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Salahaddin University- Erbil

On The Usual Metric Space $\mathbb{R}^2$

Research Project

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Certification of the Supervisor

I certify that this work was prepared under my supervision at the Department of Mathematics / College of Education / Salaheddin University- Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

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Abstract

In this work we study the topological properties of the usual metric spaces $(\mathbb{R}^2, \mathrm{u})$ and we prove that every closed and bounded set in $(\mathbb{R}^2, \mathrm{u})$ is compact, and $(\mathbb{R}^2, \mathrm{u})$ is a complete metric space, and S is closed iff $\mathrm{d}(S) \subseteq S$.
Moreover, we give some interesting examples in the usual metric space $(\mathbb{R}^2, \mathrm{u})$.

Contents

Introduction

Mathematical Analysis is the branch of Mathematics dealing with limits and related theories such as differentiation, integration, measure, sequences, series, and analytic functions (stromberg, 1965) (stillwel, 2004). These theories are usually studied in the context of real and complex numbers and functions. Analysis evolved from calculus, which involves the elementary concepts and techniques of analysis.

Analysis may be distinguished from geometry; however, it can be applied to any space of mathematical objects that has a definition of nearness (a topological space) or specific distances between objects (a metric space). Mathematical analysis is concerned with quantifying change, with a key role played by fundamental notions of continuity and approximation. This research area includes, for example: Fourier and harmonic analysis and operator theory.

Mathematical analysis formally developed in the 17th century during the Scientific Revolution (jahnke, 2003). but many of its ideas can be traced back to earlier mathematicians. Early results in analysis were implicitly present in the early days of ancient Greek mathematics. For instance, an infinite geometric sum is implicit in Zeno's paradox of the dichotomy (stillwel, 2004).

Later, Greek mathematicians such as Eudoxus and Archimedes made more explicit, but informal, use of the concepts of limits and convergence when they used the method of exhaustion to compute the area and volume of regions and solids (smith, 1958).

The explicit use of infinitesimals appears in Archimedes' The Method of Mechanical Theorems, a work rediscovered in the 20th century. In Asia, (pinto, 2004) the Chinese mathematician LiuHui used the method of exhaustion in the 3rd century AD to find the area of a circle (Dainian;Robertsonne,1966)In Indian mathematics, particular instances of arithmetic series have been found to implicitly occur in Vedic Literature as early as 2000 B.C.

Chapter one

Definitions and background

Definition 1.1: (J.R.GILES 1987)

Let X be a non-empty set A, Function $d: X \times X \to \mathbb{R}$ is a *metric space* (distance) function if

1. $d(x, y) \geq 0$; $\forall x, y \in X$.
2. $d(x, y) = 0 iff\ x = y; \forall x, y \in X$.
3. $d(x, y) = d(y, x); \forall x, y \in X$.
4. $d(x, z) \leq d(x, y) + d(y, z); \forall x, y, z \in X$.

In this case (X, d) is called a *metric space*.

Definition 1.2: (Granerg 2017)

Let (X, d) be a *metric space* and x_o (X, d), A neighborhood of a point $x_0 \in (X, d)$ of a radus $r \in R^+$ is defined to be the set of all points their distance from xo less than r and denoted by $N_r(x_O)$ that $is\ N_r(X_O) = \{x \in X; d(x_o, y_O) < r\}$.

Definition 1.3: (Granerg 2017)

Let (X, d) e a *metric space* and $S \subseteq X$, A point $X_O \in S$ is called an interior point of S , If this a neighborhood of $X_O(\ say\ N_r(X_O))$,
Such that $N_r(X_o) \subseteq S$ the setoff an interior points of S is denoted by $i(S)$.

Definition 1.4: (Arora 2005)

Let (X, d) be a *metric space* and $S \subseteq X$, A point $X_O \in S^C$ is called exterior point of S. If there is a $N_r(X_O) \in S^C$ the set of all exsterior points of S denoted by $e(S)$.

Definition 1.5: (Granerg 2017)

Let (X, d) e a *metric space* and $S \subseteq X$, then S ls called an open set; If every element of S is an interior point of S.

Definition 1.6: (Jahnke 2003)

Let (X, d) be *a metric space* and $S \subseteq X$, Then S is called closed set, If $S^C$is open set in (X, d).

Definition 1.7: (Arora 2005)

Let (X, d) e a metric space and S⊆X, A point $x_o \in X$ is called a cluster point (limit point) of S. If every neighborhood of xo contains infinite point of S. The $N_r(x_o) \cap \{S_{x_o}\} \neq \emptyset$; $\forall x \in \mathbb{R}^+$The set of cluster points of S is called a derived set of S denoted by $d(S)$ orS' .

Definition 1.8: (Cech 1969)

Let (X, d) e a metric space and $S \subseteq X$, Then closure of S denoted by S^- and defined by $S^- = S \cup d(S)$.

Definition 1.9: (Granerg 2017)

Let (X, d) e a metric space and $S \subseteq X$,Then S is called dense in it self, If $S \subseteq d(S)$.

Definition 1.10: (Arora 2005)

Let (X, d) e a *metric space* and $S \subseteq X$,Then i S s called perfect is $S = d(S)$.

Definition 1.11: (Granerg 2017)

Let (X,d) e a metric space and $S \subseteq X$. A family of open subsets of X , $\{G_\alpha : \alpha \in \Delta\}$ is said to be an open cover for S. Is $S \subseteq \bigcup_{\propto \in \Delta} G_\propto$.

Definition 1.12: (Granerg 2017)

Let (X, d) e a metric space and $S \subseteq X \; and \; \{G_\alpha : \alpha \in \Delta\}$ be an open cover of S, then the family $\{G_i, i = 1,2, \ldots n\}$ is said to be finite sub cover of $\{G_\alpha : \alpha \in \Delta\}$ for S is $S \subseteq \bigcup_{i=1}^{n} G_i$.

Definition 1.13: (Granerg 2017)

Let (X, d) e a metric space and $S \subseteq X$, then S is said to be compact set if any open cover for S has finite sub cover for S.

Definition 1.14: (Granerg 2017)

A set X is called complete if every Cauchy sequence in X is convergent in X.

Theorem1.14: (Couech _Schiar_Inequality)

Let $a_{1,}\ a_{2,}\ \ldots, a_n$ and $b_{1,}\ b_{2,} \ldots, b_n$ Be a real number, Then

$$\sum_{i=1}^{n} a_i \cdot b_i \leq \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{i=1}^{n} b_i^2}$$

Theorem1.15: (Minkowisk_ inequality): Let $a_{1,}\ a_{2,}\ \ldots, a_n$ and $b_{1,}\ b_{2,} \ldots, b_n$ Be a real number, Then

$$\sqrt{\sum_{i=1}^{n} (a_{i+} b_i)^2} \leq \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}$$

Chapter Two

Some Properties of the discrete Metric space $(\mathbb{R}^2, u)$

In this chapter we study some topological properties of the usual metric space $(\mathbb{R}^2, u)$.

We know that the usual metric space function u defined on $\mathbb{R}^2$

(u: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$) by $u((x_1, x_2),(y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

We Show that u: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a metric space function as follows

1. $u((x_1, x_2),(y_1, y_2)) \geq 0$ since $((x_i - y_i)^2 \geq 0$, Then $\sum_{i=1}^{2}(x_i - y_i)^2 \geq 0$

   then $\sqrt{\sum_{i=1}^{2}(x_i - y_i)^2} \geq 0$

   $\forall (x_1, x_2),(y_1, y_2) \in \mathbb{R}^2$ since $\sqrt{\ } \geq 0$.

2. $u((x_1, x_2),(y_1, y_2)) = 0$ $iff \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = 0$

   $(x_1 - y_1)^2 = 0$ && $(x_2 - y_2)^2 = 0$ iff

   $x_1 - y_1 = 0$ && $x_2 - y_2 = 0$ iff

   $x_1 = y_1$ && $x_2 = y_2$ iff

   $(x_1, x_2) = (y_1, y_2)$

   $\forall (x_1, x_2),(y_1, y_2) \in \mathbb{R}^2$.

3. $u((x_1, x_2),(y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ iff

   $= \sqrt{[-(y_1 - x_1)]^2 + [-(y_2 - x_2)]^2}$

   $= \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$

$u((x_1, x_2),(y_1, y_2))$, $\forall (x_1, x_2),(y_1, y_2) \in \mathbb{R}^2$.

4. $u((x_1, x_2),(z_1, z_2)) \leq u((x_1, x_2),(y_1, y_2)) + u((y_1, y_2),(z_1, z_2))$

L.H.S $u((x_1, x_2),(z_1, z_2)) = \sqrt{(x_1 - z_1)^2 + (x_2 - z_2)^2}$

$\leq \sqrt{[(x_1 - y_1) + (y_1 - z_1)]^2 + [(x_2 - y_2) + (y_2 - z_2)]^2}$

$= \sqrt{(x_{1-}\ y_1)^2 + (x_{2-}\ y_2)^2} + \sqrt{(y_{1-}\ z_1)^2 + (y_{2-}\ z_2)^2}$

{By Minkowisk_ inequality}

R.H.S = $u((x_{1\,,}x_2),(y_{1,}y_2)) + u((y_{1\,,}y_2),(z_{1,}z_2))$.

Therefore u is a *metric* function on R^2 thus ($\mathbb{R}^2, u$) is a *metric* function .

Example 2.1:

$N_r(x_o, y_o)$ ={(x,y) $\mathbb{R}^2$; $u((x_{,}y),(x_{o,}y_o)) < r$}

$N_{0.1}(0,0); = \{(y_{1,}\ y_2) \in \mathbb{R}^2 = \sqrt{(x-0)^2 + (y-0)^2} < 0.1\}$

$=\{(y_{1,}\ y_2) \in \mathbb{R}^2; \sqrt{y_{1^2} + \ y_{2^2}} \ \ < 0.1\}$

$=\{(y_{1,}\ y_2) \in \mathbb{R}^2;\ y_{1^2} + \ y_{2^2} < 0{,}01\}$.

Example 2.2: $N_r(x_o, y_o)$ ={(x,y) $\mathbb{R}^2$; $u((x_{,}y),(x_{o,}y_o)) < r$}

$N_1(3,3); = \{(y_{1,}\ y_2) \in \mathbb{R}^2 = \sqrt{(x-3)^2 + (y-3)^2} < 1\}$

$\{(y_{1,}\ y_2) \in \mathbb{R}^2\ ;\ (y_1\ -3)^2 + (y_2\ -3)^2 < 1\}$.

Example 2.3: $N_r(x_o, y_o)$ ={(x,y) $\mathbb{R}^2$; $u((x_{,}y),(x_{o,}y_o)) < r$}

$\{N_{0.5}(1,5); = (y_{1,}\ y_2) \in \mathbb{R}^2 = \sqrt{\ } < 0.5\}$

$N_{0.5}(1,5); = (y_{1,}\ y_2) \in \mathbb{R}^2;\ (x-1)^2 + (y-5)^2 < 0.25\}$.

Theorem 2.4: Every neighborhood is an open set in the usual metric space ($\mathbb{R}^2, u$).

Prove: Let $(x_{o,}\ y_o) \in \mathbb{R}^2$; then the neighborhoods of $(x_{o,}\ y_o)$ of a radius r is $N_r((x_{o,}\ y_o))$ is

$N_r((x_{o,}\ y_o))$ ={(x,y) $\mathbb{R}^2$; u($(x_{,}y),(x_{o,}y_o)) < r$}

$N_r((x_{o,}\ y_o)) = \{(x, y) \in \mathbb{R}^2 = \sqrt{(x-x_o)^2 + (y-y_o)^2} < r\}$

$N_r((x_{o,}\ y_o)) = \{(x, y) \in \mathbb{R}^2 = (x-x_o)^2 + (y-y_o)^2\ < r^2\ \}$,

To prove that $N_r((x_{o,}\ y_o))$ *is open* , let (α, β), consider $N_t((\alpha, \beta))$ where t= $min\ \{s, r-s\}\ then\ N_t((\alpha,\beta)) \subseteq N_r\big((x_{o,}\ y_o)\big)$, therefore (α,β) is an interior poin of $N_r\big((x_{o,}\ y_o)\big)$.

Example 2.5: A= $\{(0,\frac{1}{n})$; n∈ $\mathbb{Z}^+\}$

Solution: A= $\{(0,1), (0\ ,\frac{1}{2}),(0\ ,\frac{1}{3}),\dots\}$

1. A is not an open set in $((\mathbb{R}^2, u\ ),)$, since $(0,1) \in A$ which is not an interior point of A.
2. A is not closed set since A^c is not open , $A^c = \mathbb{R}^2\backslash A\ is\ not\ open$.

Note that

$d(A) = \{(0,0)\} \nsubseteq A$, therefore A is not closed by theorem $\{let(X, d)$ a metric space and $A \subseteq X$,then A is closed $iff\ d(A) \subseteq A\}$.

3. A is discrete since every $(x, y)\ is\ an\ isolated\ point$.
4. $i(A) = \emptyset$
5. $e(A) = \mathbb{R}^2(A \cup \{(0,0)\})$.
6. $b(A) = A \cup \{(0,0)\}$.
7. A is not compact since A is not closed .

Example 2.6: A= $\mathbb{R}^2$

Solution:

1. R^2 is open in , $((\mathbb{R}^2, u\ ),)$, since every element of $\mathbb{R}^2$ is an interior point , since for every (x,y)∈ $\mathbb{R}^{2,}$ N_r(x,y)⊆ $\mathbb{R}^2$.
2. $R^2\ is\ closed\ set\ , since\ (\mathbb{R}^2)^c = \emptyset\ is\ open$
3. $i(A) = \mathbb{R}^2$
4. $e(A) = \emptyset$
5. $b(A) = \emptyset$
6. d(A)= $\mathbb{R}^2$
7. $\overline{\mathbb{R}^2} = \mathbb{R}^2 \cup$d($\mathbb{R}^2$)

$= \mathbb{R}^2 \cup \mathbb{R}^2$

$= \mathbb{R}^2.$

8. $\mathbb{R}^2$ *is dense in itself since* $\mathbb{R}^2 \subseteq \mathrm{d}(\mathbb{R}^2) = \mathbb{R}^2$.
9. $\mathbb{R}^2$ is complete.

Theorem 2.7: The usual metric space $(\mathbb{R}^2, u)$ is complete.

Proof: Let $\{(x_n, y_n)\}$ be a Cauchy sequence in $(\mathbb{R}^2, u)$, we have to prove that $\{(x_n, y_n)\}$ Convergent in $((\mathbb{R}^2, u)$, let $\forall \varepsilon > 0, \exists k \in \mathbb{N}$ such that

$u((x_n, y_n).(x_m, y_m)) < \varepsilon, \forall n, m > k$

$Then \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2} < \varepsilon \ \forall n, m > k$

Then $(x_n - x_m)^2 + (y_n - y_m)^2 < \varepsilon^2 \ \forall n, m > k$

Then $(x_n - x_m)^2 < \varepsilon^2 \wedge (y_n - y_m)^2 < \varepsilon^2 \ \forall n, m > k$

Then $|(x_n - x_m| < \varepsilon \wedge \quad |(y_n - y_m| < \varepsilon \ \forall n, m > k$

This mean that $\{x_n\}$ and $\{y_n\}$ are a Cauchy sequence in$(\mathbb{R}^2, u)$, But $((\mathbb{R}^2, u)$ Is complete metric space. Then $\exists x_0, y_0 \in \mathbb{R}^2$, s.t, $x_n \to x_0 \wedge y_n \to y_0$

$|(y_n - y_0| < \frac{\varepsilon}{2}$. Then $(x_0, y_0) \in \mathbb{R}^2$

Prove that, we claim that (x_0, y_0) is the limit point of $\{(x_n, y_n)\}$. Since $x_n \to x_0$ Then $\exists k_1 \in \mathbb{N}$ such that $|(x_n - x_0| < \frac{\varepsilon}{2}$, $\forall n > k_1$. Also, since $y_n \to y_0$, $\exists k_2 \in \mathbb{N}$ such that $|(y_n - y_0| < \frac{\varepsilon}{2}$, $\forall n > k_2$. Let $k' = \max\{k_1, k_2\}$.

Then consider $u((x_n, y_n)^2, (x_0, y_0)^2) = (x_n - x_0)^2 + (y_n - y_0)^2$

$$= \frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{4} = \frac{\varepsilon^2}{4} 2 < \varepsilon.$$

So $u((x_n, y_n), (x_0, y_0) < \varepsilon$. Therefore $\{(x_n, y_n)\}$ convergent to (x_0, y_0) in $(\mathbb{R}^2, u)$.Hence $(\mathbb{R}^2, u)$ is complete.

Theorem 2.8: Every closed and bounded set in the metric space $(\mathbb{R}^2, u)$ is compact.

Proof: Let $S \subseteq \mathbb{R}^2 be\ a$ closed and bounded set then there exist $a \subseteq \mathbb{R}^2$.
Such that $[-a, a]^2$(box in $\mathbb{R}^2$) such that $S \subseteq [-a, a]^2$.
Let $\{G_\alpha: \alpha \in \Delta\}$ be an open cover for S.Suppos $\{G_\alpha: \alpha \in \Delta\}$ has no finite sub cover for S.(suppose S can not be covered by finite sub cover of $\{G_\alpha: \alpha \in \Delta\}$)
There is a closed box I_1 such that $S \subseteq I_1$, Let $\{G_\alpha: \alpha \in \Delta\}$ be an open cover for S. $S \subseteq\cup_{\alpha\in\Delta} G_\alpha$.Assume that $\{G_\alpha: \alpha \in \Delta\}$ has not finite sub cover for S.Divide I_1in two sub closed boxes I_2 and $I_2{}'$. Then at least one of $S \cap I_2$ and $S \cap I_2{}'$. Can not be covered by a finite sub cover of $\{G_\alpha: \alpha \in \Delta\}$, let $S \cap I_2$. Divide I_2 in two sub closed boxes I_3 and $I_3{}'$ then at least one $\cap\ I_3$ and $S \cap I_3{}'$can not covered by finite subcover of $\{G_\alpha: \alpha \in \Delta\} let\ S \cap I_3$.continues in this setps we get the infinite familly $\{S \cap I_n\}$which can not be covered by finite sub cover of $\{G_\alpha: \alpha \in \Delta\}$ and we get

1. $\{ I_n \}$ is a set of closed boxes.
2. $I_n \supseteq I_{n+1} \forall_{n\in\mathbb{N}}$.
3. $| I_n | \to 0$ as $n \to \infty$, where $| I_n|$ is area of $I_n \forall_{n\in\mathbb{N}}$

So by Nested interval theorem [Let $\{ I_n\}$ be sequence of closed interval. Such that $I_n \supseteq I_{n+1} \forall_{n\in\mathbb{N}}$ then $\cap_{n\in\mathbb{N}} I_n \neq \emptyset$ if $\{| I_n |\}$ converges to 0, then $\cap_{n\in N} I_n$ Consists of only one point]. then $\cap_{n\in N}$ consists of one one point let $\cap_{n\in N}$ $I_n = \{x_0\}$. we clean that x_0 is a cluster point of S.Then for every open set V we have $V \cap (S - x_0) \neq \emptyset$, since $\exists k \in \mathbb{N}$. Such that $I_k \cap S \subseteq V$. But S is closed. Then $x_0 \in S$(since $d(S) \subseteq S$) there exist $I_n \cap S$ contains x_0 , $\exists G_\alpha x_0$ in $\{G_\alpha: \alpha \in \Delta\}$ since $\subseteq\cup_{\alpha\in\Delta} G_\alpha$ and $G_\alpha x_0 \supseteq I_n \cap S\ \forall_{n>k}$. So $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ is a finite subcover of $\{G_\alpha: \alpha \in \Delta\}$ which covers S is a compact set.

Theorem 2.19: In the usual metric space $(\mathbb{R}^2, u)$, a set $S \subseteq \mathbb{R}^2$is closed $iff\ d(S) \subseteq S$.

Proof: Let S be a closed set in the usual metric space $(\mathbb{R}^2, u)$,S o S^c is an open set in $(\mathbb{R}^2, u)$by the definition of closed set we have then all elements of

$S^c, therefore$ for every $(x_0,y_0) \in S^c$ there are interior points of S^c . a neighborhoods of (x_0,y_0) (Say $N_r(x_0,y_0)$), such that $N_r\big((x_0,y_0)\big) \subseteq S^c$ then $N_r(x_0,y_0)) \cap (\mathrm{S} - \{x_0\}) = \emptyset$, So (x_0,y_0) is not a cluster point of S , then $d(S) \subseteq S$. Conversely , assume that $d(S) \subseteq S$ this means that all cluster points of S are in S. So, all pointe of S^c are not cluster points of S then for each $(x,y) \subseteq S$ therte is a nbd of (x,y) $(Say\ N_\varepsilon(x,y)$) such that $N_\varepsilon(x_0,y_0)) \cap (\mathrm{S} - \{x\}) = \emptyset$ so $N_\varepsilon(x,y) \subseteq S$. That is $(x,y)$is an interior point of S^c. So, S^c is open, therefore S is closed.

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پوختە

ئێمە لەم ئیشەی خۆمان کە لە topological properties خوێندومان لە usual metric spaces

وە ئیمە ئەمانەمان پێشان داوە prove that every closed and bounded set in$(\mathbb{R}^2, \mathrm{u})$ is compact, and also $(\mathbb{R}^2, \mathrm{u})$ is a complete, and S is closed iff $d(S) \subseteq S$. وە هەروەها چەند نموونەیەکمان لە usual metric spaces $(\mathbb{R}^2, \mathrm{u})$ پێشانمان داوە .