زانكۆی سەلاحەدین - هەولێر
Salahaddin University-Erbil

The mean distance of the Möbius Kantor special graph MK_{16n}

Research Project

Submitted to the Department of Mathematics in partial fulfillment of the requirements for the degree of BSc. in MATHEMATIC

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Certification of The Supervisor

I certify that this work was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University- Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

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Acknowledgment

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My thanks appreciations go to Mathematical Department and all my valuable teachers.

Abstract

In this work we extend the Möbius-Kantor special graph MK of order 16 into $MK_{16\mathrm{n}}$ of order $16n$, for positive integer $n \geq 1$.Then we find the mean distance of MK and $MK_{16\mathrm{n}}$ which are $\mu(MK) = \frac{34}{15}$ and $\mu(MK_{16\mathrm{n}}) = \frac{n^2+20n-32}{16(n-1)}$ respectively.

Moreover, we study some graphical properties of $MK_{16\mathrm{n}}$,such as: regularity, girth, circumference, eccentricity, radius, diameter, non-Hamiltonian, non-Eulerian, non-tree.

Table of Contents

List of Figures

CHAPTER ONE

INTRODUCTION

In an old city of Eastern Prussia, named Königsberg, there was a river, called River Pregel, flowing through its center. In the 18th century, there were seven bridges over the river connecting the two islands (B and D) and two opposite banks (A and C) as shown in Figure 1.1.

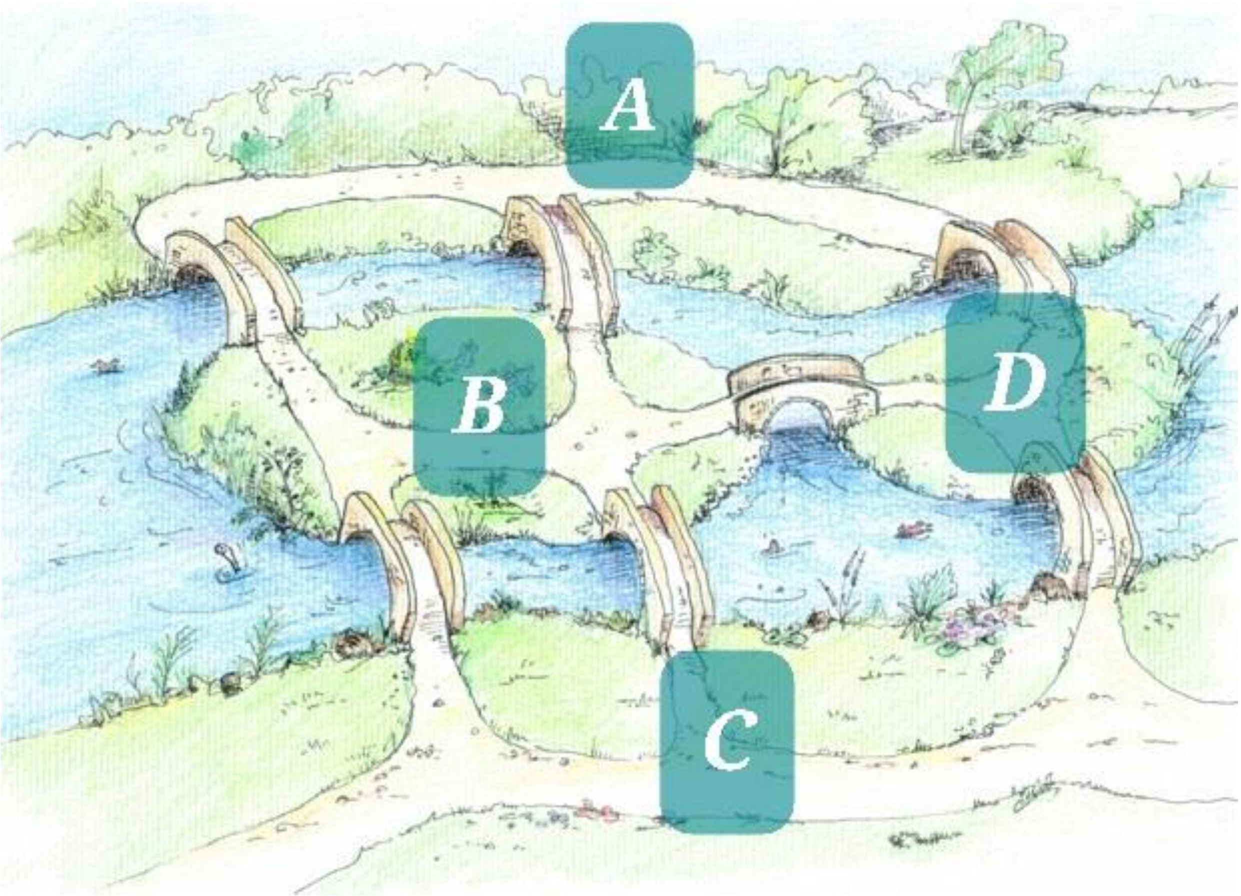


Figure 1.1 The bridges in Königsberg.

It was said that the people in the city had always amused themselves with the following problem:

Starting with any one of the four places A, B, C or D as shown in Figure 1.1, is it possible to have a walk which passes through each of the seven bridges once and only once, and return to where you started? No one could find such a walk; and after a number of tries, people believed that it is simply not possible, but no one could prove it either. (Koh, et al., 2007) (Pro.Tim, 2020) (Swati & Dr.Chinta, 2021)

Leonhard Euler, the greatest mathematician that Switzerland has ever produced, was told of the problem. He noticed that the problem was very much different in nature from the problems in traditional geometry, and instead of considering the original problem, he studied it's much more general version which encompassed any number of islands or banks, and any number of bridges connecting them. His finding was contained in the article [E] (the English translation of its title is: The solution of a problem to the geometry of position) published in 1736. As a direct consequence of his finding, he deduced the impossibility of having such a walk in the Königsberg bridge problem. This was historically the first time a proof was given from the mathematical point of view. (Koh, et al., 2007) (Pro.Tim, 2020) (Swati & Dr.Chinta, 2021)

Graph theory is the most useful subject in all branches of Mathematics and widely applied in subjects like Computer Technology, Communication Science, Electrical Engineering, Physics, Architecture, Operation Research, Economics, Sociology, Genetics etc. Graph Theory is the study of graphs which are mathematical structures used to model pairwise relation between objects. It is a bridge connecting mathematics with different branches of Science. It is a branch of discrete Mathematics. The graph is a way to express the information in picture form. Graph theory is increasing area as it is applied to areas of Mathematics, Science and Technology. From the last century, many researchers attracted towards Graph theory. (Pradnya, 2022) (Gowda, et al., 2021) (Swati & Dr.Chinta, 2021)

Graph theory provided healthy atmosphere for research of provable technique in discrete Mathematics for researchers. Many applications are studied by Graph Theory in the computing, Industrial, natural and social sciences. Graph theory is a vast area with many applications to real life which helps the researchers to get more ideas to manage the problems in the real-life situation. It has tremendous application in modern science and engineering. (Pradnya, 2022) (Gowda, et al., 2021) (Swati & Dr.Chinta, 2021)

The concepts and ideas of graph theory are widely used in various branches of science. Now without knowledge of graph we use graph theory concepts in our daily life. For example, when we have to go to a place which is connecting with our starting point by different ways then we use the shortest road to arrive the destination earlier. Here if we consider this problem in terms of graph theory the two places can be considered as vertices and roads are as edges. If we also assume the direction of travel, then the graph is directed. In similar way we can use concepts of graph theory in various situations. (Pradnya, 2022) (Gowda, et al., 2021) (Swati & Dr.Chinta, 2021)

In Mathematics Graph Theory is used in Operation Research: Graph theory is dynamic tool in combinatorial operation research. The most popular and successful applications of network in OR is the planning and scheduling of large complicated project. Game theory is applied to the problems in engineering, economics and war science to find optimal way to perform certain tasks in competitive environment. The method of finite game is represented by biograph. Many important OR problems are solved using graphs. A Graph is used to model the transportation of commodity from one destination to another. A network called transport network. The main objective is to maximize and minimize the cost within the prescribed flow. (Pradnya, 2022) (Gowda, et al., 2021) (Swati & Dr.Chinta, 2021)

The applications of Graph theory in Daily life are such GPS or Google Map: To find the shortest distance from one destination to another we use GPS map. Here destination is vertices and their connections are edges. The optimal route is obtained by this software. School and colleges are used this technique to pick up students from their stop to school. Each stop is considered as vertex and the route is considered as an edge. The concept of Hamiltonian path is used. Social

Network: We connect with friends and family via social media or video get viral. Here user is a vertex and other connected users are edge. Therefore, video get viral when reached to certain connection. Google Search: By using google we search the web pages. Pages on the internet are linked by the hyperlink. Here page is a vertex and the link between two pages is an edge. Here connected graph concept is used. (Pradnya, 2022) (Gowda, et al., 2021) (Swati & Dr.Chinta, 2021)

CHAPTER TWO

BACKGROUND

Definition 2.1: (CHartrand, et al., 2016) A *graph* G is a finite non empty set V of objects called *vertices* (the singular is *vertex*) together with a possibly empty set E of un order pairs of distinct vertices of G called *edges*.

Definition 2.2: (John, et al., 2008) The *order* of a graph G is defined to be the cardinality of the set of vertices of G which is denoted by $|V(G)|$.

Definition 2.3: (John, et al., 2008)The *size* of a graph G is defined to be the cardinality of edges of G which is denoted by $|E(G)|$.

Definition 2.4: (John, et al., 2008) The *distance* from a vertex u to a vertex v in a graph G is the length (number of edges) of a shortest $u - v$ path in G which is denoted by $d(u, v)$.

Definition 2.5: (P.Dankkelmann, 1997) (S.Sivasubramanian, 2009) Let G be a finite connected graph of order p then the *mean* (*average*) *distance of* G is denoted by $\mu(G)$ and defined by $\mu(G) = \frac{\delta(G)}{p(p-1)}$, where p is the order of G.

Definition 2.6: (R.Sharafdini & T.Reti, 2020)Let G be a connected graph, and $v \in V(G)$, the *transmission of a vertex* v is denoted by $\sigma(v)$ and defined by $\sigma(v) = \sum_{u \in V(G)} d(v, u)$.

Definition 2.7: (P.Dankkelmann, 1997) Let G be a connected graph. Then the *transmission of* G is denoted by $\sigma(G)$ and defined by $\sigma(G) = \sum_{u \in V(G)} \delta(u)$.

Definition 2.8: (Pradnya, 2022) A *walk* in a graph G is defined to be an alternating sequence of vertices and edges with finite length.

Definition 2.9: (Pradnya, 2022) An open walk in which no vertex appears more than once is called a *path*.

Definition 2.10: (Pradnya, 2022) A closed walk in which no vertex appear more than once is called a *cycle.*

Definition 2.11: (John, et al., 2008) The *length* of a walk (or path, or cycle) is its number of edges, counting repetitions.

Definition 2.12: (CHartrand, et al., 2016) A walk whose initial and terminal vertices are distinct is an *open walk.*

Definition 2.13: (CHartrand, et al., 2016) Two vertices u and v in a graph G are called *connected* if G contains a $u - v$ path. The graph G is itself connected if every two vertices of G are connected.

Definition 2.14: (CHartrand, et al., 2016) The *eccentricity* $e(v)$ of a vertex v in a connected graph G is the distance between v and a vertex farthest from v in G.

Definition 2.15: (CHartrand, et al., 2016) The *diameter* $diam(G)$ of a connected graph G is the largest eccentricity among the vertices of G.

Definition 2.16: (CHartrand, et al., 2016) The *radius* $rad(G)$ is the smallest eccentricity among the vertices of G.

Definition 2.17: (CHartrand, et al., 2016) A walk whose initial and terminal vertices are same is a *closed walk.*

Definition 2.18: (CHartrand, et al., 2016) The length of a smallest cycle in a graph G (containing cycles) is the *Girth of* G, denoted by $g(G)$.

Definition 2.19: (CHartrand, et al., 2016) The length of a longest cycle is the *circumference of* G, denoted by $c(G)$.

Definition 2.20: (Pradnya, 2022) A *bipartite graph* or biograph is a graph whose vertex set can be divided into two disjoint and independent sets V_1 and V_2 such that every edge connects a vertex in V_1 to one in V_2.

Definition 2.21: (John, et al., 2008) A graph G is *regular* if every vertex of G has the same degree.

Definition 2.22: (W.Weisstein, 1995) *cubic graphs*, also called trivalent graphs, are graphs all of whose nodes have degree 3 (i.e., 3-regular graphs).

Definition 2.23: (Kenneth, 2014) A graph is *Eulerian* if it has a closed walk that contains every edge exactly once.

Definition 2.24: (CHartrand, et al., 2016) A graph that contains a Hamiltonian cycle is itself called *Hamiltonian.*

Definition 2.25: (CHartrand, et al., 2016) A *tree* is a connected acyclic graph.

Definition 2.26: (Sanger & Wales, 2022) The *Möbius–Kantor graph* is a symmetric bipartite cubic graph with 16 vertices and 24 edges named after August Ferdinand Möbius and Seligmann Kantor, as given in Figure 2.1 below.

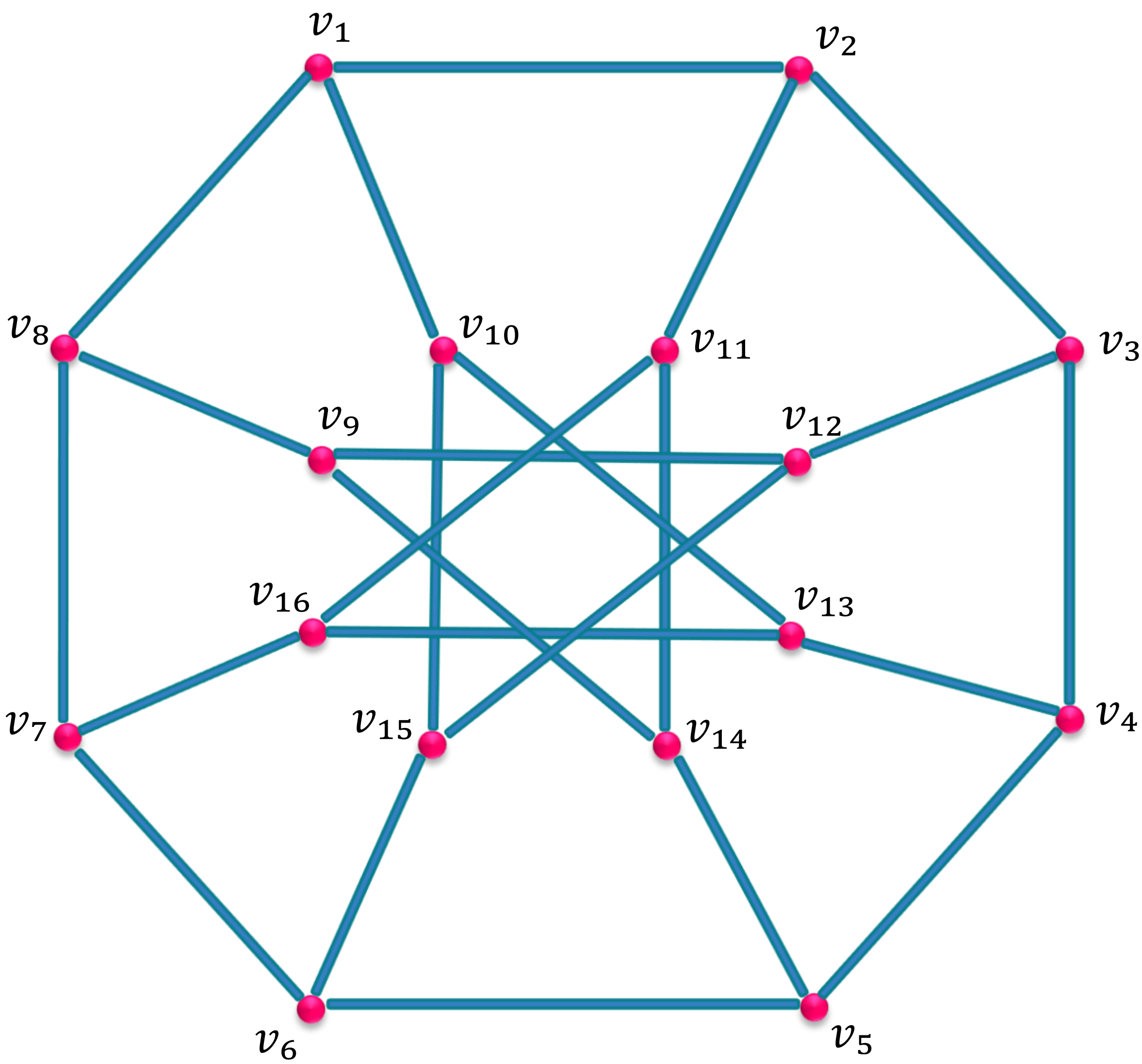


Figure 2.1 Mobius – Kantor MK graph

CHAPTER THREE

THE MEAN DISTANCE OF MÖBIUS-KANTOR SPECIAL GRAPH AND ITS EXTENSION WITH SOME PROPERTIES

3.1 The Mean distance of the Möbius-Kantor special graph MK

Theorem 3.1.1: The mean distance of the Möbius-Kantor special graph MK is $\mu(MK) = \frac{34}{15}$.

Proof: The transmission of any vertex v in the Möbius-Kantor graph is $\sigma(v)$ and

$\sigma(v) = \sum_{u\epsilon v(MK)}^{16} d(v,u) = 2(1+2+3)+4+1+4(2)+3(3) = 34$

Since all vertices of the Mobius-Kantor graph are symmetric in adjacency and the order of MK is 16, then

$\sigma(MK) = \sum_{i=1}^{16} \sigma(v_i) = \sum_{i=1}^{16} 34 = (16)(34)$.

So $\mu(MK) = \frac{\sigma(MK)}{16(15)} = \frac{16(34)}{16(15)} = \frac{34}{15}$.

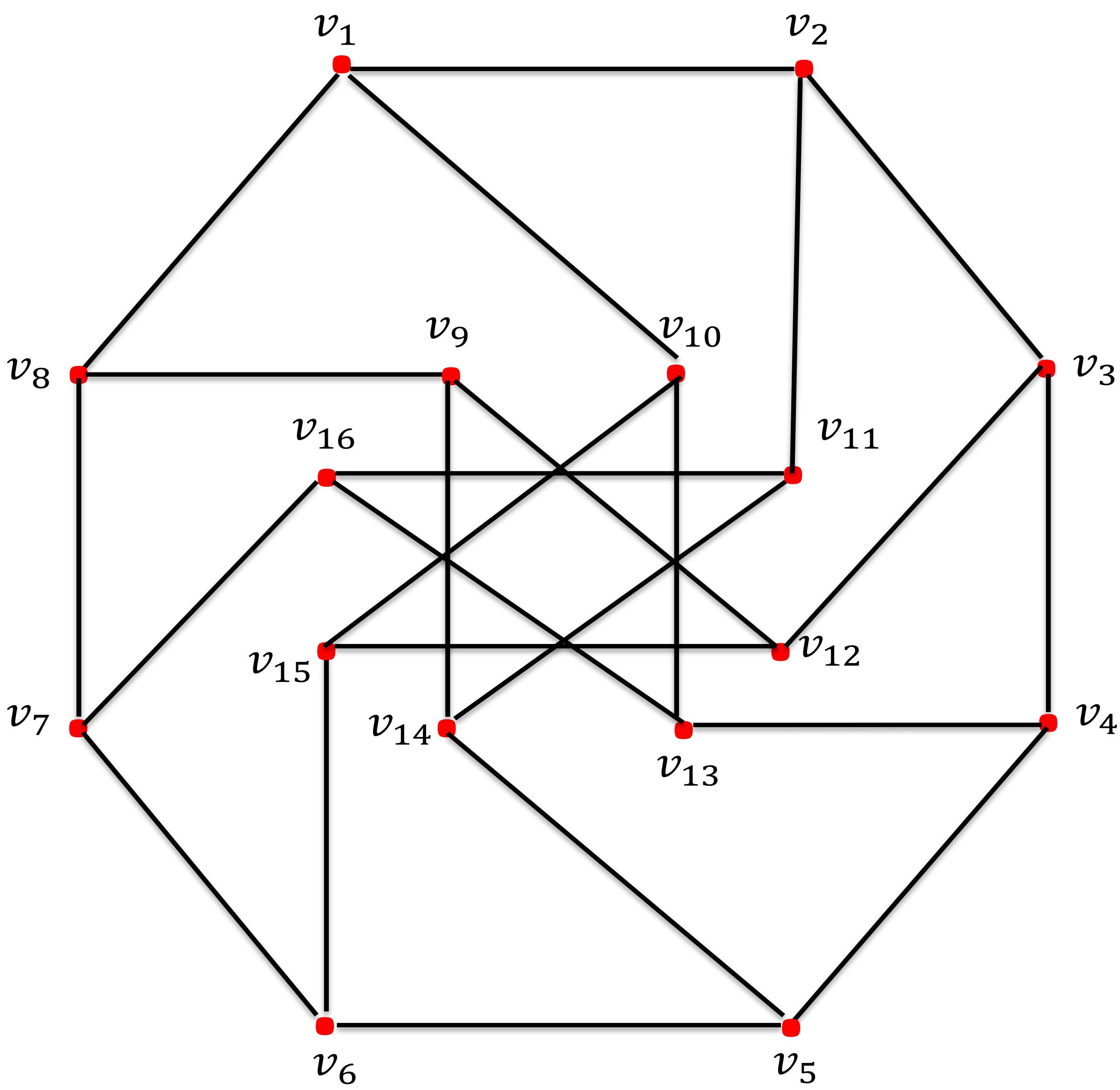


Figure 3.1 Möbius – Kantor MK graph

3.2 The Extension of Möbius-Kantor special graph (MK_{16m})

Definition 3.2.1: The extension of Möbius-Kantor special graph MK is a new special graph named *extended Möbius-Kantor graph* (MK_{n}) of order $n = 16m$ and size $24m$ for any positive integer $m > 1$, which is a 3-regular graph of girth $8m$, as given in Figure 3.2.1 below.

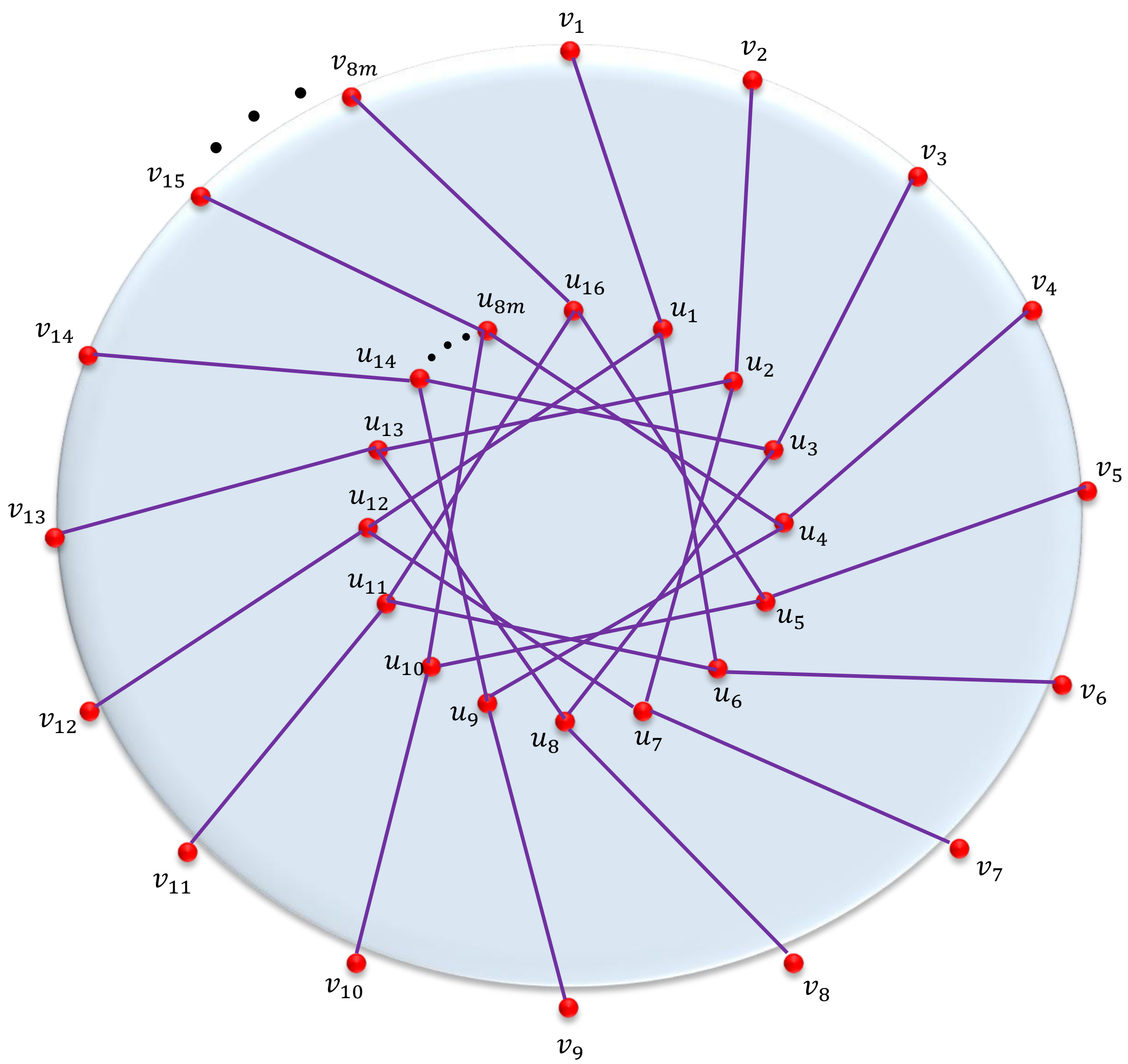
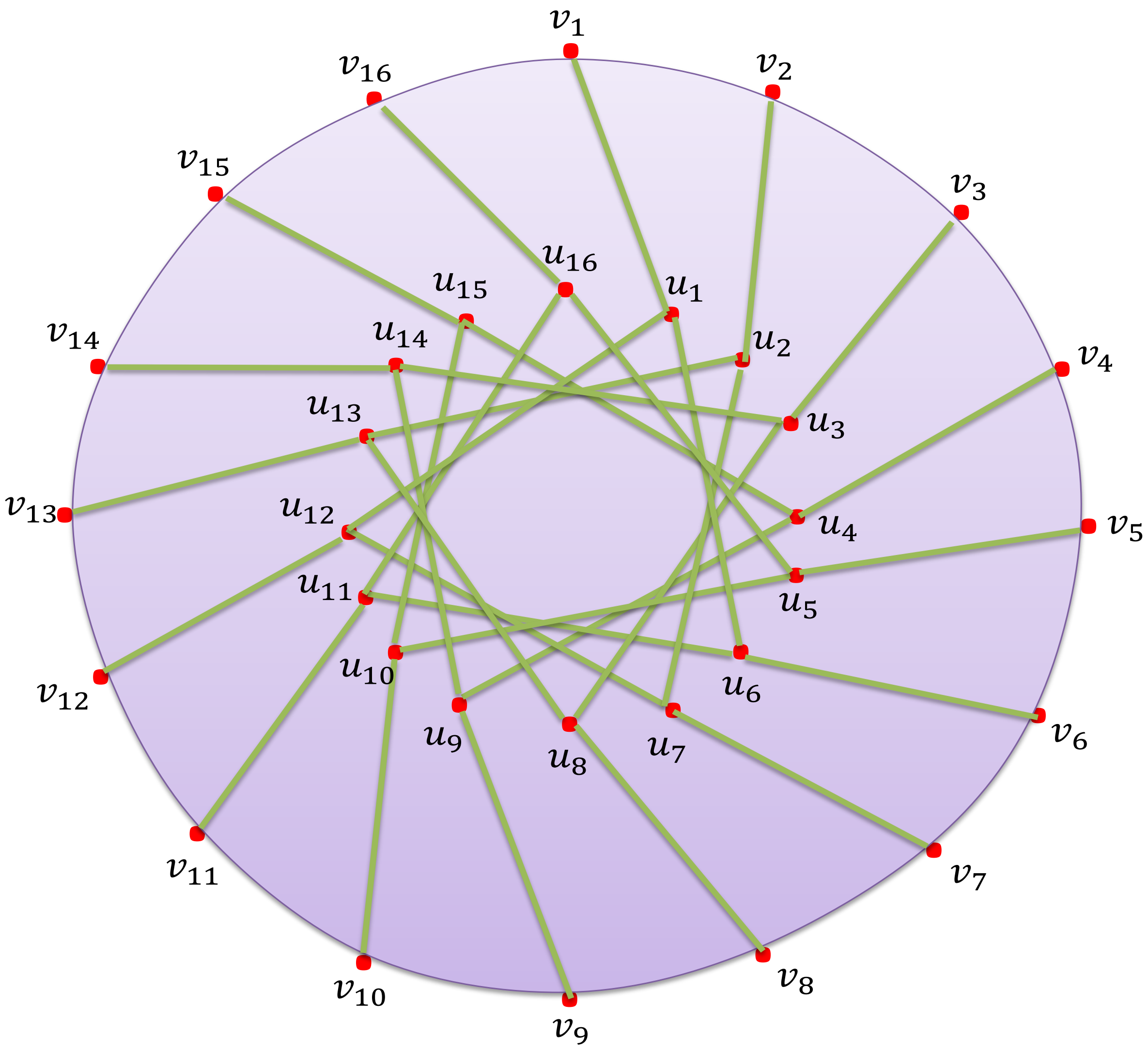


Figure 3.2.1 of extended Möbius-Kantor special graph MK_{n} where $n = 16m$.

Example 3.2.2: The following graph is the extended Möbius-Kantor special graph MK_{32}, where n=32, that is m=2.



Theorem3.2.3: The mean distance of the extended Möbius-Kantor graph MK_n, where $n = 16m$ is $\mu(MK_n) = \frac{n^2+20n-32}{16(n-1)}$.

Proof: Note that the transmission of a vertex v_1 in the graph MK_n is

$$\sigma(v_1) = 2\left(1+2+3+\cdots+\frac{n}{8}\right) + 2\left(3+4+\cdots+\frac{n}{8}+\left(\frac{n}{8}+1\right)\right) + \left(\frac{n}{4}+2\right) + 1 + 2\left(2+3+4+\cdots+\frac{n}{8}+\left(\frac{n}{8}+1\right)\right) + 2\left(2+3+4+\cdots+\frac{n}{8}\right) + (\frac{n}{8}+1)$$

$$= 2\sum_{i=1}^{\frac{n}{8}} i + 2\sum_{i=1}^{\frac{n}{8}+1} i + 2\sum_{i=1}^{\frac{n}{8}+1} i + 2\sum_{i=1}^{\frac{n}{8}} i + \left(\frac{n}{8}+2\right) + \left(\frac{n}{8}+1\right) - 9$$

$$= 2\left(\frac{n}{8}\right)\left(\frac{n}{8}+1\right) + 2\left(\frac{n}{8}+1\right)\left(\frac{n}{8}+2\right) + \frac{2n}{8} - 6$$

$$\sigma(v_1) = \frac{n^2}{16} + \frac{5n}{4} - 2$$

Since all vertices of MK_n are symmetric with the adjacency of the vertices of MK_n, then the transmission of the special generalized graph MK_n is $\sigma(MK_n) = \sum_{v\epsilon V(MK_n)} \sigma(v) = n(\frac{n^2}{16} + \frac{5n}{4} - 2)$

So, the average distance of the generalized Möbius-Kantor special graph $\mathrm{MK_n}$ is

$$\mu(MK_n) = \frac{\mathrm{n}\left(\frac{\mathrm{n}^2}{16}+\frac{5\mathrm{n}}{4}-2\right)}{\mathrm{n}(\mathrm{n}-1)} = \frac{\frac{\mathrm{n}^2}{16}+\frac{5\mathrm{n}}{4}-2}{\mathrm{n}-1} .$$

Remark 3.2.4: In Definitions 3.2.1, if we set $m = 1$,then we obtain Möbius-Kantor graph MK.

3.3 Some Graphical properties of the extended Möbius-Kantor special graph (MK_n), where $n = 16m$, for $m \in \mathbb{Z}^+$

3.3.1 Regularity of MK_n

The graph MK_n is a 3-regular graph, since $deg(v) = 3$, $\forall v \in V(MK_n)$.

3.3.2 Girth of MK_n

The girth of the graph MK_n is $g(MK_n) = 6$.

3.3.3 Circumference of MK_n

The circumference of MK_n is $C(MK_n) = \frac{1}{2}n = 8m$.

3.3.4 Eccentricity of vertices of MK_n

The eccentricity of the vertices of the graph MK_n is $e(v) = \frac{n}{8} + 1$, for all $v \in V(MKn)$.

3.3.5 Radius of MK_n

The radius of MK_n is $rad(MK_n) = \frac{n}{8} + 1$.

3.3.6 Diameter of MK_n

The diameter of the graph MK_n is $\text{diam.}(MK_n) = \frac{n}{8} + 1$.

3.3.7 Non-Hamiltonian of MK_n

The graph MK_n is not a Hamiltonian graph, since MK_n has no cycle contains all edges of MK_n.

3.3.8 Non-Eulerian of MK_{n}

The graph MK_{n} is not Eulerian graph, since MK_{n} contain vertices of odd degree.

3.3.9 Non-Tree of MK_{n}

The graph MK_{n} is not a tree graph, since MK_n contain a cycle.

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پوختە

لەم کارەدا هەڵساین بە فراوانکردنی گرافی تایبەت بەناوی (Möbius–Kantor) گراف بۆ گرافی فراوانکراو بەناوی (Extended Möbius-Kantor) گراف ($MK_{16\mathrm{n}}$) کە ژمارەی سەرەکانی لە 16 وە دەگۆڕێت بۆ $16n$ بۆ ژمارەی تەواوی موجەب n ، کە تێیدا تێکڕای دووری هەردوو گراف دەدۆزینەوە و دەکاتە $\frac{34}{15}$ و $\frac{n^2+20n-32}{16(n-1)}$ بەدواییەکەوە.

هەروەها ئێمە دیراسەی چەند سیفەتێکی گرافی ئەم گرافە تایبەتە فراوانکراوە ($MK_{16\mathrm{n}}$)مان کردووە وەك ڕێکی ، کەمەرە ، چێوە ، جیاوازی سەنتەری ، نیوەتیرە ، تیرە ، نا هاملتۆنی ، نا ئۆیلەری ، نابێتە داری.

الخلاصة

في هذا العمل قمنا بتوسيع البيان الخاص (Möbius–Kantor graph) إلى البيان الموسع (Extended Möbius-Kantor graph) البيان ($MK_{16\mathrm{n}}$) حيث عدد رؤسه من 16 إلى $16n$ لكل عدد صحيح موجب n , فيها نجد معدل المسافة للبيانين و هما $\frac{34}{15}$ و $\frac{n^2+20n-32}{16(n-1)}$ على التوالي.

بالأضافة إلى ذلك , قمنا بدراسة بعض الصفات البيانية للبيان الموسع $MK_{16\mathrm{n}}$, مثل : الأنتظام , المحيط , الخصر, الأختلاف المركزي , نصف القطر , القطر , لا هاملتوني , لا أويليري و ليست شجرة.