زانكۆی سةلاحةدين-هةولير

Salahaddin Unlversily – Erbil

On the Metric space lcm minus gcd of natural numbers

Submitted to the department of Mathematics in partial fulfillment of the Requirements for the degree of BSC. in Mathematics

Prepared by:

Rayan Jawhar Yasin

Supervised by:

Assist. Prof. Dr. Rashad Rashid Haji

April- 2023

CERTIFICATION OF THE SUPERVISORS

I certify that this work was prepared under my supervision at the Department of Mathematics/College of Education/Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

Signature: 

Supervisor: Dr. Rashad Rashid Haji

Scientific grade: Assist. Professor

Date: / / 2023

In view of the available recommendations, I forward this work for debate by the examining committee.

Signature: 

Name: Dr. Rashad Rashid Haji

Scientific grade: Assist. Professor

Chairman of the Mathematics Department

Date: / / 2023

ACKNOWLEDGMENT

Primarily, I would like to thanks my god for helping me to complete this

research with success.

Then I would like to express special thanks to my supervisor Dr. Rashad Rashid Haji, whose valuable to guidance has been the once helped me to completing my research.

Words can only inadequately express my gratitude to my supervisor for patiently helping me to think clearly and consistently by discussing every point of this dissertation with me.

In addition, I would like to thank my family, and colleague and library staff whose support has helped me to conceive this research.

TABLE OF CONTENT

ABSTRACT

In this work we study the topological properties of the metric spaces $(\mathbb{N}, d)$ (the Metric space Lcm minus gcd of natural numbers).

We prove that every set in $(\mathbb{N}, d)$ is open, closed, and discrete. Moreover, we prove that every subset of $\mathbb{N}$ is compact if and only if it is finite. Also we prove that the boundary and derived of every set in $(\mathbb{N}, d)$ is empty.

Furthermore, we give some interesting examples in this metric space.

INTRODUCTION

Analysis is the branch of mathematics dealing with continuous functions, limits and related theories such as differentiation, integration, measure, infinite sequences, series, and analytic functions. These theories are usually studied in the context of real and complex numbers and functions analysis evolved from calculus. Which involves the elementary concepts and techniques of analysis.

Analysis may be distinguished from geometry; however it can be applied to any space of mathematical objects that has definition of nearness a (topological space) or specific between objects a (metric space).

In the early 1900s, the usual approach to mathematics was far less abstract and axiomatic. Hence, at the time, various spaces that mathematicians studied (such as function spaces as we have studied a bit of in this class) had different notions of convergence. Each space has it’s own notion of the word, which was studied in its own respect. There were some similarities between these notions, but there was no general understanding of the term.

Then, in 1906, Fréchet introduced the idea of metric spaces in his Ph.D. Dissertation. (Dote 2022)

Metric spaces provide a notion of distance and a framework with which to formally study mathematical concepts such as continuity and convergence, and other related ideas. Many metrics can be chosen for a given set, and our most common notions of distance satisfy the conditions to be a metric. Any norm on a vector space induces a metric on that vector space and it is in these types of metric spaces that we are often most interested for study of signals and systems. (al 2022)

CHAPTER ONE

Background

Definition 1.1: (Dhananjay Gopal 2021)

Let X be a non empty set. A function$d: X \times X \to \mathbb{R}$, is called a *metric* (*distance*) function if:

1. $d(x, y) \geq 0, \ \forall x, y \in X$

2. $d(x, y) = 0$ If and only if $x = y, \ \ \forall x, y, z \in X$

3. $d(x, y) = d(y, x) \quad \forall x, y, z \in X$

4. $d(x, z) \leq d(x, y) + d(y, z) \qquad \forall x, y, z \in X$

In this case (X, d) is called a *metric space*.

Definition 1.2: (Dhananjay Gopal 2021)

Let (X, d) be a metric space and $x_0 \in X$. A neighborhood of a point x_0 of all points their distance from x_0 less that R and denoted by $N_r(x_0) = \{x \in X: d(x, x_0) < R\}$

Definition 1.3: (Dhananjay Gopal 2021)

Let (X, d) be a metric space and S sub set of X A point $x_0 \in S$ is called an *interior point* of S if there is a nbd of x_0 (say $N_r(x_0)$ such that $N_r(x_0)$ sub set of S the set of all interior point of S) denoted by $i(S)$.

Definition 1.4: (Dhananjay Gopal 2021)

Let (X, d) be a metric space and S sub set of X A point $x_0 \in \ S^c$ is called *exterior point* of S if there is a $N_r(x_0) \in \ S^c$ the set of all exterior point of S is denoted by $e(S)$.

Definition 1.5: (Dhananjay Gopal 2021) (Alexander S. Kravchuk 2007)

Let (X, d) be a metric space and S sub set of X A point $x_0 \in S$ is called a *boundary* of S is every nbd of x_0 contains at least a point of S and at least a point of S^c the set of an boundary points S is denoted by $b(S)$.

Definition 1.6: (Alexander S. Kravchuk 2007)

Let (X, d) be a metric space and S sub set of X then S is called an *open set* If every element of S is an interior element.

Definition1.7: (Alexander S. Kravchuk 2007)

Let (X, d) be a metric space and S sub set of X then S is *closed set* If S^c is open in (X, d)

Definition 1.8: (Dhananjay Gopal 2021)

Let (X, d) be a metric space and S sub set of X A point $x_0 \in X$ is called a *cluster point* of S is every nbd of x_0 contains infinite points of S, that is $N_r(x_0) \cap \{S - \{x_0\}\} = \emptyset$ for all $x \in \mathbb{R}$ set of an cluster points of S denoted by $d(S)$.

Definition 1.9: (Alexander S. Kravchuk 2007)

Let (X, d) be a metric space and S sub set of X then *closure* of S is denoted by $\overline{S}$ and defined by $\overline{S} = S \cup d(S)$.

Definition 1.10: (Dhananjay Gopal 2021)

Let $d(S)$ be a metric space and S sub set of X, A point $x_0 \in S$ is called an *isolated point* of S if x_0 is not a cluster point of S. If all point of S are isolated then S is called a discrete set.

Definition 1.11: (Alexander S. Kravchuk 2007)

Let (X, d) be a metric space and S sub set of X. then S is called dense initself. if $S \subset d(S)$

Definition 1.12: (Dhananjay Gopal 2021)

Let (X, d) be a metric space and S sub set of X. then S is called *perfect* if $S = d(S)$

Definition 1.13: (Dhananjay Gopal 2021)

Let (X, d) be a metric space and S sub set of X, then we say that S is a *compact* set if every open cover for S has a finite sub cover for S.

Definition 1.14: (Dhananjay Gopal 2021)

A metric space (X, d) is called *complete* if every Cauchy sequence in is (X, d) convergent to a point in (X, d).

Definition 1.15: (Dhananjay Gopal 2021)

Let (X, d) be a metric space and S sub set of X,$x_0 \in S$ and then we say that S is a *bounded* set in (X, d) if ther exsist $M \in \mathbb{R}$ such that $d(x, x_0) \leq M$for all $x \in S$ or $d(x, x_0) \leq M$ for all $x, y \in S$

Theorem 1.16: In the metric space $(\mathbb{N}, d)$, every set is open.

Proof: seen p.7

Theorem 1.17: In the metric space $(\mathbb{N}, d)$, every set is closed.

Proof: seen P.8

Theorem 1.18: In the metric space $(\mathbb{N}, d)$, the derived set of S sub set of $\mathbb{N}$ is empty, that is $d(S) = \emptyset$.

Proof: seen P.9

Theorem 1.19: In the metric space $(\mathbb{N}, d)$, the closure of every set S sub set of $\mathbb{N}$ is S itself that is, $\overline{S}$= S

Proof: seen P.10

Theorem 1.20: Let S be any nonempty set in the metric space $(\mathbb{N}, d)$ then any element of S is isolated.

Proof: seen P.10

Theorem 1.21: every nonempty set S in the metric space $(\mathbb{N}, d)$ is discrete.

Proof: seen P.10

Theorem1.22: A set S in the metric space $(\mathbb{N}, d)$ is compact if and only if S is finite

Proof: seen P.8

Chapter Two

On the Metric space Lcm minus gcd of natural numbers

Definition 2.1: Let $d: \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R}$ defined by $d(n,m) = Lcm(n,m) - \gcd(n,m)$. Then d is a metric function on $\mathbb{N}$. So the order pair $(\mathbb{N}, d)$ is a metric space called Lcm minus gcd metric space of natural numbers. As shown below

1. $d(n,m) \geq 0 \ , \qquad \forall n,m \in \mathbb{N}$

   $since\ Lcm(n,m) \geq \gcd(n,m)$

   $so\quad Lcm(n,m) - \gcd(n,m) \geq 0$

   $Then\ d(n,m) \geq 0 \ , \quad \forall n,m \in \mathbb{N}$

2. $d(n,m) = 0\ if\ and\ only\ if\ n = m$

   $d(n,m) = 0$

   $\Leftrightarrow Lcm(n,m) - \gcd(n,m) = 0$

   $\Leftrightarrow Lcm(n,m) = \gcd(n,m)$

   $\Leftrightarrow n = m, \quad where\ n,m \in \mathbb{N}$

3. $d(n,m) = Lcm(n,m) - \gcd(n,m)$

   $Lcm(m,n) - \gcd(m,n)$

   $d(m,n) \qquad\qquad \forall n,m \in \mathbb{N}$

4. $for\ any\ n,m,k \in \mathbb{N}\ we\ note\ that$

   $d(n,k) = Lcm(n,k) - \gcd(n,k)$

   $\leq Lcm(n,m) + Lcm(m,k) - [\gcd(n,m) + \gcd(m,k)$

   $= Lcm(n,m) - \gcd(n,m) + Lcm(m,k) - \gcd(m,k)$

   $= d(n,m) + d(m,k)$

   $since\ Lcm(n,k) \leq Lcm(n,m) + Lcm(m,k) \quad \forall\ n,m,k \in \mathbb{N}$

   $And\ \gcd(n,m) + \gcd(m,k) \leq \gcd(n,k) \qquad \forall\ n,m,k \in \mathbb{N}$

   $Then\ - [\gcd(n,m) + \gcd(m,k) \geq \gcd(n,k) \quad \forall\ n,m,k \in \mathbb{N}$

   $therefore\ (,d)\ is\ a\ metric\ space.$

Example2.2: The neighborhood of the point 4 of radius 0.5 is, $N_{0.5}(4)$

$N_{0.5}(4) = (x \in X: d(x, 4) < 0.5\}$

$= \{x \in \mathbb{N};\ \mathrm{Lcm(x, 4)} - \mathrm{gcd(x, 4)}\}$

$= \{x \in \mathbb{N}\ ; \mathrm{x} = 4\}$

$= \{4\}$

Example2.3: The neighborhood of the point 4 of radius 1 is, $N_1(4)$

$N_1(4) = (x \in X: d(x, 4) < 1\}$

$= \{x \in \mathbb{N}; \mathrm{Lcm(x, 4)} - \mathrm{gcd(x, 4)} < 1\}$

$= \{x \in \mathbb{N}\ ; \mathrm{x} = 4\}$

$= \{4\}$

Example2.4: The neighborhood of the point 4 of radius 2 is, $N_2(4)$

$N_2(4) = (x \in X: d(x, 4) < 2\}$

$= \{x \in \mathbb{N}; \mathrm{Lcm(x, 4)} - \mathrm{gcd(x, 4)} < 2\}$

$= \{x \in \mathbb{N}\ ; \mathrm{x} = 4\}$

$= \{4\}$

Example2.5: The neighborhood of the point 4 of radius 3 is, $N_3(4)$

$N_3(4) = (x \in X: d(x, 4) < 3\}$

$= \{x \in \mathbb{N}; \mathrm{Lcm(x, 4)} - \mathrm{gcd(x, 4)} < 3\}$

$= \{x \in \mathbb{N}\ ; \mathrm{x} = 4\ , \mathrm{x} = 2\}$

$= \{2,4\}$

Theorem 2.3: In the metric space $(\mathbb{N}, d)$ every set is open.

Proof: Let $S \subset \mathbb{N}$

If $S = \emptyset$, then $\emptyset$ is an open. Since $\emptyset$ does not contain X, where x is not interior.

Let $S \neq \emptyset$, and $x \in S$, then $N_1(x) = \{x\} \subset S$.

Then x is an interior point of S.

Hence S is open set.

Theorem 2.4: In the metric space $d(\mathbb{N}, d)$ every set is closed.

Proof: Let S be any set in $(\mathbb{N}, d)$ then $S^c \subset \mathbb{N}$.

By theorem 2.3 every set in $(\mathbb{N}, d)$ is open. Then $S^c$is an open set in $(\mathbb{N}, d)$

So $\left(S^{c^c}\right) = S$ is closed in $(\mathbb{N}, d)$ [by def. of closed set]

Example 2.5: The set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ in the metric space $(\mathbb{N}, d)$.

Then A is open set of the metric space$(\mathbb{N}, d)$, since by Theorem [In the metric space $(\mathbb{N}, d)$ every set is open]. Then A is closed. Since $A^c = \mathbb{N} - A$ is open.

Every element of A is interior. Since A is open. Then the exterior point of A is $\mathbb{N}/A$. The boundary set of S is $b(A) = \emptyset$. Then A is not dense itself. Since $A \not\subseteq d(A)$

The A is not perfect. Since $A \neq d(A)$. And the derived set of S is $d(A) = \emptyset$.

Every point in A is isolated point since A is not cluster point. Then closure of A is $\overline{A} = A \cup \emptyset$ then $\overline{A} = A$ [since$d(A) = \emptyset$]

Theorem 2.6: A set S in the metric space $(\mathbb{N}, d)$is compact if and only if S is finite

Proof: to prove that S is compact.

We take an open cover $\{G_\alpha : \alpha \in \Delta\}\, for\ S$.

Then $S \subset \bigcup_{a \in \Delta} G_a$ then $\{a_1, a_2, a_3, \dots, a_n\} \subset \bigcup_{a \in \Delta} G_a$

Then $\exists$ G_{α_i} in $\{G_\alpha : \alpha \in \Delta\}$ such that $a_i \in G_{\alpha_i}$, $\forall i = 1,2, \dots, n$

Then $\bigcup_{i=1}^{n} \{a_i\} \subset \bigcup_{i=1}^{n} G_{a_i}$

Then $S \subset \bigcup_{i=1}^{n} G_{\alpha_i}$ that is $\{G_{\alpha_i}: i = 1,2, \dots, n\}$ is a finite sub cover of $\{G_\alpha: \alpha \in \Delta\}$

Covering S. hence S is compact.

Conversely: Let $S = \{a_1, a_2, a_3, \dots\}$ be an infinite set in the metric space $(\mathbb{N}, d)$; then we take the open cover $\{N_1(a_i)\ ; i = 1, 2, 3, \dots.\}$ for S

Since $N_1(a_i) = \{\ a_i\}$, $\forall i = 1,2,3 \dots$

Then $S \subseteq \bigcup_{i=1}^{\infty} N_1\,(a_i)$, $\forall i = 1,2,3 \dots$

We see that the open cover $\{N_1(a_i)\ ; i = 1, 2, 3, \dots.\}$ has no finite sub cover for S

Thus S is not compact.

Therefore only finite sets in $(\mathbb{N}, d)$ are compact set.

Example 2.7: consider the metric space $(\mathbb{N}, d)$ and the sequence

$$a_n = \begin{cases} 2, & n \leq 100 \\ 6, & n > 100 \end{cases}$$

$a_n = \{2, 2, \dots, 2, 6, 6, \dots\}$

$N_{0.5}(6) = \{6\}$ contain infinite terms of $\{a_n\}$ except finite terms.

Therefore $a_n \longrightarrow 6$.

Example 2.8: consider the metric space $(\mathbb{N}, d)$ and the sequence

$$b_n = \begin{cases} 2, & n \leq 100 \\ n, & n > 100 \end{cases}$$

Is not converge.

Theorem 2.9: In the metric space $(\mathbb{N}, d)$,the derived set of S sub set of $\mathbb{N}$ is empty, that is $d(s) = \emptyset$.

Proof: Let $S \subseteq \mathbb{N}$ then for every $m \in \mathbb{N}$, we have $N_{0.5}(m) =\ \{m\}$

So $N_{0.5}(m) \cap (S - \{m\}) = \emptyset$

Hence, m is not a cluster point of S.

Theorem 2.10: In the metric space $(\mathbb{N}, d)$,the closure of every set S sub set of $\mathbb{N}$ is S itself that is, $\overline{S} = S$

Proof: Let $S \subseteq \mathbb{N}$. Then we know that $\overline{S} = S \cup d(s)$

But $d(S) = \emptyset$ by Theorem 2.9 then $\overline{S} = S$.

Theorem 2.11: Let S be any nonempty set in the metric space $(\mathbb{N}, d)$ then any element of S is isolated.

Proof: Let $x_0 \in S$. Then x_0 is not a cluster point of S,

Since $\exists\ a$ neighbor hood (say $N_{0.1}(x_0))\ of\ x_0\ of\ the\ form\ N_{0.1}(x_0) = \{x_0\}$

So $N_{0.1}(x_0) \cap (s - \{x_0\}) = \emptyset$.

Therefore x_0 is an isolated of S.

Theorem 2.12: every nonempty set S in the metric space $(\mathbb{N}, d)$ is discrete.

Proof: Let $S \subseteq \mathbb{N}$ be a nonempty set and $x_0 \in S$. Since x_0 is not a cluster point of S. $(d(S) = \emptyset)$, then $x_0 \in S$ is an isolated point of S.

Hence S is discrete.

Example 2.13: Let S be the set of prime numbers of $\mathbb{N}$. Then $S = \{2, 3, 5, 7, 11, 13, 17, ...\}$. Then S is an open set of the metric space $(\mathbb{N}, d)$, since for each $m \in S$, there is a neighborhood (say $N_{0.5}(m)$) such that $N_{0.5}(m) = \{m\} \subseteq S$.

Every element of S is interior. Since S is open

Also, S is closed, since $S^c = \mathbb{N} - S$ is an open set.

The boundary set of S is $b(S) = \emptyset$. And the derived set of S is $d(S) = \emptyset$.

That is no element of $\mathbb{N}$ is a cluster point of S. So every element of S is isolated, then S is a discrete set.

Reference

al, Richard Baraniuk et. 2022. *LibreTexts.* May 23. Accessed 11 18, 2023. https://eng.libretexts.org/Bookshelves/Electrical_Engineering/Signal_Processing_and_Modeling/Signals_and_Systems_(Baraniuk_et_al.)/16%3A_Appendix_C-_Analysis_Topics_Overview/16.01%3A_Introduction_to_Metric_Spaces.

Alexander S. Kravchuk, Pekka J.Neittaanmaki. 2007. *Varitional and Quasi-Variational Inequalities in Mechanics.* Netherlands: Springer .

Dhananjay Gopal, Aniruddha Deshmukh, bhay S. Ranadive and Shubham Yadav. 2021. *An Introduction to Metric Space.* Boca Ranto London New York: CRC Press.

Dote, Paige. 2022. *Introduction to Metric Space.* January 20. Accessed 11 18, 2023. https://web.mit.edu/paigeb/www/18.S097/Lect_6.pdf.

پوختە:

ئیمە لەم کارەدا لە تایبەتمەندییە تەپۆلۆژیەکانی بۆشای دووریەکان دەکۆلینەوە (بۆشای دووری بچوکترین چەند جارەی هاوبەش کەم گەورەترین کۆلکەی هاوبەش لە ژمارە سروشتیەکان)

لەم کارەدا ئێمە نیشانی دەدەین کە ئەو نەخشەیەی لە سەر شێوەی جیاوازی بچوکترین چەند جارەی هاوبەش و گەورەتری کۆلکەی هاوبەش لە سەر ژمارە سروشتیەکان دەبێتە نەخشەی دووری بۆیە

(N, d) دەبیتە نەخشەی دووری. کەتێیدا دەسەلمینین هەر کۆمەڵیەک لە

N کراوە، داخراو، جیاکراوە و کۆمپاکتە بە مەرجیک کۆمەڵەکە کۆتای بیت.

جگە لەوەش لە بۆشای دووری چەند نموونەێک دەهێنینەوە.