y_1' + v'(x) --- (3)$

Subst. both (2) & (3) in D.E.(1) we obtain Bernolli's equ. as follows

$(y_1' + v') + p(x)(y_1 + v) + Q(x)(y_1 + v)^2 = f(x)$

$y_1' + p(x)y_1 + Q(x)y_1^2 + v' + p(x)v + 2y_1 v\, Q(x) + v^2 q(x) = f(x)$

$f(x) + v' + (p(x) + 2y_1 Q(x))v + q(x)v^2 = f(x)$

$v' + (p(x) + 2y_1 Q(x))v = -Q(x)v^2$

Is Bernoulli's equ. , we can solve a Bernoulli's equ. by using Bernoulli's method

Example: solve: $x^2 y' = 3x^2 y^2 + xy - 1$ & $y_1 = \frac{1}{3x}$ is a particular solution

Solution: $[x^2 y' - xy - 3x^2 y^2 = -1 --- (1)] \div x^2$

$y' - \frac{1}{x} y - 3y^2 = \frac{-1}{x^2}$

$y' + p(x)y + Q(x)y^2 = f(x)$

$\frac{-1}{3x^2} - \frac{1}{x}\frac{1}{3x} - 3(\frac{1}{3x})^2 = \frac{-1}{x^2} \Rightarrow \frac{-1}{3x^2} - \frac{1}{3x^2} - \frac{1}{3x^2} = \frac{-1}{x^2} \Rightarrow \frac{-1}{x^2} = \frac{-1}{x^2}$

$\therefore y_1 = \frac{1}{3x}$ is a particular solution of D.E.(1)

Suppose $y_G = y_1 + v(x)$

$y = \frac{1}{3x} + v(x) --- (4)$

$y' = \frac{-1}{3x^2} + v'(x) --- (5)$

Subst. both equ. (4) & (5) in D.E.(1) , we obtain the Bernoulli's equ.

$-\frac{1}{3x^2} + v' - \frac{1}{x}(\frac{1}{3x} + v) - 3(\frac{1}{3x} + v)^2 = \frac{-1}{x^2}$

$$-\frac{1}{3x^2}+v'-\frac{1}{3x^2}-\frac{1}{x}v-\frac{1}{3x^2}-\frac{2}{x}v-3v^2=\frac{-1}{x^2}$$

$$-\frac{1}{x^2}+v'-\frac{3}{x}v-3v^2=\frac{-1}{x^2}$$

$v'-\frac{3}{x}v=3v^2---(6)$ is Bernoulli's equ.

$$v'v^{-2}-\frac{3}{x}v^{-1}=3---(7)$$

Suppose $v^{-1}=u---(8)$

$$-v^{-2}v'=u'---(9)$$

Subst. equ. (8) & (9) in D.E (7) , we obtain the F.O.L.D.E

$-u'-\frac{3}{x}u=3$ is F.O.L.D.E.

$$e^{\int p(x)dx}=e^{3\int\frac{1}{x}dx}=e^{3lin\,x}=x^3$$

$$u=x^{-3}[\int(-3)x^3dx+c]\Rightarrow u=\frac{-3}{4}x=cx^{-3}---(10)$$

Subst. equ. (8) in equ. (10)

$$v^{-1}=\frac{-3}{4}x+cx^{-3}$$

$$v^{-1}=\frac{4cx^{-3}-3x}{4}\Rightarrow v=\frac{4}{4cx^{-3}-3x}$$

$$y_G=\frac{1}{3x}+v$$

$y_G=\frac{1}{3x}+\frac{4}{4cx^{-3}-3x}$ is the general solution of Riccatti's equ.

H.W: solve the following D.E.: (1) $y'=y^2-\frac{2}{x^2}\ \&\ y_1=\frac{1}{x}$ is a particular solution.

(2) $3xy'+y+x^2y^4=o$

(3) $y'+y=y^2(\cos x+\sin x)$

Simulatonous Ordinary Differential Equation

Is a set of equations which contains only one independent variable and the number of equations are equal to the number of the dependent variables , as fallows:

$$\frac{dx}{dt}=f(x,t)---(1)$$

$$\frac{dy}{dt}=g(x,y,t)---(2)$$

We can solving system (*) by choosing the D.E in which contains only one dependent variable say equ.(1) & solving by integration immediately or by previous ways , we get the value of dependent variable and subst. in equ. (2) , and solving by previous way , we get the G. solution of system (*)

Example: solve: $\frac{dx}{x-t}=\frac{dy}{x+y}=\frac{dt}{2t}$

Solution :

$$\frac{dx}{x-t}=\frac{dt}{2t}\Rightarrow\frac{dx}{dt}=\frac{x-t}{2t}---(1)$$

$$\frac{dy}{x+y}=\frac{dt}{2t}\Rightarrow\frac{dy}{dt}=\frac{x+y}{2t}---(2)$$

(A)

To find the general Solution, we can choose the first equ.

$$\frac{dx}{dt}=\frac{x-t}{2t}$$

$$\frac{dx}{dt}-\frac{x}{2t}=\frac{-1}{2} \text{ [FOLDE]}$$

$$\frac{dx}{dt}+p(t)x=Q(t)$$

$$p(t)=\frac{-1}{2t}\ ,\quad Q(t)=-\frac{1}{2}$$

$$x=\sqrt{t}\,[\int(\frac{-1}{2})\frac{1}{\sqrt{t}}dt+c]$$

$$x=t^{\frac{1}{2}}\left[-\frac{1}{2}\frac{\sqrt{t}}{\frac{1}{2}}+c\right]\Rightarrow x=-t+ct^{\frac{1}{2}}$$

$x=-t+ct^{\frac{1}{2}}$ subst. the value of (x) in equ. (2) in system A

$$\frac{dy}{dt}=\frac{(ct^{\frac{1}{2}}-t)+y}{2t}$$

$$\frac{dy}{dt}=\frac{ct^{\frac{-1}{2}}}{2}-\frac{1}{2}+\frac{y}{2t}$$

$$\frac{dy}{dt}-\frac{1}{2t}y=\frac{ct^{\frac{-1}{2}}}{2}-\frac{1}{2} \text{ is F.O.L.D.E}$$

H.W./ Solve : $x^2y'=x^2y^2+xy-3$ where $y_1=1/x$ be the particular solution.

CHAPTER THREE

Existence & Uniqueness Solution of D.E in the First Order and First Degree

Theorem: Existence & Uniqueness Theorem of First Order O.D.E

If f is a function of variables x, y, f continuous in R in witch defined in $|x-x_0|<a$ and $|y-y_0|<b$ s.t $a,b>0$ & $\frac{\partial f}{\partial y}$ $is\,cont.fun.in R$ then D.E $\frac{dy}{dx}=f(x,y)$ has a unique & cantinous solution $y=\varphi(x)$ $path\ in\ (x_0,y_0), \forall x,y\in R.$

Example: Show that $\frac{dy}{dx}=2x$ satisfy the conditions of (Existence & Uniqueness Theorem)

Example: Is the D.E $y'=\frac{3}{2}\sqrt{y}$ satisfy the conditions of (Exis. & uniq. theorem)? How?

Solution: $f(x,y)=\frac{3}{2}\sqrt{y}$ is $cont^n$. function in R such that $y\geq 0$.

$\frac{\partial f}{\partial y}=\frac{3}{2}(\frac{1}{2\sqrt{y}})=\frac{3}{4\sqrt{y}}$ is $cont^n$. function in R such that $y>0$.

$\frac{dy}{dx}=\frac{3}{2}\sqrt{y}\Rightarrow\frac{dy}{\sqrt{y}}=\frac{3}{2}dx$ Such that $y>0$,

$\frac{y^{\frac{1}{2}}}{\frac{1}{2}}=\frac{3}{2}x+c\Rightarrow y^{\frac{1}{2}}=\frac{3}{4}x+\frac{c}{2}$

$y=(\frac{3}{4}x+\frac{c}{2})^2$ is a solution of D.E $y'=\frac{3}{2}\sqrt{y}$ such that $y>0$ $\therefore$ the D.E $y'=\frac{3}{2}\sqrt{y}$ satisfy the conditions of existence & uniqueness theorem such that ($y>0$), but if y=0 then the solution doesn't exist (or not uniqe)$\therefore$ the D.E $y'=\frac{3}{2}\sqrt{y}$ doesn't satisfy the conditions of existence & uniqueness theorem where (y≤0) .

H.W// Show that $y'=3x^2-5$ satisfy the conditions of existence & uniqueness theorem.

The singular points

Def: (1) Is a points in (x,y) plane in which doesn't contains one or more necessary conditions in existence & uniqueness theorem (doesn't satisfied)

Def: (2) A point (xo,yo) is called singular point if :

(1) The solution is not exist in (xo,yo)

(2) The solution is not uniqe in (xo,yo)

(3) The solution is not continuos in (xo,yo)

Example: Find the set of singular point or singular solution of D.E $y'=\sqrt{y}$ if there exist.

Solution: $f(x,y)=\sqrt{y}$ is $cont^n$. in R such that $y\geq 0$

$\frac{\partial f}{\partial y}=\frac{1}{2}y^{\frac{-1}{2}}\Rightarrow\frac{\partial f}{\partial y}=\frac{1}{2\sqrt{y}}$ is cont. in R s.t $y>0$ $\frac{dy}{dx}=\sqrt{y}\Rightarrow\frac{dy}{\sqrt{y}}=dx$ such that $y>0\Rightarrow\frac{y^{\frac{1}{2}}}{\frac{1}{2}}=x+c\Rightarrow$

$y^{\frac{1}{2}}=\frac{x+c}{2}\Rightarrow y=(\frac{x+c}{2})^2$ is a solution of $y'=\sqrt{y}$ such that ($y>0$), but if y=0 doesn't exist and in $(y\prec 0)$.

$\therefore y\leq 0$ be singular solution. doesn't satisfy the condition of existence & uniqueness theorem

H.W: Example: Find the general solution and the singular solution of the D.E $y'=\frac{y-2}{x-1}$ if there exists and find the singular set?

Some definitions:

Def: (Open set)

A set D is called open set if for every point (x, y) in D is the center of a rectangle which contained in D.

Def: (Open subset)

A subset D of the plane is open if for any point (xo,yo) in D ,there exist a positive number a,b such that for any point (x,y) which satisfies $|x-xo|\le a \,\&\, |y-yo|\le b$ belong to D.

Example: Plane (RXR) is open set

Example: $D_1=\{(x,y); x^2+y^2<4\}$ is open set

Example: $D_2=\{(x,y); 0<x\le 1 \,\&\, 0\le y\le 3\}$ is not open set

Example: $D_3=\{the\ line\ is\ not\ open\ set\}$

Theorem: Existences and uniqness of first order D.E U is a solution of

$y'=f(x,y)---(1)$, which satisfy the initial condition $u(xo)=yo$ iff $u(x)=yo+\int_{xo}^{x} f(s,u(s))ds.---(2)$

Proof: Suppose that U. is a solution of D.E (1),we must prove hat $u(x)=yo+\int_{xo}^{x} f(s,u(s))ds$,since U is a solution of D.E.(1) defined on an interval (I) satisfying $u(x_0)=yo \Rightarrow u'=f(x,y)---(3)$ by integration equ(3) from (xo)in to (x) ,we get

$$\int_{xo}^{x} u'(s)ds=\int_{xo}^{x} f(s,u(s))ds$$

$$\Rightarrow u(x)-u(xo)=\int_{xo}^{x} f(s,u(s))ds$$

$$\Rightarrow u(x)=u(xo)+\int_{xo}^{x} f(s,u(s))ds$$

$$\therefore u(x)=yo+\int_{xo}^{x} f(s,u(s))ds \text{ is proved}$$

Conversely//Suppose that u is any function in which satisfy this equation $u(x)=u(xo)+\int_{xo}^{x} f(s,u(s))ds$,we must prove that (u) is a solution of $y'=f(x,y)$ and satisfy the initial condition $u(xo)=yo$ in an interval (I). Let $(x=xo)$ in equ(2) then $u(xo)=yo$ then u satisfy the initial condition . Note that the integral is always cont. function , so that a solution of equ(2) is continuos since u & F are $cont^n$.function , then $f(s,u(s))$ be a $cont^n$ function[by fundamental theorem in calculus of D.E $y'=f(x,y)$]. Let F be a $cont^n$. function on an (I) let $xo\in I$ and u be a function defined by $u(x)=\int_{xo}^{x} f(s)ds$ $\forall x\in I$, then u is differentiable and $u'(x)=f(x,u(x))$ we mean that (u) is solution of $y'=f(x,y)---(1)$ in which satisfy $u(xo)=yo$

$\therefore$ u is a solution of D.E---(1)

Lipschitz function:

If f and $\frac{\partial f}{\partial y}$ are continuos function on the closed region (Q), choose $(x,y_1),(x,y_2)$ in this closed region, by using the mean value theorem in calculus $|f(x,y_1)-f(x,y_2)|=\left|\frac{\partial f(x,y_3)}{\partial y}\right||y_1-y_2|$ such that $y_1<y_3<y_2$.

but $(\frac{\partial f}{\partial y})$ be a $cont^n$. in closed and bounded region Q.

Now $\exists$ anon-negative (positive) number (k) such that $\left|\frac{\partial f(x,y_3)}{\partial y}\right|\le K$ $\forall (x,y_3)\in Q$

Now then $|f(x,y_1)-f(x,y_2)|\le K|y_1-y_2|---(A)$ since $\frac{\partial f}{\partial y}$ be $cont^n$. in Q, then F satisfy :

$|f(x,y_1)-f(x,y_2)|\le K|y_1-y_2|$ $\forall$ two points (x,y_1) and (x,y_2) in Q .

Def: (Lipschitz function)

A function $f(x,y)$ defined and continuous in a domain (D) of the xy-plane satisfies the Lipschitz conditionally (or is Lipschitz condition) written $F\in lipy$, if $\exists$ a constant K such that $|f(x,y_1)-f(x,y_2)|\le K|y_1-y_2|---(1)$ for every pair of points $(x,y_1),(x,y_2)\in D$, the least value of (K) for which (1) holds is called the Lipschitz constant of $f(x,y)$ in D.

Example: Prove that f is a Lipschtiz in closed and bounded region Q such that $Q=\{(x,y);|x|\le 2\,and\,|y|\le 1\}$ in which f is defined as $f(x,y)=\sin(xy)+e^{y^2}$

Solution: Since $f(x,y)=\sin(xy)+e^{y^2}$ is continuous function in R , then $\frac{\partial f}{\partial y}=x\cos(xy)+2y\,e^{y^2}$ is continuous function

$$\Rightarrow\left|\frac{\partial f}{\partial y}\right|=\left|x\cos(xy)+2ye^{y^2}\right| \quad \Rightarrow\left|\frac{\partial f}{\partial y}\right|\le|x\cos(xy)|+\left|2ye^{y^2}\right| \qquad \{by\ |a+b|\le|a|+|b|\}$$

$$\Rightarrow\left|\frac{\partial f}{\partial y}\right|\le|x|\,|\cos(xy)|+2|y|\cdot\left|e^{y^2}\right|, \qquad \{by\ |a\cdot b|=|a|\cdot|b|\}$$

$$\Rightarrow\left|\frac{\partial f}{\partial y}\right|\le 2\cos(2)+2\cdot 1\cdot e \quad \Rightarrow\frac{\partial f}{\partial y}\le 2(\cos(2)+e)$$

$K=2$ is constant Lipschitz

H.W: Example: Is $f(x,y)=x^2y$ lip. Function in R s.t $R=\{(x,y);|x|\le 1\,and\,|y|\le 1\}$

Theorem:-There exist a positive number (k) such that $|f(x,y_1)-f(x,y_2)|\le K|y_1-y_2|$, for any two points $(x,y_1),(x,y_2)$ in the rectangle Q.

Proof: -choose two points $(x,y_1),(x,y_2)\in Q$ by the mean value theorem there exist (y_3) such that $y_1<y_3<y_2$ satisfy $|f(x,y_1)-f(x,y_2)|\le\left|\frac{\partial f(x,y_3)}{\partial y}\right|\cdot|y_1-y_2|$ {by def^n of lip F} since F is $cont^n$. in Q, the F is defined and bounded $\therefore$ $\exists$ a positive number (K) such that $\left|\frac{\partial f(x,y_3)}{\partial y}\right|\le K$.

$|f(x,y_1)-f(x,y_2)|\le K|y_1-y_2|$ then there exist a positive number (K) express a constant lip condition .

Def: ((norm)) ($\|\ \|$)

Let ((I)) be a bounded and closed interval and(u) be a $cont^n$. function on (I) then the norm of u denoted by $\|u\|$ and defined by $\|u\|=\max|u|$, $\forall x\in I$

Properties of norm ($\|\ \|$):

If u and ϑ are two $cont^n$. function on (I) then:

(1) $\|u-\vartheta\|=\max|u(x)-\vartheta(x)|$, $\forall x\in I$

(2) $\|u\|=0\leftrightarrow u(x)=0$

(3) $\|u+\vartheta\|\le\|u\|+\|\vartheta\|$

(4) $\|u(x)\|\ge|u(x)|$

Uniqueness theorem of first order O.D.E:

Let d be a number which satisfies $(0<d<\frac{1}{k})$ where (K) is a positive Lipschitz constant for f . if (u) and(ϑ) are solution of $y'=f(x,y)---(1)$ defined on the interval I : $|x-xo|\le d$ and if $(u(xo)=\vartheta(xo))$ then $u=\vartheta$

Proof: suppose $u(xo)=\vartheta(xo)=yo$ {by exist & uniqness theorem}

$$u(x)=yo+\int_{xo}^{x} f(s,u(s))ds \text{ and}$$

$$\vartheta(x)=yo+\int_{xo}^{x} f(s,\vartheta(s))ds$$

$$u(x)-\vartheta(x)=\int_{xo}^{x}[f(s,u(s))-f(s,\vartheta(s))]ds$$

$$|u(x)-\vartheta(x)|=\left|\int_{xo}^{x}[f(s,u(s))-f(s,\vartheta(s))]ds\right|$$

$$|u(x)-\vartheta(x)|\le\left|\int_{xo}^{x}|f(s,u(s))-f(s,\vartheta(s))|\,ds\right|---(2)$$

By using lip. Condition, since $(s,u(s))$ and $(s,\vartheta(s))\in Q$, then

$|f(s,u(s))-f(s,\vartheta(s))|\le K|u(s)-\vartheta(s)|---(3)$ subst. equ.(3)in equ.(2), we obtain

$$|u(x)-\vartheta(x)|\le\left|\int_{xo}^{x}K|u(s)-\vartheta(s)|ds\right|$$

$$|u(x)-\vartheta(x)|\le K|u(s)-\vartheta(s)|\cdot|x-xo|$$

$$|u(x)-\vartheta(x)|\le K\,d\,|u(s)-\vartheta(s)|---(4)$$

$$\max_{\forall x\in I}|u(x)-\vartheta(x)|\le K\,d\max_{\forall x\in I}|u(s)-\vartheta(s)|$$

$$\|u(x)-\vartheta(x)\|\le K\,d\,\|u-\vartheta\|---(5)$$

Suppose $u\ne\vartheta$ $\|u-\vartheta\|>0$

Divide equ. (5) by $\|u-\vartheta\|$, we get

$1\le Kd$

But $0<d<\frac{1}{K}\Rightarrow Kd<1$

This is contradiction

$\therefore u=\vartheta$

CHAPTER FOUR

Reduction of Higher Order Ordinary D.E.

Consider a D.E of higher order is $F(x,y,y',y'',...,y^{(n)'}) = 0----(1)$ can be reduced into first order by using substitution. We can study this D.E of second order, then equ. (1) becomes $g(x,y,y',y'') = 0----(2)$ equ(2) can be solved by two cases:

Cases 1:If the dependent variable (y) does not (explicity) or (appears) in the D.E (2), then equ(2), becomes $g(x,y',y'')=0----(3)$

Suppose $\frac{dy}{dx}=y'=p----(4)$

$$\frac{d^2y}{dx^2}=y''=p'=\frac{dp}{dx}----(4)$$

$$y''=\frac{dp}{dx}----(5)$$

Subst. both equ (4) and (5) in D.E (3) we get a relation between (x,p)

$g_2(x,p,\frac{dp}{dx})=0\ ----(6)$ can be solving by previous way and we get (p) subst. (p) by $(\frac{dy}{dx})$ and by integration immediately we get the g.solution

Cases 2:If the independent variable (x) does not (explicity) or(appears) in the D.E (2) ,then equ(2), becomes $g(y,y',y'')=0----(*)$

Suppose $\frac{dy}{dx}=y'=p----(**)$

$$\frac{d^2y}{dx^2}=y''=p'=\frac{dp}{dx}\Rightarrow y''=\frac{dp}{dx}.\frac{dy}{dy}\Rightarrow y''=\frac{dp}{dy}.\frac{dy}{dx}=p.\frac{dp}{dy}----(***)\ \text{(by chain rule)}$$

Subst. both equ (**) and (***) in D.E (*) we get a relation between (y,p)

$g^{**}(x,p,p\frac{dp}{dy})=0\ ----(****)$ can be solving by previous way and we get (p) subst. (p) by $(\frac{dy}{dx})$ and by integration immediately we get the g.solution

Examples: solve the following D.E.

1) $xy''+(y')^3=0$ ----------(1)

2) $yy''+(y')^2=0----(*)$

3) $x^2y''-(y')^2-2xy'=0$ H.W

4) $y'''-y''=1$

5) $2y''-(y')^2=0$ H.W.

Solution 1) since (y) not appears.

$\frac{dy}{dx}=y'=p$

$p'=y''=\frac{dp}{dx}$ ——(2) Subst. equ (2) in equ (1) we get $xp'+p^3=0$

$\{xp' + p^3 = 0 \quad S.D.E\} \quad \frac{dy}{xp^3}$

$\int \frac{dp}{p^3} + \int \frac{dx}{x} = 0 \Rightarrow \{\frac{1}{-2} p^{-2} + lin\, x = c\}.-2$

$\frac{1}{p^2} - 2lin\, x = -2c \qquad$ Suppose $c_1 = -2c$

$\frac{1}{p^2} + lin\, x^{-2} = c_1 \Rightarrow \frac{1}{p^2} = c_1 - lin\, x^{-2} \Rightarrow p = \frac{1}{\sqrt{c_1 - lin\, x^{-2}}} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{c_1 - lin\, x^{-2}}}$

$y = \int \frac{1}{\sqrt{c_1 - lin\, x^{-2}}} dx$

Solution 2) since (x) not appears in equ.(*)

Then Suppose $\frac{dy}{dx} = y' = p$

$\frac{d^2y}{dx^2} = y'' = p' = \frac{dp}{dx}$ ——— (**)

$y'' = p.\frac{dp}{dy} ----(***)$

Subst. both equ (**) and (***) in D.E (*) we get a relation between (y,p)

$\{ \; yp\frac{dp}{dy} + p^2 = 0 \; \} S.D.E \qquad \frac{dy}{p^2 y}$

$\int \frac{dp}{p} + \int \frac{dy}{y} = 0$

$lin\, p + lin\, y = c$

$lin(p.y) = c$

$p.y = e^c \qquad$ let $c_1 = e^c$

$p.y = c_1$

$ydy = c_1 dx$

Solution 4) since x *and* y don's not appears in D.E (1), then solving by any cases. Suppose we can solving by first case.

$y''' - y'' = 1 ----(1)$

$\frac{dy}{dx} = y'' = p$

$p' = y''' = \frac{dp}{dx}$ ——— (2)

$\frac{dp}{dx} - p = 1$

$\frac{dp}{dx} = 1 + p \qquad S.D.E$

$\frac{dp}{1+p} = dx$

$lin(1+p) = x + c$

$1 + p = ke^x \qquad let\; e^c = k$

$p = ke^x - 1$

$y'' = ke^x - 1 \qquad ----(3) \; (y\; does\; n't\; appears)$

$$\frac{dy}{dx}=y'=p$$
$$p'=y''=\frac{dp}{dx} \qquad (4)$$

Subst equ (4) in D.E (3), we get

$$\frac{dp}{dx}=ke^{x}-1$$
$$\int dp=\int(ke^{x}-1)dx$$
$$p=ke^{x}-x+c$$
$$dy=(ke^{x}-x+c)\,dx$$
$$y=ke^{x}-\frac{x^{2}}{2}+xc+c_{1}$$ be the g.solution of D.E (1)

Example: solve H.W

1) $xy''+y'=3x^{2}-x$

2) $(1+x^{2})y''-2xy'=2x$

3) $y''+2y(1+y')^{2}=0$

4) $y''+\left(y'\right)^{2}+y=0$

Higher Degree of ordinary D.E

O.D.E of the first order but of the higher degree. The general form of higher degree of O.D.E is $a_0p^{n}+a_1p^{n-1}+a_2p^{n-2}+---+a_{n-1}p+a_n=0$ where $a_0,a_1,---,a_n$ are constant and $(a_0\neq 0)$---(1). To solve equ.(1) there exist three cases:

Case1: equ. (1) Solvable for (p). if equ.(1) can be written of the form

$$(p-f_1(x,y))(p-f_2(x,y))\dots(p-f_n(x,y))=0$$
$$p-f_1(x,y)=0\Rightarrow p=f_1(x,y)\Rightarrow\frac{dy}{dx}=f_1(x,y)\Rightarrow g_1(x,y)=c$$
$$p-f_2(x,y)=0\Rightarrow p=f_2(x,y)\Rightarrow\frac{dy}{dx}=f_2(x,y)\Rightarrow g_2(x,y)=c$$

Continue in this way, we get $g_n(x,y)=c$.

Then the general solution of this case is $[g_1(x,y)-c][g_2(x,y)-c]\dots[g_n(x,y)-c]=0$;where c is constant.

Example: solve: $(\frac{dy}{dx})^{2}+x\frac{dy}{dx}+y\frac{dy}{dx}+xy=0---(1)$

Solution: Suppose $\frac{dy}{dx}=p\Rightarrow p^{2}+xp+yp+xy=0\Rightarrow p(p+x)+y(p+x)=0$

$(p+x)(p+y)=0$, or $p+x=0\Rightarrow dy+x\,dx=0\Rightarrow y+\frac{x^{2}}{2}=c$

and $p+y=0\Rightarrow dy/y+dx=0\Rightarrow lin\,y+x=c$

The general solution of D.E.O is $(y+\frac{x^{2}}{2}-c)(lin\,y+x-c)=0$ where c is a constant.

Solves the follows equation. (H.W)

(1) $yp^{2}+(x-y)p-x=0$

(2) $x\,y\,p^{2}+(x^{2}+xy+y^{2})p+x^{2}+xy=0$

Case 2: equ.(1) solvable for y

Equ.(1): $a_0 p^n + a_1 p^{n-1} + ... + a_{n-1} p + an = 0$ can be written of the form $F(x, y, p) = 0 \Rightarrow$

$y = F(x, p) ---(2)$ differentiation equ.(2) with respect to x and subst. $\frac{dy}{dx} = p$ we obtain a relation between

x, p can be solved by previous way and obtained the value of p.

Finally subst. the value of p in equ.(2) , we get the g. solution, and the singular

Example: 1) solve: $3p^5 - py + 2 = 0$

Solution: $3p^5 - py + 2 = 0 ---(1)$

$$py = 3p^5 + 2 \Rightarrow y = 3p^4 + \frac{2}{p} ---(2)$$

$$\frac{dy}{dx} = p = 12p^3 \frac{dp}{dx} + \frac{-2\frac{dp}{dx}}{p^2} \Rightarrow p = 12p^3 \frac{dp}{dx} - \frac{2}{p^2}\frac{dp}{dx} \Rightarrow 12p^3 \frac{dp}{dx} - \frac{2}{p^2}\frac{dp}{dx} - p = 0$$

$$2p\frac{dp}{dx}(6p^2 - \frac{1}{p^3}) - p = 0]*\frac{1}{p} \Rightarrow [2(6p^2 - \frac{1}{p3})\frac{dp}{dx} - 1 = 0]*dx \Rightarrow 2(6p^2 - \frac{1}{p^3})dp - dx = 0$$

2) Solve: $y = 2xp + \tan^{-1}(xp^2)$

Solution: $y = 2xp + \tan^{-1}(xp^2) ---(1)$

$$\frac{dy}{dx} = p = 2(x\frac{dp}{dx} + p) + \frac{1}{1 + (xp^2)^2} \cdot (2px\frac{dp}{dx} + p^2)$$

$$p = 2x\frac{dp}{dx} + 2p + \frac{2xp\frac{dp}{dx} + p^2}{1 + x^2 p^4} \Rightarrow p + 2x\frac{dp}{dx} + \frac{2xp\frac{dp}{dx} + p^2}{1 + x^2 p^4} = 0$$

$$p(1 + \frac{p}{(1 + x^2 p^4)}) + 2x\frac{dp}{dx}(1 + \frac{p}{1 + x^2 p^4}) = 0 \Rightarrow (1 + \frac{p}{(1 + x^2 p^4)})(p + 2x\frac{dp}{dx}) = 0$$

$$p + 2x\frac{dp}{dx} = 0]\frac{dx}{xp} \Rightarrow \int \frac{dx}{x} + 2\int \frac{dp}{p} = 0 \Rightarrow lin\, x + 2lin\, p = c_1 \Rightarrow lin\, x + lin\, p^2 = c_1$$

$$lin(xp^2) = c_1 \Rightarrow xp^2 = e^{c1} = c \Rightarrow p = \sqrt{\frac{c}{x}}$$

$y = 2x\sqrt{\frac{c}{x}} + \tan^{-1}(c)$ is the g. solution or $1 + \frac{p}{1 + x^2 p^4} = 0 \Rightarrow x^2 p^4 + p = 0$

H.W: solve the following D.E

(1) $xp^2 + x = 2yp$

(2) $y + px = x^4 p^2$

(3) $y = (2 + p)x + p^2$

Case 3: equ.(1) solvable for (x) equ.(1) $a_0 p^n + a_1 p^{n-1} + ---- + a_{n-1} p + an = 0$ can be written of the form $F(x, y, p) = 0 \Rightarrow x = F(y, p) ---(2)$ differentiation equ.(2) with respect to y and subst. $\frac{dx}{dy} = \frac{1}{p}$ we obtain the a relation between y, p can be solved by previous way and obtain the value of p. finally subst. the value of p in equ.(2) we get the g. solution .

Example: solve: $y = 3px + 6y^2 p^2 ---(1)$

Solution: since equ.(1) solvable for x, then

$3px = y - 6y^2 p^2$

$3x = \frac{y}{p} - 6y^2 p$

$3\frac{dx}{dy} = 3\cdot\frac{1}{p} = \frac{p - y\,dp/dy}{p^2} - 6y^2\,dp/dy - 12py \Rightarrow \frac{3}{p} = \frac{1}{p} - \frac{y}{p^2}dp/dy - 6y^2\,dp/dy - 12yp$

$[\frac{2}{p} + \frac{y}{p^2}dp/dy - 6y^2\,dp/dy - 12yp = 0]*p^2 \Rightarrow (2p + y\,dp/dy) + (6y^2p^2\,dp/dy + 12yp^3) = 0$

$(2p + y\,dp/dy) + 6yp^2(y\,dp/dy + 2p) = 0 \Rightarrow (2p + y\,dp/dy)(1 + 6yp^2) = 0$

$2p + y\,dp/dy = 0]*\frac{dy}{py} \Rightarrow 2\frac{dy}{y} + \frac{dp}{p} = 0 \Rightarrow lin\,y^2 + lin\,p = c \Rightarrow lin(y^2p) = c$

and $y^2p = c_1 \Rightarrow p = c_1/y^2 \Rightarrow y = 3\frac{c_1}{y^2}x + 6y^2\frac{c_1^2}{y^4}$ be the general solution.

or $1 + 6yp^2 = 0 \Rightarrow \quad 6yp^2 = -1 \Rightarrow p^2 = \frac{-1}{6y} \Rightarrow p = \pm\sqrt{\frac{-1}{6y}} \notin \mathbb{R}$

H.W: solve (1) $p^3 - p(y-3) + x = o$

(2) $3py + 6x^2p^2 = x$

(3) $3px + 6x^2p = xy$

Clairt's Equation: an equation of the form $y = px + f(p) --- (1)$ where f is a function of (p) only is called clairut's equation and we can obtained the general solution by subst. p by c as follows: $y = cx + f(c)$ is the g. solution of equ.(1) where c is a constant differentiation equ.(1) with respect to x and subst. $\frac{dy}{dx} = p$, we obtain $dy/dx = p = p + x\,dp/dx + f'(p)dp/dx$

$\frac{dp}{dx}(x + f'(p)) = 0$

$dp/dx = 0 \Rightarrow dp = 0 \Rightarrow p = c, \therefore y = cx + f(c)$

or $x + f'(p) = 0 \Rightarrow f(p) = x^2/2 + c$.

Example: solve: $y = px + \sqrt{4 + p^2} --- (1)$

Solution: since equ.(1) is clairut's equ. Then $p = c$

$y = cx + \sqrt{4 + c^2}$

Example: solve: $p = \sin(y - xp)$

Solution: $p = \sin(y - xp) --- (1)$

$\sin^{-1} p = \sin^{-1}\sin(y - xp) \Rightarrow \sin^{-1} p = (y - xp) \Rightarrow y - xp = \sin^{-1} p$

$y = xp + \sin^{-1} p --- (2)$ is clairut's equ.

$\frac{dy}{dx} = p = p + x\,dp/dx + \frac{1}{\sqrt{1-p^2}}\cdot dp/dx \Rightarrow dp/dx\,(x + \frac{1}{\sqrt{1-p^2}}) = 0 \Rightarrow dp/dx = 0 \Rightarrow dp = 0 \Rightarrow p = c$

$\therefore y = cx + \sin^{-1} c$ is the general solution

or $x + \frac{1}{\sqrt{1-p^2}} = 0 \Rightarrow \frac{1}{\sqrt{1-p^2}} = -x \Rightarrow \sqrt{1-p^2} = \frac{-1}{x} \Rightarrow 1 - p^2 = \frac{1}{x^2} \Rightarrow 1 - \frac{1}{x^2} = p^2$

$\Rightarrow p = \pm\sqrt{1 - \frac{1}{x^2}}$ be a singular solution .

H.W: solve the following D.E.

(1) $(y - px)^2 = 1 + p^2$

(2) $p^3 - y + xp = 0$

(3) $y - p - p^2 = xp$

Higher order ordinary D.E with constant coefficients:

The general form of higher order O.D.E is

$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2} \frac{d^{n-2} y}{dx^{n-2}} + ---- + a_3 \frac{d^3 y}{dx^3} + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{d^1 y}{dx^1} + a_0 y = Q(x) --- (1)$ where $a_1, a_2, ---, a_n$ are constants.

An $a_n y'^{(n)} + a_{n-1} y'^{(n-1)} + a_{n-2} y'^{(n-2)} + ---- + a_3 y''' + a_2 y'' + a_1 y' + a_0 y = Q(x) --- (1)$

If $Q(x) \neq 0$, then equ.(1) is called non-Homog. D.E with constant coefficient if $Q(x) = 0$,then equ.(1) becomes(an $\frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ---- + a_1 \frac{dy}{dx} + a_0 y = 0) --- (2)$ is called a H.D.E with c.c .

If at least one of the coefficient $a_0, a_1, ---, a_n$ is a function of x only then equ.(1) becomes $a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + ---- + a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y = Q(x) --- (3)$ equ.(3) is called NON.H.D.E with variable coefficients or if $Q(x) = 0$ in equ.(3) , then equ.(3) becomes

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + ---- + a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y = 0 --- (4)$$

Equ.(4) is called a H.D.E with V.C.

Linearly dependent and linearly independent solutions (functions)

Definition: a functions $(y_1, y_2, ---, y_n)$ are called a L.D.F (on an interval $I \subseteq R$) if $\exists$ aconstants $c_1, c_2, c_3, ..., c_n$ s.t at least one of them does not equal to zero and $c_1 y_1 + c_2 y_2 + ---- + c_n y_n = 0$

Definition: A functions $y_1, y_2, ---, y_n$ are called a L.I.S on an interval ($I \subseteq R$) if there exist $c_1 = c_2 = c_3 = ---- = c_n = 0$ s.t $c_1 y_1 + c_2 y_2 + ---- + c_n y_n = 0$ (all of constants are equal to zero) .

Example:Is e^x, e^{-x} $L.D.S$?

Solution $y_1 = e^x, y_2 = e^{-x}$

$\exists\, c_1, c_2$ s.t

$c_1 y_1 + c_2 y_2 = 0$

$c_1 e^x + c_2 e^{-x} = 0$ ----- (1)

$c_1 e^x - c_2 e^{-x} = 0$ ----- (2)

$2c_1 e^x = 0 ; 2 \neq 0 \,\&\, e^x \neq 0 \quad \therefore c_1 = 0$

$c_2 e^{-x} = 0 ; e^x \neq 0 \quad \forall x \in 0 \Rightarrow c_1 = 0$

$c_1 = c_2 = 0 \quad \therefore e^x, e^{-x}$ are not $L.D.S$ but e^x, e^{-x} are $L.I.S$ .solution in R

Example: Is e^x, x $L.D.S$ or $L.I.S$?

Solution: $y_1 = e^x, y_2 = x$

$\exists\, c_1, c_2$ s.t

$c_1 y_1 + c_2 y_2 = 0$

$c_1 e^x + c_2 x = 0$ ----- (1)

$-c_1 e^x - c_2 = 0$ ----- (2)

$c_2 (x - 1) = 0$

If $c_2 = 0 \,\&\, x - 1 \neq 0 \Rightarrow x \neq 1$

Subst $c_2 = 0$ in equ(1) ; $c_1 e^x = 0 \Rightarrow e^x \neq 0 \;\forall\; x \in R$ then $c_1 = 0$

$c_1 = c_2 = 0 \quad \therefore e^x, x$ are not $L.D.S$ on interval (R / x=1) (OR e^x ,x are L.I.S. at x≠1)

If $c_2 \neq 0 \& x-1=0 \Rightarrow x=1 \therefore \ e^x, x \ are \ L.D.S$ on interval (x=1)

Examples: solve the following differential equations.

1) $cos\, x, \sin x \ are \ L.D.S \ or \ L.I.S$

2) $e^{4x}, -2x^{4x}$

3) x, e^{2x}, e^{-2x}

Wronskian functions

The Wronskian for (n-th) differentiable functions on the interval (I) will be defined by the determinant of $(y_1, y_2, ..., y_n)$ and denoted by W(y1,y2,---yn) $= \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \ddots & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)'} & \cdots & y_n^{(n-1)'} \end{vmatrix}$ The necessary and sufficient conditions that $(y_1, y_2, ..., y_n)$ be L.I.F is that $w(y_1, y_2, ..., y_n) \neq 0$ other wise $w(y_1, y_2, ..., y_n) = 0$ is called L.D.F

Example: e^x, e^{-x} Is L.I.F by using the Wronskian method

Solution: $w(y_1, y_2) = \begin{vmatrix} y_{11} & y_1 \\ y_2' & y_2' \end{vmatrix}$

$w(e^x, e^{-x}) = \begin{vmatrix} e^x & e^{-x} \\ e^x & e^{-x} \end{vmatrix} = -e^{x-x} - e^{-x-x} = -2 \neq 0$

$\therefore$ e^x, e^{-x} are L.I.F at R

Example: Prove that: e^{4x} , $-2e^{4x}$ are L.D.F by using Wronskian method

Solution:

$w(e^{4x}, -2e^{4x}) = \begin{vmatrix} e^{4x} & -2e^{4x} \\ 4e^{4x} & -8e^{4x} \end{vmatrix} \Rightarrow w(e^{4x}, -2e^{4x}) = -8e^{8x} + 8e^{8x}$

$w(e^{4x}, -2e^{4x}) = 0$

Since $w(e^{4x}, -2e^{4x}) = 0$ then either $c_1 \neq 0$ or $c_2 \neq 0$ then e^{4x} , $-2e^{4x}$ is L.D.S on R .

Example H.W: 1) e^x, e^{-x}, $x\, e^{-x}$

2) sinx, cosx

Operators: (D)

Is a differentiation of any dependent variable with respect to independent variable.

$$D = \frac{d}{dx}, D^2 = \frac{d^2}{dx^2}, D^3 = \frac{d^3}{dx^3}, ---, D^n = \frac{d^n}{dx^n}$$

Or $Dy = \frac{dy}{dx}, D^2y = \frac{d^2y}{dx^2}, D^3y = \frac{d^3y}{dx^3}, ---, D^ny = \frac{d^ny}{dx^n}$

Properties:

(1) $D^n D^m (f(x)) = D^n D^m (f(x)) = D^{n+m} f(x)$

(2) $(D^n + D^m) f(x) = (D^n + D^m) f(x) = D^m f(x) + D^n f(x)$

(3) $D(f+g)_{(x)} = Df(x) + Dg(x)$

(4)if C is constant then $D(Cf(x)) = C\,Df(x)$

(5) $(D-a)(D-b)y = (D-b)(D-a)y$ where a,b are constants .

(6) $(D-a(x))(D-b)y \neq (D-b)(D-a(x))y$ where at least one of them (a) or (b) is a function of (x) only.

Ex: $(D-2)(D-3)e^x = (D-3)(D-2)e^x$.

Ex: $(D-2x)(D-3)e^x \neq (D-3)(D-2x)e^x$

Proof (5) : $(D-a)(D-b)y = D^2y - Dby - aDy + aby$

$= D^2y - bDy - aDy + aby ---(*)$

$(D-b)(D-a)y = D^2y - Day - bDy + aby$

$= D^2y - aDy - bDy + aby ---(**)$

$\therefore (D-a)(D-b)y = (D-b)(D-a)y$ Where a,b are constants .

Proof (6): $(D-a(x))(D-b)y = D^2y - Dby - a(x)Dy + a(x)by$

$(D-a(x))(D-b)y = D^2y - bDy - a(x)Dy + a(x)by ---(1)$

$(D-b)(D-a(x))y = D^2y - D(a(x)y) - bDy + ba(x)y$

$(D-b)(D-a(x))y = D^2y - a(x)Dy - yDa(x) - bDy + ba(x)y ---(2)$

Since equ(1) $\neq$ equ(2) then

$(D-a(x))(D-b)y \neq (D-b)(D-a(x))y$

Note: we can written higher order O.D.E with constant coefficient or with the variable coefficient (Homog.or.non-Homog) by using the operator

$$a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ---- + a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = Q(x) ---(1)$$

$$a_nD^ny + a_{n-1}D^{n-1}y + ---- + a_2D^2y + a_1Dy + a_0y = Q(x)$$

$$(a^nD^ny + a_{n-1}D^{n-1}y + ---- + a_2D^2y + a_1Dy)y = Q(x) ---(*)$$

$f(D)y = Q(x)$ Where

$f(D) = a_nD^n + a_{n-1}D^{n-1} + ---- + a_2D^2 + a_1D + a_0$

$f(D)y = 0 ---(***)$ Homog.L.D.E

With C.C. $(y \neq 0) \Rightarrow f(D) = 0$ the $f(D)$ is called the characteristic equation.

$\left[a_n(x)D^n + a_{n-1}(x)D^{n-1} + ---- + a_2(x)D^2 + a_1(x)D + a_0(x)\right]y = Q(x) ---(**)$ non-Homg. with V.C

Or $\left[a_n(x)D^n + a_{n-1}(x)D^{n-1} + ---- + a_2(x)D^2 + a_1(x)D + a_0(x)\right]y = 0 ---(***)$ Homog. With V.C.

Reduction of higher order O.D.E in to first order D.E with c.c.

$$a_nD^n + a_{n-1}D^{n-1} + ---- + a_3D^3 + a_2(D^2 + a_1D + a_0)y = Q(x) ---(1)$$

Where $a_0, a_1, ---, a_n$ are constants and Q is a function of (x) only we study the second order and non-Homog.D.E with C.C as follows:

$(D^2 + aD + b)y = Q(x) ---(1)$

$f(D)y = 0 \Rightarrow y \neq 0 \Rightarrow f(D) = 0$

$(D-m_1)(D-m_2)y = Q(x) ---(2)$

Where $\boxed{b = m_1m_2}$ and $\boxed{a = -(m_1+m_2)}$

$$-m_1 D$$
$$-m_2 D$$
$$-(m_1+m_2)D$$

Suppose $(D-m_2)y=u(x)---(3)$

Subst. eqn.(3) in eqn.(2), we get a.F..L.D.E

$$(D-m_1)u=Q(x)$$
$$Du-m_1 u=Q(x)$$
$$\frac{du}{dx}-m_1 u=Q(x)\Rightarrow p(x)=-m_1\Rightarrow Q(x)=Q(x)\Rightarrow \boxed{y=u}$$
$$u'+p(x)u=Q(x)$$
$$I.f=e^{-m_1\int dx}=e^{-m_1 x}$$
$$u.e^{-m_1 x}=\int Q(x)e^{-m_1 x}\,dx+c_1$$
$$u=e^{m_1 x}[\int Q(x)e^{-m_1 x}\,dx+c_1]---(4)$$

Subst. equ (3) in equ.(4), we get L.F.O.D.E.

$$\frac{dy}{dx}-m_2 y=e^{m_1 x}[\int Q(x)e^{-m_1 x}\,dx+c_1]$$
$$I.f=e^{-m_2 x}$$
$$y.e^{-m_2 x}=\{\int e^{m_1 x}[\int Q(x)e^{m_1 x}\,dx+c_1]e^{-m_2 x}\,dx+c_2$$
$$y=e^{m_2 x}\{\int e^{m_1 x}[\int Q(x)e^{-m_1 x}\,dx+c_1]e^{-m_2 x}\,dx+c_2\}$$

be the g. solution where c_1, c_2 are constants.

Ex.: solve: $y''+y'-2y=e^x---(1)$

Solution: $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=e^x---(1)$

$$D^2y+Dy-2y=e^x---(1)$$
$$(D^2+D-2)y=e^x---(1)$$
$$(D+2)(D-1)y=e^x---(2)$$

Suppose $(D-1)y=u(x)---(3)$

Subst. equ.(3) in equ.(2) , we get $(D+2)u=e^x$ is (F.O.L.D.E)

$$Du+2u=e^x$$
$$\frac{du}{dx}+2u=e^x\,(p(x)=2,Q(x)=e^x)$$
$$I.f=e^{\int p(x)dx}=e^{2\int dx}=e^{2x}$$
$$\boxed{I.f=e^{2x}}$$
$$u.e^{2x}=\int e^x e^{2x}\,dx+c_1$$
$$u=e^{-2x}[\int e^{3x}\frac{3}{2}dx+c_1]$$
$$u=e^{-2x}[\frac{1}{3}e^{3x}+c_1]$$ where c is constant

$$u=\frac{1}{3}e^{x}+c_1e^{-2x}\ ---(4)$$

Subst, equ(4) in equ.(3) , we get

$$(D-1)y=\frac{1}{3}e^{x}+c_1e^{-2x}$$

$$\frac{dy}{dx}-y=\frac{1}{3}e^{x}+c_1e^{-2x}$$ (L.F.O.D.E)

$$(p(x)=-1,Q(x)=\frac{1}{3}e^{x}+c_1e^{-2x})$$

$$I.f=e^{-x}$$

$$y.e^{-x}=\int(\frac{1}{3}e^{x}+c_1e^{-2x})\cdot e^{-x}\,dx+c_2$$

$$y=e^{x}\cdot[\int(\frac{1}{3}e^{x}+c_1e^{-2x})\cdot e^{-x}\,dx+c_2]$$

$$y=e^{x}\cdot(\frac{1}{3}x-\frac{c_1}{3}e^{-3x}+c_2)$$

$$y=e^{x}(\frac{1}{3}x-\frac{1}{3}c_1e^{-3x}+c_2)$$

$y=\frac{1}{3}xe^{x}-\frac{1}{3}c_1e^{-2x}+c_2e^{x}$ is the g. solution where c_2 is constant.

Example: $(D^3-D)y=2\cos^2 x$

Solution: $(D^3-D)y=2\cos^2 x\ ---(1)$

$$D(D^2-1)y=2\cos^2 x$$

$$D(D-1)(D+1)y=2\cos^2 x\ ---(2)$$

Suppose $(D-1)(D+1)y=u\ ---(3)$

Subst equ.(3) in equ.(2) , we get .

$$Du=2\cos^2 x$$

$\frac{du}{dx}=2\cos^2 x$ is S.D.E

$$\int du=2\int\cos^2 x\,dx$$

$$u=2\int(\frac{1}{2}+\frac{1}{2}\cos 2x)dx$$

$$u=x+\frac{1}{2}\sin 2x+c_1\ ---(4)$$

Subst equ.(4) in equ.(3) , we get

$$(D-1)(D+1)y=x+\frac{1}{2}\sin 2x+c_1\ ---(5)$$

Suppose $(D+1)y=\vartheta\ ---(6)$

Subst equ.(6) in equ.(5) , we get

$(D-1)\vartheta=x+\frac{1}{2}\sin 2x+c_1$ L.F.O.D.E

$$\frac{d\vartheta}{dx}-\vartheta=x+\frac{1}{2}\sin 2x+c_1$$

$$\vartheta e^{-x}=\int(x+\frac{1}{2}\sin 2x+c_1)e^{-x}\,dx+c_2$$

$$\vartheta e^{-x}=\int x\,e^{-x}\,dx+\frac{1}{2}\int\sin(2x)e^{-x}dx+c_1\int e^{-x}\,dx+c_2$$

$$\vartheta e^{-x}=[\int x\,e^{-x}\,dx+\frac{1}{2}\int e^{-x}\sin 2x dx+c_1\int e^{-x}\,dx+c_2]---(7)$$

Subst equ.(6) in equ.(7) , we get

$$(D-1)y=e^{x}[\int x\,e^{-x}\,dx+\frac{1}{2}\int e^{-x}\sin 2x\,dx+c_1\int dx+c_2]$$

L.F.O.D.E

Solve: H.W: $(D^2+4D+4)y=e^{-2x}\sec^2 x$

Homogenous linear D.E. with C.C.

The general from of higher-order O.D.E with C.C. is

$$(a_nD^n+a_{n-1}D^{n-1}+----+a_3D^3+a_2D^2+a_1D+a_0)y=0$$

We can study the L.H.D.E of the second order with C.C.

$$y''+ay'+by=0---(1)$$

$$\left(D^2+aD+b\right)y=0---(1)$$

Note: if $\left(D^2+aD+b\right)y=Q(x)--$

Is non-Homog. L.D.E. with C.C. we can find the general solution of equ.(*) such that contains the complementary solution of $\left(D^2+aD+b\right)y=0---(**)$ H.D.E with C.C. in which denoted by (yc) , and the particular solution of equ.(*) (non-Homog),denoted by(y_p),then $\boxed{y_6=y_c+y_p}$.

We study the H.D.E. with C.C.

$$\left(D^2+aD+b\right)y=0---(1)$$

$f(D)y=0\Rightarrow f(D)=0$ is a characteristic equ.

$$f(D)=D^2+aD+b=0$$

$$f(m)=m^2+am+b=0$$

$$\boxed{m_{1,2}=\frac{-a\mp\sqrt{a^2-4b}}{2}}---(2)$$

Or $f(D)y=0$

$$\left(D^2+aD+b\right)y=0$$

$\left(D-m_1\right)\left(D-m_2\right)y=0---(3)$ Such that

m_1,m_2 are roots

$$m_1m_2=b$$

$$-(m_1+m_2)=a$$

Suppose $\left(D-m_2\right)y=u---(4)$

Subst equ.(4) in equ.(3) , we get S.D.E.

$(D-m_1)u=0$

$Du-m_1u=0$

$\frac{du}{dx}-m_1u=0$

$\frac{du}{u}-m_1dx=0$

$\ln u-m_1x=C$,where c is constant

$\ln u=C+m_1x$

$u=e^c\cdot e^{m_1x}$ $\boxed{e^c=c*}$ $\rightarrow$ where c* is constant

$u=c*r^{m_1x}---(5)$

Subst equ.(4) in equ.(5) , we get

$(D-m_2)y=c*e^{m_1x}$

$\frac{du}{dx}-m_2y=c*e^{m_1x}---(6)$ L.F.O.D.E

$p(x)=-m_2$

$Q(x)=c*e^{m_1x}$

$y\cdot e^{-m_2x}=\int c*e^{m_1x}e^{-m_2x}dx+c_1$

$y=e^{+m_2x}[c*\int e^{(m_1-m_2)x}dx+c]_1---(7)$

We find the relation between two equ.(2)&(3). There exist three cases:

Case1: if $(a^2-4b>0)$ then there exists two distinct (different) real roots $(m_1\neq m_2\in IR)$

$y=e^{m_2x}[c*\int e^{(m_1-m_2)x}dx+c_1]$

$y=e^{m_2x}[\frac{c*}{m_1-m_2}e^{(m_1-m_2)x}+c_1]$

$y=\frac{c*}{m_1-m_2}e^{m_1x}+c_1e^{m2x}$

$y=c_2e^{m_1x}+c_1e^{m2x}$, where $c_2=\frac{c*}{m_1-m_2}$

$\boxed{y_c=c_1e^{m_1x}+c_2e^{m_2x}}$

Be a complementary solution of H.L.D.E with C.C in which contains two real distinct roots.

Case 2:

If $(a^2-4b=0)$ then there exist two (repeated) real roots

$(m_1=m_2=m\in IR)$. In equ.(7)

$y=e^{mx}[c*\int e^{(m-m)x}dx+c_1]$

$y=e^{mx}[c*\int e^{(m-m)x}dx+c_1]$

$y=e^{mx}(c*x+c_1)$

$yc=e^{mx}(c_1+c_2x)$ where $c*=c_2$

Or

$yc = c_1 e^{mx} + c_2 x\, e^{mx}$

Or

$yc = (c_1 x + c_2) e^{mx}$

Be a complementary solution of H.L.D.E with C.C in which contains two equal roots (repeated real roots).

Case 3: if $\left(a^2 - 4b < 0\right)$ then there exist two complex roots

$$m_{1,2} = a \pm ib \begin{pmatrix} m_1 = a + ib \\ m_2 = a - ib \end{pmatrix}$$

$b \neq 0, m_{1,2} \in C$

In first case

$yc = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$yc = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}$

$yc = c_1 e^{ax} e^{ibx} + c_2 e^{ax} \cdot e^{-ibx}$

By Euler's formula

$$\boxed{\begin{aligned} e^{ix} &= \cos x + i \sin x \\ e^{-ix} &= \cos x - i \sin x \end{aligned}}$$

$yc = e^{ax} [c_1 (\cos bx + i \sin bx) + c_2 (\cos bx - i \sin bx)]$

$yc = e^{ax} [(c_1 + c_2) \cos bx + (ic_1 + ic_2) \sin bx]$

$yc = e^{ax} [K_1 \cos bx + K_2 \sin bx]$

Where $c_1 + c_2 = K$ and $ic_1 + ic_2 = K_2$

be a complementary solution of H.L.D.E with C.C in which contains two complex roots .

Or

$yc = K_1 e^{ax} \cos bx + K_2 e^{ax} \sin bx$

Examples:

Solve the following H.D.E.(find the g. solution or the complementary solution)

(1) $y'' + 5y' + 6y = 0$

(2) $(D - 3)^2 y = 0$

(3) $(D^2 + D + 1) y = 0$

(4) $(D - 2)(D - 3)(D - 1)^4 y = 0$

(5) $(D^3 + 4D^2 + D - 6) y = 0$

(6) $(D^2 + 5D + 6)(D^2 + D + 1) y = 0$

(7) $(y'''' - 4y'' + 4) y = 0$

(8) $(D^3 - 1)^2 y = 0$

Solution (1): $(D^2 + 5D + 6) y = 0$

$m^2 + 5m + 6 = 0$

$(m + 3)(m + 2) = 0$

$m = -3, m = -2 \in R$

$m_1 \neq m_2 \in R$

$-2 \neq -3 \in R$

$yc = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\boxed{yc = c_1 e^{-3x} + c_2 e^{-2x}}$

Solution (2): $(D-3)^2 y = 0$

$(m-3)^2 = 0$

$m = 3,3$

$yc = e^{mx}(c_1 x + c_2)$

$yc = e^{3x}(c_1 x + c_2)$

Or

$yc = e^{3x}(c_1 + c_2 x)$

Solution (*): $(D-3)^3 y = 0$

$(m-3)^3 = 0$

$yc = e^{3x}(c_1 + c_2 x + c_3 x^2)$

Solution (3): $m^2 + m + 1 = 0$

$A = 1, B = 1, C = 1$

$$m_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4\cdot 1\cdot 1}}{2\cdot 1}$$

$\therefore m_{1,2} = a \pm ib \quad \therefore a = \dfrac{-1}{2}$

$b = \dfrac{\sqrt{3}}{2}$

$yc = e^{ax}(K_1 \cos bx + K_2 \sin bx)$

$$yc = e^{\frac{-1}{2}x}[K_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + K_2 \sin\left(\frac{\sqrt{3}}{2}x\right)]$$

Solution (4): $(D-2)(D-3)(D-1)^4 y = 0$

$(m-2)(m-3)(m-1)^4 = 0$

$m_1 = 2 \in R$

$m_2 = 3 \in R$

$m_3 = -1, -1, -1, -1 \in R$

Or

$m_3 = m_4 = m_5 = m_6 = -1$

$yc = c_1 e^{2x} + c_2 e^{3x} + e^{-x}(c_3 + c_4 x + c_5 x^2 + c_6 x^3)$

Solution (5): $(D^3 + 4D^2 + D - 6)y = 0$

$(D+3)(D^2 + D - 2)y = 0$

$(D+3)(D+2)(D-1)y = 0$

$(m+3)(m+2)(m-1) = 0$

$\Rightarrow m + 3 = 0 \Rightarrow m_1 = -3$

$\Rightarrow m + 2 = 0 \Rightarrow m_2 = -2$

$\Rightarrow m-1=0 \Rightarrow m_3=1$

$\therefore yc=c_1e^{-3x}+c_2e^{-2x}+c_3e^{x}$

Solution (6): $(D^2+5D+6)(D^2+D+1)y=0$

$(D+3)(D+2)(D^2+D+1)y=0$

$(m+3)(m+2)(m^2+m+1)=0$

$\Rightarrow m+3=0 \Rightarrow m_1=-3$

$\Rightarrow m+2=0 \Rightarrow m_2=-2$

$\Rightarrow m^2+m+1=0 \Rightarrow m_{3,4}=\dfrac{-1\pm\sqrt{1^2-4\cdot1\cdot1}}{2}$

$\Rightarrow m_{3,4}=\dfrac{-1}{2}\mp i\,\dfrac{\sqrt{3}}{2}$

$\therefore yc=c_1e^{-3x}+c_2e^{-2x}+e^{\frac{-1}{2}x}[K_1\cos\left(\dfrac{\sqrt{3}}{2}x\right)+K_2\sin\left(\dfrac{\sqrt{3}}{2}x\right)]$

Solution (7): $(y''''-4y''+4y)=0$

$D^4y-4D^2y+4Y=0$

$(D^2-2)^2y=0$

$(m^2-2)^2=0$

$(m^2-2)(m^2-2)=0$

$m^2-2=0 \Rightarrow m^2=2 \Rightarrow m_{1,2}=\mp\sqrt{2}$

$m^2-2=0 \Rightarrow m^2=2 \Rightarrow m_{3,4}=\mp\sqrt{2}$

$yc=e^{\sqrt{2}x}(c_1+c_2x)+e^{-\sqrt{2}x}(c_3x+c_4)$

Or

$yc=(c_1e^{\sqrt{2}x}+c_2e^{-\sqrt{2}x})+x(c_3e^{\sqrt{2}x}+c_4e^{-\sqrt{2}x})$

Solution (8): $(D^3-1)y=0$

$(D-1)(D^2+D+1)y=0$

$(m-1)(m^2+m+1)=0$

$m-1=0 \Rightarrow m_1=1 \in IR$

$m^2+m+1=0 \Rightarrow m_{2,3}=\dfrac{-1}{2}\mp i\,\dfrac{\sqrt{3}}{2} \in \not{c}$

$yc=c_1e^{x}+e^{\frac{-1}{2}x}[c_2\cos\left(\dfrac{\sqrt{3}}{2}x\right)+c_3\sin\left(\dfrac{\sqrt{3}}{2}x\right)]$

Solution (9): $(D^3-1)^2y=0$

$(m-1)^2(m^2+m+1)^2=0$

$m-1=0 \Rightarrow m_{1,2}=-1,1 \in R$

$$\boxed{y c_1 = e^{x}\left(c_1 x + c_2\right)}$$

$$m_{\substack{3,4\\5,6}} = \frac{-1}{2} \pm i\,\frac{\sqrt{3}}{2} \in \mathbb{C}$$

$$y c_2 = e^{\frac{-1}{2}x}\left[k_1\cos\left(\frac{\sqrt{3}}{2}x\right) + k_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + x e^{\frac{-1}{2}x}\left[k_3\cos\left(\frac{\sqrt{3}}{2}x\right) + k_4\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$y c = e^{x}\left(c_1 x + c_2\right) + e^{\frac{-1}{2}x}\left[k_1\cos\left(\frac{\sqrt{3}}{2}x\right) + k_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + x e^{\frac{-1}{2}x}\left[k_3\cos\left(\frac{\sqrt{3}}{2}x\right) + k_4\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$ H.W:

solve

(1) $y''' - y'' - 8y' + 12y = 0$

(2) $(D3 + 3D^2 + 3D + 1)y = 0$

(3) $(D^3 + 2D^2 - 5D - 6)y = 0$

(4) $\left(y'''' - 16y\right)^2 = 0$

(5) $\left(D+1\right)^3\left(D^2 + 2D + 1\right)\left(D^2 - D + 2\right)^3\left(D+7\right)\left(D-5\right)y = 0$

How to find a particular solution of L. Non-Homog D.E. with C.C.:

Consider $(a_n D^n + a_{n-1}D^{n-1} + - - - - + a_3 D^3 + a_2 D^2 + a_1 D + a_0)y = Q(x)$

BE A L.NON-H. D.E With C.C, where $a_0, a_1, - - -, a_n$ Constants and Q is a function of (x) only.

There exist three methods:

(1) the variation of parameters method
(2) the operators method
(3) Un determinant coefficients method

(1) The variation of parameters method:

We study non-H.L.D.E of second order with C.C.:

$$y'' + ay' + by = Q(x) - - - (1)$$

$$D^2 + aD + by = Q(x) - - - (1)$$

To find the general solution of D.E (1), we obtain (yc) and (yp)

$y_G = y_c + y_p$ to find (yc) (be a complementary solution) of equ.(1) suppose

$$\left(D^2 + aD + b\right)y = 0 - - - (2)$$

$$\boxed{m_2 + am + b = 0}$$

$y_c = c_1 y_1 + c_2 y_2 - - - (3)$ (where y_1, y_2 are two linearly independent. Solutions) since y_1, y_2 are two linearly independent solution. Solution then

$$\begin{pmatrix} y_1'' + a y_1' + b y_1 = 0 \\ y_2'' + a y_2' + b y_2 = 0 \end{pmatrix} - - - (*)$$

We can find a particular solution of D.E(1) by changing two arbitrary constants c_1, c_2 in to $v_1(x), v_2(x)$

$y_P = v_1 y_1 + v_2 y_2 - - - (4)$ be a particular solution of D.E(1)

$$y_p = v_1 y_1' + y_1 v_1' + v_2 y_2' + v_2' y_2$$

Suppose $\boxed{y_1v_1' + y_2v_2' = 0}$ $---(A)$

$$y'p = v_1y_1' + v_2y_2' \quad ----(5)$$

$$y''p = v_1y_1'' + y_1'v_1' + v_2y_2'' + y_2'v_2' \quad ---(6)$$

Subst. equ (4, 5 and 6) in D.E (1), we get

$$v_1y_1'' + y_1'v_1' + v_2y_2'' + y_2'v_2' + a(v_1y_1' + v_2y_2') + b(v_1y_1 + v_2y_2) = Q(x)$$

$$v_1(y_1'' + ay_1' + by_1) + v_2(y_2'' + ay_2' + by_2) + v_1'y_1' + v_2'y_2' = Q(x)$$

$$v_1(0) + v_2(0) + v_1'y_1' + v_2'y_2' = Q(x)$$

$$v_1'y_1' + v_2'y_2' = Q(x) \quad ----(B)$$

We can solve both equ. A and B by grammer's method.

$$v_1'y_1' + v_2'y_2' = Q(x) \quad ----(B)$$

$$v_1'y_1' + v_2'y_2' = 0 \quad ----(A)$$

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ Q(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \Rightarrow v_1' = \frac{-y_2Q(x)}{w(y_1, y_2)}$$

$$v_1 = \int \frac{-y_2Q(x)}{w(y_1, y_2)} dx$$

And

$$v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & Q(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \Rightarrow v_2' = \frac{y_1Q(x)}{w(y_1, y_2)}$$

$$v_2 = \int \frac{y_1Q(x)}{w(y_1, y_2)} dx$$

$$y_P = \left(-\int \frac{y_2Q(x)}{w(y_1, y_2)} dx\right)y_1 + \left(\int \frac{y_1Q(x)}{w(y_1, y_2)} dx\right)y_2$$

$$y_G = y_C + y_P$$

Example: - solve $y'' + y' = e^x$

Solution:- $(D^2 + D)\ y = e^x$

$$(D^2 + D)\ y = 0$$

$$m^2 + m = 0$$

$$m(m+1) = 0$$

$$m_1 = 0\ , m_2 = -1$$

$$y_c = c_1e^{0x} + c_2e^{-x}$$

$$y_c = c_1 1 + c_2e^{-x}$$

$$y_P = v_1y_1 + v_2y_2$$

$$y_1 = 1$$

$$y_2 = e^{-x}$$

$$y_P = v_1 1 + v_2 e^{-x}$$

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$w(1, e^{-x}) = \begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{vmatrix} = -e^{-x}$$

$$v_1 = \int \frac{y_1 Q(x)}{w(y_1, y_2)} dx \Rightarrow v_1 = \int \frac{-e^{-x} e^{x}}{-e^{-x}} dx \Rightarrow v_1 = e^{x}$$

$$v_2 = \int \frac{y_1 Q(x)}{w(y_1, y_2)} dx \Rightarrow \quad v_2 = \int \frac{e^{x}}{-e^{-x}} dx \Rightarrow v_2 = -\int e^{2x} dx \quad v_2 = -\frac{1}{2} e^{2x}$$

$$y_p = e^{x} - \frac{1}{2} e^{x} \Rightarrow y_p = \frac{1}{2} e^{x}$$

$$y_G = y_C + y_P \Rightarrow y_G = c_1 1 + c_2 e^{-x} + \frac{1}{2} e^{x}$$

H.W.
Solve the following D.Es.

1) $y'' + 4y = \sec 2x$

2) $y'' + y = \tan x$

3) $(D^2 + 4)y = \sec^2 2x$

4) $y'' + y' + y = x^2 e^{-x}$

Operator's method

To study n-th order O.D.E with C.C by using operators method (D) dependent on the type of Q(x) and by using some theorems.

Theorem: - 1) $f(D)e^{bx} = f(b)e^{bx}$

2) $f(D^2)\cos bx = f(-b^2)\cos bx$ where b is constant.

3) $f(D^2)\sin bx = f(-b^2)\sin bx$

4) $f(D)\{e^{bx} y\} = e^{bx}\{f(D+b)y\}$

Proof (1):- $f(D)e^{bx} = (P_n D^n + P_{n-1} D^{n-1} + \dots + P_3 D^3 + P_2 D^2 + P_1 D^1 + P_0)e^{bx}$ where $p_0, p_1, \dots, p_n$ are constant

$$f(D)e^{bx} = (P_n(D^n e^{bx}) + P_{n-1}(D^{n-1} e^{bx}) + \dots + P_3(D^3 e^{bx}) + P_2(D^2 e^{bx}) + P_1(D^1 e^{bx}) + P_0 e^{bx})$$

$$(D^1 e^{bx}) = b e^{bx}$$

$$(D^2 e^{bx}) = D(De^{bx})$$
$$= D(be^{bx})$$
$$= b(De^{bx})$$
$$= b^2 e^{bx}$$

$$(D^3 e^{bx}) = D(D^2 e^{bx})$$
$$= D(b^2 e^{bx})$$
$$= b^2(De^{bx})$$
$$= b^3 e^{bx}$$

.
.
.

$(D^{n-1}e^{bx})=b^{n-1}e^{bx}$

$(D^{n}e^{bx})=b^{n}e^{bx}$

$f\ (D)e^{bx}=(P_n(b^n e^{bx})+P_{n-1}(b^{n-1}e^{bx})+...+P_3(b^3e^{bx})+P_2(b^2e^{bx})+P_1(be^{bx})+P_0e^{bx})$

$f\ (D)e^{bx}=(P_nb^n+P_{n-1}b^{n-1}+...+P_3b^3+P_2b^2+P_1b+P_0)e^{bx}$

$f(D)e^{bx}=f(b)e^{bx}$

Type of Q(x)

1) e^{bx}
2) $\cos bx$ or $\sin bx$
3) x^m
4) $e^{bx}\cos bx$ or $e^{bx}\sin bx$ or $e^{bx}\ x^m$
5) $x^m\cos bx$ or $x^m\sin bx$
6) $e^{bx}\ x^m\cos bx$ or $e^{bx}\ x^m\sin bx$

There exist six cases.

Case1:- If Q(x) $=e^{bx}$ then the particular solution is[$f(D)y=Q(x)\Rightarrow f(D)y=e^{bx}$]

$$y_p=\frac{1}{f(D)}Q(x)\Rightarrow y_p=\frac{1}{f(D)}e^{bx}$$

i) If $f(b)\neq 0$ then $y_p=\frac{1}{f(D)}e^{bx}$ (By using theorem $f(D)e^{bx}=f(b)e^{bx}$)

Example 1): - solve: $y''+2y'+y=e^x$

Solution: $y''+2y'+y=e^x$

$$(D^2+2D+1)y=e^x$$

$$(D+1)^2y=e^x$$

Suppose $(D+1)^2y=0$

$$(m+1)^2=0$$

$$m=-1\,,-1$$

$$y_c=e^{-x}(c_1+c_2x)$$

$$y_p=\frac{1}{f(D)}e^x$$

$$y_p=\frac{1}{(D^2+2D+1)}e^x$$

$$y_p = \frac{1}{(1^2 + 2*1+1)} e^x$$

$$y_p = \frac{1}{4} e^{bx}$$

$$y_G = e^{-x}(c_1 + c_2 x) + \frac{1}{4} e^x$$

Example 2):- solve: $y'' + y' + 4y = e^{-x}$

$$(D^2 + D + 4)y = e^x$$

$$m^2 + m + 4 = 0$$

$$m_{1,2} = \frac{-1 \pm \sqrt{1^2 - 16}}{2}$$

$$m_{1,2} = \frac{-1}{2} \pm i\frac{\sqrt{15}}{2}$$

$$y_c = e^{-\frac{1}{2}x}[k_1 \cos(\frac{\sqrt{5}}{2}x) + k_2 \sin(\frac{\sqrt{5}}{2}x)]$$

$$y_p = \frac{1}{f(D)} e^{bx}$$

$$y_p = \frac{1}{(D^2 + D + 4)} e^{-x}$$

$$y_p = \frac{1}{((-1)^2 + (-1) + 4)} e^{-x}$$

$$y_p = \frac{1}{4} e^{-x}$$

$$y_G = e^{-\frac{1}{2}x}[k_1 \cos(\frac{\sqrt{5}}{2}x) + k_2 \sin(\frac{\sqrt{5}}{2}x)] + \frac{1}{4} e^{-x}$$

H.W: solves the following.

1) $y'' + y' - 2y = e^{5x}$

2) $y''' - y'' = e^{3x}$

ii) If $f(b) = 0$ then $y_p = \frac{1}{f(b) = 0} e^{bx}$ undefined, and $y_p = \frac{1}{(D-b)^r g(D)} e^{bx}$

Where b is a root in the complementary solution and (r) is the number of repeated root g(D) the remainder terms in f(D).

By using theorem $f(D)\{e^{bx} y\} = e^{bx}\{f(D+b)y\}$

$$y_p = e^{bx} \frac{1}{(D-b+b)^r g(D+b)}\{1\}$$

$$y_p = e^{bx} \frac{1}{(D)^r g(D+b)}$$

$$y_p = \frac{e^{bx}}{g(b)}(D)^{-r} \quad \text{, where } g(D+b) = g(b)$$

$$y_p = \frac{e^{bx}}{g(b)} \frac{x^r}{r!} \qquad , \text{where } (D)^{-r} = \frac{x^r}{r!}$$

It is particular solution.

Examples:- $y''' - y'' = e^x$

Solution:- $(D^3 - D^2)y = e^x$

$$(D^3 - D^2)y = 0$$

$$D^2(D-1)y = 0$$

$$m^2(m-1) = 0 \Rightarrow m_{1,2} = 0, m_3 = 1$$

$$y_c = (c_1 + c_2 x) + c_3 e^x$$

$$y_p = \frac{1}{f(D)} Q(x)$$

$$y_p = \frac{1}{D^3 - D^2} e^x$$

$$y_p = \frac{1}{(D-b)^r g(D)} e^{bx}$$

$$y_p = e^x \frac{1}{(D-1+1)^r g(D+1)}$$

$$y_p = \frac{e^x}{D(D+1)^2}$$

$$y_p = \frac{e^x}{D*1} \Rightarrow y_p = e^x D^{-1}$$

$$y_p = e^x x$$

$$y_G = (c_1 + c_2 x) + c_3 e^x + e^x x$$

Example:- $(D+2)^3(D-1)y = e^{-2x}$

Solution:- $(m+2)^3(m-1) = 0$

$$m_{1,2,3} = -2 \, , m_4 = 1$$

$$y_c = e^{-2x}(c_1 + c_2 x + c_3 x^2) + c_4 e^x$$

$$y_p = \frac{1}{(D+2)^3(D-1)} e^{-2x}$$

$$y_p = e^{-2x} \frac{1}{(D)^3(D-3)}$$

$$y_p = \frac{e^{-2x}}{-3} D^{-3}$$

$$y_p = \frac{e^{-2x}}{-3} \frac{x^3}{3!}$$

$$y_G = e^{-2x}(c_1 + c_2 x + c_3 x^2) + c_4 e^x + \frac{e^{-2x}}{-3} \frac{x^3}{3!}$$

H.W: Solve the following solution.

1) $(D-1)(D+2)(D-3)y = e^{3x}$

$2)(D-1)(D-2)^2 y=(e^{2x}+2e^{x}+3e^{-x})$

Case 2:- If $Q(x)=\cos ax \ or \ \sin ax$ then $f(D^2)y=\cos ax \ or \ \sin ax$

$$\Rightarrow y_p=\frac{1}{f(D^2)}\cos ax \ or \ \sin ax$$

By using theorem $f(D^2)\cos ax=f(-a^2)\cos ax$ or $f(D^2)\sin ax=f(-a^2)\sin ax$

There exist two branches

i)If $f(-a^2)\neq 0$ then $y_p=\dfrac{1}{f(-a^2)}\cos ax \ or \ \sin ax$

Example:- $y''-4y=\cos 2x$

Solution:- $(D^2-4)y=$

$$m^2-4=0\Rightarrow m=\pm 2$$

$$y_c=c_1e^{2x}+c_2e^{-2x}$$

$$y_p=\frac{1}{D^2-4}\cos 2x$$

$$y_p=\frac{1}{-2^2-4}\cos 2x$$

$$y_p=\frac{-1}{8}\cos 2x$$

$$y_G=c_1e^{2x}+c_2e^{-2x}+\frac{-1}{8}\cos 2x$$

Example:- $(D^2+3D-4)y=\sin 2x$

Solution:- $(D^2+3D-4)y=0$

$$m^2+3m-4=0$$
$$(m+4)(m-1)=0$$
$$m_1=-4 \ , m_2=1$$

$$y_c=c_1e^{-4x}+c_2e^{x}$$

$$y_p=\frac{1}{D^2+3D-4}\sin 2x$$

$$y_p=\frac{1}{-2^2+3D-4}\sin 2x$$

$$y_p=\frac{1}{3D-8}\sin 2x$$

$$y_p=\frac{3D+8}{9D^2-64}\sin 2x$$

$$y_p=\frac{3D+8}{9(-2^2)-64}\sin 2x$$

$$y_p=\frac{3D+8}{-100}\sin 2x$$

$$y_p = \frac{-1}{100}(3D\sin 2x + 8\sin 2x)$$

$$y_p = (\frac{-6}{100}\cos 2x + \frac{-8}{100}\sin 2x)$$

$$y_G = c_1 e^{-4x} + c_2 e^{x} + (\frac{-6}{100}\cos 2x + \frac{-8}{100}\sin 2x)$$

H.W: Solve the following solution.

$$y'' - 9y = 5cos 3x$$

ii) If $f(-a^2) = 0$ then $y_p = \frac{1}{f(-a^2)} cosax \ or \ \sin ax$

can be changing $cosax \ or \ \sin ax$ into Euler's formula

$$e^{ix} = cosx + i \sin x \ or \ (e^{iax} = cosax + i \sin ax)$$

$y_p = \frac{1}{f(D^2)} e^{aix}$ of the 1-st cases

Can be solved by first cases. Finally e^{aix} changed into $(cosax + i \sin ax)$ then choose the real part if the problem contains $(cosax)$ or choose the imaginary part if the problem contains $(\sin ax)$

Example:- $(D^2 + 9)y = \sin 3x$

Solution:- $(D^2 + 9)y = 0$

$$m^2 + 9 = 0$$

$$m^2 = -9 \Rightarrow m = \pm i3$$

$$y_c = [k_1 cos 3x + k_2 \sin 3x]$$

$$y_p = \frac{1}{(D^2 + 9)}\sin 3x$$

$$y_p = \frac{1}{(-3^2 + 9)}\sin 3x$$

$$y_p = \frac{1}{0}\sin 3x \qquad uudefined$$

$$e^{i3x} = cos 3x + i \sin 3x$$

$$y_p = \frac{1}{D^2 + 9} e^{3ix}$$

$$y_p = \frac{1}{(D + 3i)(D + 3i)} e^{3ix}$$

$$y_p = \frac{e^{3ix}}{6i}\frac{x}{1!}$$

$$y_p = \frac{1}{6i}\frac{x}{1!}(cos 3x + i \sin 3x)$$

$$y_p = \frac{x}{6i} cos 3x - \frac{x}{6}\sin 3x$$

$$y_p = \frac{x}{6i} cos 3x$$

$$y_G = k_1 cos 3x + k_2 \sin 3x + (\frac{x}{6i} cos 3x)$$

Case 3:- If $Q(x) =$ polynomials function i.e. $Q(x) = x^m \Rightarrow y_p = \frac{1}{f(D)} x^m$ can be solved by using the series

$$\frac{1}{1-D} = (1-D)^{-1} = 1 + D + D^2 + D^3 + \ldots$$

Or $\frac{1}{1+D} = (1+D)^{-1} = 1 - D + D^2 - D^3 + \ldots$

Example:- solve : $y'' + 2y' + 3y = x^3$

Solution:- $(D^2 + 2D + 3)y = 0$

$$m^2 + 2m + 3 = 0$$

$$m_{1,2} = \frac{-2 \pm \sqrt{-8}}{2}$$

$$m_{1,2} = \frac{-2}{2} \pm \frac{2i\sqrt{2}}{2}$$

$$m_{1,2} = -1 \pm i\sqrt{2}$$

$$y_c = e^{-x}(k_1 \cos(\sqrt{2}x) + k_2 \sin(\sqrt{2}x))$$

$$y_p = \frac{1}{D^2 + 2D + 3} x^3$$

$$y_p = \frac{1}{3[1 + (\frac{D^2}{3} + 2\frac{D}{3})]} x^3$$

$$y_p = \frac{1}{3}[1 + (\frac{D^2}{3} + 2\frac{D}{3})]^{-1} x^3$$

$$y_p = \frac{1}{3}[1 - (\frac{D^2}{3} + 2\frac{D}{3}) + (\frac{D^2}{3} + 2\frac{D}{3})^2 + (\frac{D^2}{3} + 2\frac{D}{3})^3] x^3$$

$$y_p = \frac{x^3}{3} - \frac{1}{9}D^2x^3 - \frac{2}{9}Dx^3 + \frac{1}{3}(\frac{D^4}{9} + \frac{4}{9}D^3 + \frac{4}{9}D^3 + \frac{2}{9}D^2)x^3$$

$$y_p = \frac{x^3}{3} - \frac{6}{9}x - \frac{2}{3}x^2 + \frac{24}{27}x - \frac{48}{81}$$ is particular solution .

$$y_G = e^{-x}(k_1 \cos(\sqrt{2}x) + k_2 \sin(\sqrt{2}x)) + \frac{x^3}{3} - \frac{6}{9}x - \frac{2}{3}x^2 + \frac{24}{27}x - \frac{48}{81}$$

Case 4:- if $Q(x) = e^{ax} \cos bx$ or $e^{ax} \sin bx$ or $Q(x) = e^{ax} x^m$ then

$y_p = \frac{1}{f(D)} e^{ax} cosbx$ or $(\sin bx)$ or $y_p = \frac{1}{f(D)} e^{ax} x^m$ can be solved by using the theorem

$$f(D)\{e^{bx}y\} = e^{bx} f(D+y)$$

$y_p = e^{ax} \frac{1}{f(D+a)} cosbx$ or $(\sin bx)$ can be solving by second cases or $y_p = e^{ax} \frac{1}{f(D+a)} x^m$ can be solved by third cases

Example: - solve: $(D^2 - 4)y = e^{3x} \sin 2x$

Solution:- $(D^2-4)y=0$

$$m^2-4=0\Rightarrow m_{1,2}=\pm 2$$

$$y_c=c_1e^{2x}+c_2e^{-2x}$$

$$y_p=e^{3x}\frac{1}{(D+3)^2-4}\sin 2x$$

$$y_p=e^{3x}\frac{1}{D^2+6D+9-4}\sin 2x$$

$$y_p=e^{3x}\frac{1}{D^2+6D+5}\sin 2x$$

$$y_p=e^{3x}\frac{1}{-2^2+6D+5}\sin 2x$$

$$y_p=e^{3x}\frac{1}{1+6D}\sin 2x$$

$$y_p=e^{3x}\frac{1-6D}{1-36D^2}\sin 2x$$

$$y_p=e^{3x}\frac{1-6D}{1-36(-(2)^2)}\sin 2x$$

$$y_p=\frac{e^{3x}}{145}\sin 2x-\frac{12e^{3x}}{145}\cos 2x$$

Case 5:- If $Q(x)=x^m\cos bx\, or\,(x^m\sin bx)$ then $y_p=\frac{1}{f(D)}x^m\cos bx\, or\,(x^m\sin bx)$

We can solved by changing sinbx or cosbx into Euler's formula $(e^{ib}=\cos bx+i\sin bx)$

$$y_p=\frac{1}{f(D)}x^m e^{ib}\ \text{----*}$$

$y_p=e^{ibx}\frac{1}{f(D+ib)}x^m$ can be solving by third cases. Finally we can changing e^{ibx} into $\cos bx+i\sin bx$ and chooses the real part if equ (*) contain (cosbx),but if equ(*) contain (sinbx) ,then choose the imaginary part.

Example:- $(D^2+3D+2)y=x\sin 2x$

Solution:- y_c = H.W

$$y_p=\frac{1}{(D^2+3D+2)}x\sin 2x$$

$$y_p=\frac{1}{(D^2+3D+2)}xe^{i2x}$$

$$y_p=e^{i2x}\frac{1}{(D+2i)^2+3(D+2i)+2)}x$$

$$y_p = e^{i2x}\frac{1}{D^2 + 4Di - 4 + 3D + 6i + 2}x$$

$$y_p = e^{i2x}\frac{1}{D^2 + 4Di - 4 + 3D + 6i + 2}x$$

$$y_p = e^{i2x}\frac{1}{D^2 + 4Di + 3D + 6i - 2}x$$

$$y_p = e^{i2x}\frac{1}{(6i-2)[1+(\frac{D^2}{6i-2}+\frac{4Di}{6i-2}+\frac{3D}{6i-2}]}x$$

$$y_p = \frac{e^{i2x}}{(6i-2)}[1-(\frac{D^2}{6i-2}+\frac{4Di}{6i-2}+\frac{3D}{6i-2})]x$$

$$y_p = \frac{e^{i2x}}{(6i-2)}[x - \frac{4i}{(6i-2)} - \frac{3}{(6i-2)}]$$

$$y_p = \frac{(cos\,2x + i\sin 2x)}{(6i-2)}[x - \frac{4i}{(6i-2)} - \frac{3}{(6i-2)}]$$