زانکۆی سەلاحەدین - هەولێر
Salahaddin University-Erbil

The average distance of the complete bipartite graph

Research Project

Submitted to the department of Mathematics in partial fulfillment of the requirements for the degree of BSc. in MATHEMATIC

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April – 2023

Certification of the Supervisor

I certify that this work was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University- Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

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Scientific grade**: Assistant Professor**

Date**: / 4 /2023**

In view of available recommendations, I forward this word for debate by the examining committee.

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Acknowledgment

First of all, I would like to thanks God for helping me to complete this project with success.

Secondly, I would like to express my special thanks to my supervisor Assist. Prof. Dr. Rashad Rashid Haji, it has been great honor to be his student.

Is a great pleasure for me to undertake this project I have taken efforts however it would not have been possible without the support and help of many individuals.

Also, I would like to express my gratitude towards my parents and my friends.

My thanks appreciations go to Mathematical Department and all my valuable teachers.

Abstract

In this work we find the mean (average) distance of the complete bipartite graph $k_{n,m}$ which is $\mu(k_{m,n}) = \frac{2n(n-1)+2m(m-1)+2nm}{(n+m)(n+m-1)}$ and the mean distance of $k_{n,n}$ which is $\boxed{\mu\,(k_{n,n}) = \frac{3n-1}{2n-1}.}$

As a special case of $k_{n,m}$. Moreover, we study some graphical properties of $k_{n,m}$.

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CHAPTER ONE

INTRODUCTION

A complete bipartite graph is a type of bipartite graph in which all vertices in one set are connected to all vertices in the other set. This is represented by Km, n. where m and n are the number of vertices in two different sets.

A complete bipartite graph has several properties, including maximal matching of size $min(m, n)$, and the adjacency matrix of a complete bipartite graph has eigenvalues $\sqrt{mn}, -\sqrt{mn}$, and 0. Also, all complete bipartite graphs are modular graphs. Commonly used full bipartite graphs include utility graphs, claws, and star graphs. Complete bipartite graphs have many applications in real-world scenarios and play an important role in the Foucault pendulum (Yao et al., 2022).

A bipartite graph is a graph type whose vertices are split into two disjoint sets such that no two adjacent vertices are in the same set. This means that if you split your vertices into two sets (say set A and set B), all edges will run between vertices in set A and vertices in set B, but not between vertices within each set. means Bipartite graphs are useful in a variety of applications, such as: B. To model relationships between two different types of objects, or to identify potential matches between members of two different groups. (Shang, 2015)

Bipartite graphs are especially useful in computer science where they can be used to model a variety of real-world problems such as social networks and recommender systems. (Yao et al., 2022) For example, a social network analysis application can consider users as Group A and topics or groups of interest as Group B. Benefit from it

It is also used in scheduling problems where jobs need to be assigned to workers based on certain constraints.

Moreover, bipartite graphs are important in the field of mathematics as they serve as a fundamental concept in graph theory and are used to prove various theorems and statements. Overall, bipartite graphs have a wide range of practical and theoretical applications and are important tools in several research areas. A bipartite graph is a type of graph whose vertices are split into two disjoint sets that have many real-world applications in computer science, planning problems, and graph theory. (Parker et al., 2021)

CHAPTER TWO

Literature Review

Definition 2.1:(Behzad & Chartrand, 1979) A *graphs* G is a finite non empty set of objects called *vertices* (the singular word is *vertex*) together with a (Possibly empty) set of un order pairs of distinct vertices of G called edges.

Definition 2.2: (Behzad & Chartrand, 1979) The cardinality or the number of the vertex set of a graph G is called the *order* of G, denoted by $p(G)$ or simply p.

Definition 2.3: (Behzad & Chartrand, 1979) The cardinality or the number of the edge set of a graph G is called the *size* of G, is denoted by $q(G)$ or simply q.

Definition 2.4: (Ping Zhang, L. L. 2015) in graph theory, the distance between two vertices of a graph G is defined as the length of the shortest path between those vertices. The length of a path is the number of edges in the path, so the distance between two vertices is the minimum number of edges that must be traversed to get from one vertex to the other

Definition 2.5: (P. Dankelmann, 1997) Let G be a *connected* graph, and $v \in V(G)$, the *transmission* of v is denoted by $\sigma(v)$ and defined by $\sigma(v) = \sum_{u \epsilon V(G)} d(u, v)$.

Definition 2.7: (P. Dankelmann, 1997) Let G be a *connected* graph, then the *transmission* of a graph G is denoted by $\sigma(G)$ and defined by $\sigma(G) = \sum_{v \epsilon V(G)} \sigma(v)$.

Definition 2.8: (P. Dankelmann, 1997) Let G be a connected graph, then the *mean distance* (*average distance*) of G is denoted by $\mu(G)$ and defined by $\mu(G) = \frac{\sigma(G)}{p(p-1)}$,where p is the order of G .That is ,The *average distance* of a graph G is the average value of the distance between all pairs of vertices in G .

CHAPTER THREE

THE AVERAGE DISTANCE OF THE COMPLETE BIPARTITE GRAPH

Definition 3.1: (Wayne,G&Ortrud,R. 2010) A graph G is a complete bipartite graph if $V(G)$ can be partitioned into two sets U and W (called partite sets again) so that uw is an edge of G if and only if $u\ 2U$ and $w\ 2W$. If $jUj = s$ and $jWj = t$, then this complete bipartite graph has order $s + t$ and size st and is denoted by $Ks; t\ (or\ Kt; s)$. The complete bipartite graph $K1; t$ is called a star. The complete bipartite graphs $K1; 3, K2; 2, K2$;3 and $K3; 3$ are shown in Figure 1.17. Observe that $K2; 2 = C4$. The star $K1; 3$ is sometimes referred to as a claw.

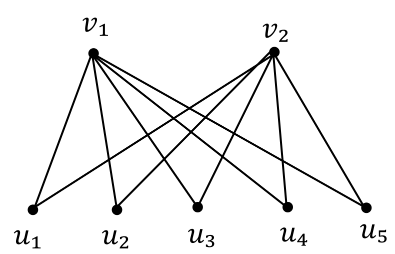


Figure 3.1 Complete Bipartite Graph

Theorem 3.1.2.

The mean distance of the complete bipartite graph Kn, m is

$\mu(k_{n,m}) = \frac{2n(n-1)+2m(m-1)+2nm}{(n+m)(n+m-1)}$ for n, m >1

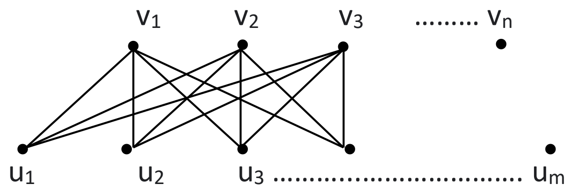


Figure 3.1 Complete Bipartite Graph Kn, m

Proof:

$$\sigma(v_1) = \sum_{i=1}^{n} d(v_1, v_n) + \sum_{i=1}^{m} d(v_1, u_m)$$

$$\sum_{i=1}^{n} d(v_1, v_n) = \underbrace{2+2+2+\ldots\ldots+2}_{n\text{-1 times}} = 2(n-1)$$

$$\sum_{i=1}^{m} d(v_1, u_m) = \underbrace{1 + 1 + 1 + \cdots \ldots + 1}_{m \text{ times}} = \text{m}$$

$$\sigma(v_1) = \sum_{i=1}^{n} d(v_1, v_n) + \sum_{i=1}^{m} d(v_1, u_m)$$

$$= 2(n-1) + m$$

$$\sigma(v_2) = \sum_{i=1}^{n} d(v_2, v_n) + \sum_{i=1}^{m} d(v_2, u_m)$$

$$\sum_{i=1}^{n} d(v_2, v_n) = 2+2+\underbrace{2+\ldots\ldots+2}_{n\text{-1 times}} = 2(n-1)$$

$$\sum_{i=1}^{m} d(v_2, u_m) = 1+\underbrace{1+1+\ldots\ldots+1}_{m\text{ times}} = m$$

$$\sigma(v_2) = \sum_{i=1}^{n} d(v_2, v_n) + \sum_{i=1}^{m} d(v_2, u_m)$$

$$= 2(n-1) + m$$

$$\sigma(v_3) = \sum_{i=1}^{n} d(v_3, v_n) + \sum_{i=1}^{m} d(v_3, u_m)$$

$$\sum_{i=1}^{n} d(v_3, v_n) = 2+2+\underbrace{2+\ldots\ldots+2}_{n\text{-1 times}} = 2(n-1)$$

$$\sum_{i=1}^{m} d(v_3, u_m) = 1+1+\underbrace{1+1+\ldots\ldots+1}_{m\text{ times}} = m$$

$$\sigma(v_3) = \sum_{i=1}^{n} d(v_3, v_n) + \sum_{i=1}^{m} d(v_3, u_m)$$

$$= 2(n-1) + m = \sigma(V4) = \sigma(V5) = \ldots\ldots = \sigma(Vn)$$

$$\sigma(u_1) = \sum_{i=1}^{m} d(u_1, u_m) + \sum_{i=1}^{n} d(u_1, v_n)$$

$$\sum_{i=1}^{m} d(u_1, u_m) = 2+2+\underbrace{2+\ldots\ldots+2}_{m\text{-1 times}} = 2(m-1)$$

$$\sum_{i=1}^{n} d(u_1, u_m)) = 1+1+\underbrace{1+\ldots\ldots+1}_{n\text{ times}} = n$$

$$\sigma(u_1) = \sum_{i=1}^{m} d(u_1, u_m) + \sum_{i=1}^{n} d(u_1, v_n)$$

$$= 2(m-1) + n$$

$$\sigma(u_2) = \sum_{i=1}^{m} d(u_2, u_m) + \sum_{i=1}^{n} d(u_2, v_n)$$

$$\sum_{i=1}^{m} d(u_2, u_m) = 2+2+\underbrace{2+\ldots\ldots+2}_{m\text{-1 times}} = 2(\mathrm{m}-1)$$

$$\sum_{i=1}^{n} d(u_2, v_n) = 1+1+\underbrace{1+\ldots...+1}_{n\text{ times}} = \mathrm{n}$$

$$\sigma(u_2) = \sum_{i=1}^{m} d(u_2, u_m) + \sum_{i=1}^{n} d(u_2, v_n)$$

$$= 2(m-1) + n$$

$$\sigma(u_3) = \sum_{i=1}^{m} d(u_3, u_m) + \sum_{i=1}^{n} d(u_3, v_n)$$

$$\sum_{i=1}^{m} d(u_3, u_m) = 2+2+\underbrace{2+\ldots\ldots+2}_{m\text{-1 times}} = 2(\mathrm{m}-1)$$

$$\sum_{i=1}^{n} d(u_3, v_n) = 1+1+\underbrace{1+\ldots...+1}_{n\text{ times}} = \mathrm{n}$$

$$\sigma(u_3) = \sum_{i=1}^{m} d(u_3, u_m) + \sum_{i=1}^{n} d(u_3, v_n)$$

$$\sigma(u_3) = 2(m-1) + n = \sigma(u_4) = \sigma(u_5) = \ldots\ldots\ldots.. = \sigma(u_n)$$

$$\mu\left(k_{m,n}\right) = \sum_{i=1}^{n} \sigma(v_i) + \sum_{i=1}^{m} \sigma(u_i)$$

$$= n[2(n-1)+m] + m[2(m-1)+n]$$
$$= 2n(n-1) + 2m(m-1) + 2nm$$

$$\mu(k_{m,n}) \ = \frac{\sigma\ (k_{m,n})}{(n+m)\ (n+m-1)}$$

$$= \frac{2n\ (\ n-1)\ +\ 2m\ (m-1)\ +\ 2nm}{(n+m\ )\ (n+m-1)}$$

Remark 3.2: the mean distance of the complete bipartite graph $k_{n,n}$ is:

$$\mu(k_{n,n}) \ = \frac{3n-1}{2n-1}$$

Proof: set $m \ = \ n$ in theorem we obtain that

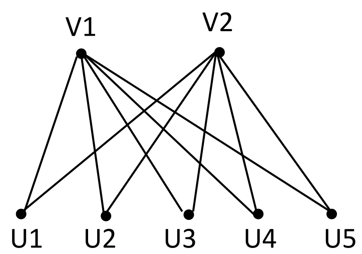
$$\mu(k_{n,n}) \ = \ \frac{2n(n-1) + \ 2n\ (n-1) + \ 2n^2}{2n\ (2n - \ 1\ )}$$

$$= \frac{4n\ (\ n - \ 1\ )\ +\ 2n^2}{2n\ (\ 2n\ \ -1\ )}$$

$$= \frac{2(n-1)+n}{2n-1}$$

$$= \frac{3n-1}{2n-1}$$

Example 3.3: The mean distance of the complete bipartite graph k 2,5 can be found as follows



$$\mu\ (k\ 2,5)$$

$$\sigma(v_1) = \sum_{i=1}^{2} d(v_1, v_i) + \sum_{i=1}^{5} d(v_1, u_i)$$

$$\sum_{i=1}^{2} d(v_1, v_2) = 2$$

$$\sum_{i=1}^{5} d(v_1, u_i\ ) = 1 + 1 + 1 + 1 + 1$$

$$\sigma(v_1) = \sum_{i=1}^{2} d(v_1, v_i) + \sum_{i=1}^{5} d(v_1, u_i)\ = 2 + 5 = 7$$

$$\sigma(v_1) = \sum_{i=1}^{2} d(v_2, v_i) + \sum_{i=1}^{5} d(v_2, u_i)$$

$$\sum_{i=1}^{2} d(v_2, v_i) = 2$$

$$\sum_{i=1}^{5} d(v_2, u_i) = 1 + 1 + 1 + 1 + 1 = 5$$

$$\sum_{i=1}^{2} d(v_2, v_i) + \sum_{i=1}^{5} d(v_2, u_i) = 2 + 5 = 7$$

$$\sigma(u_1) = \sum_{i=1}^{5} d(u_1, u_i) + \sum_{i=1}^{2} d(u_1, v_i)$$

$$\sum_{i=1}^{5} d(u_1, u_i) = 2 + 2 + 2 + 2 = 8$$

$$\sum_{i=1}^{2} d(u_1, v_i) = 1 + 1 = 2$$

$$\sigma(u_1) = \sum_{i=1}^{5} d(u_1, u_i) + \sum_{i=1}^{2} d(u_1, v_i) = 8 + 2 = 10$$

$$\sigma(u_2) = \sum_{i=1}^{5} d(u_2, u_i) + \sum_{i=1}^{2} d(u_2, v_i)$$

$$\sum_{i=1}^{5} d(u_2, u_i) = 2 + 2 + 2 + 2 = 8$$

$$\sum_{i=1}^{2} d(u_2, v_i) = 1 + 1 = 2$$

$$\sigma(u_2) = \sum_{i=1}^{5} d(u_2, u_i) + \sum_{i=1}^{2} d(u_2, v_i) = 8 + 2 = 10$$

$$\sigma(u_3) = \sum_{i=1}^{5} d(u_3, u_i) + \sum_{i=1}^{2} d(u_3, v_i)$$

$$\sum_{i=1}^{5} d(u_3, u_i) = 2 + 2 + 2 + 2 = 8$$

$$\sum_{i=1}^{2} d(u_3, v_i) = 1 + 1 = 2$$

$$\sigma(u_3) = \sum_{i=1}^{5} d(u_3, u_i) + \sum_{i=1}^{2} d(u_3, v_i) = 8 + 2 = 10$$

$$\sigma(u_4) = \sum_{i=1}^{5} d(u_4, u_i) + \sum_{i=1}^{2} d(u_4, v_i)$$

$$\sum_{i=1}^{5} d(u_4, u_i) = 2 + 2 + 2 + 2 = 8$$

$$\sum_{i=1}^{2} d(u_4, v_i) = 1 + 1 = 2$$

$$\sigma(u_4) = \sum_{i=1}^{5} d(u_4, u_i) + \sum_{i=1}^{2} d(u_4, v_i) = 8 + 2 = 10$$

$$\sigma(u_5) = \sum_{i=1}^{5} d(u_5, u_i) + \sum_{i=1}^{2} d(u_5, v_i) = 8 + 2 = 10$$

$$\sigma\left(k_{2,5}\right) = \sum_{i=1}^{2} \sigma(v_i) + \sum_{i=1}^{5} \sigma(u_1)$$

$$\sum_{i=1}^{2} \sigma(v_1) = 7 + 7 = 14$$

$$\sum_{i=1}^{5} \sigma(u_i) = 10 + 10 + 10 + 10 + 10 = 50$$

$$\sigma(k_{2,5}) = \sum_{i=1}^{2} \sigma(v_i) + \sum_{i=1}^{5} \sigma(u_i) = 14 + 50 = 64$$

$$\mu(k_{2,5}) = \frac{\sigma(k_{2,5})}{p(p-1)}$$

$$\frac{64}{7(6)} = \frac{64}{42} = \frac{32}{21}$$

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پوختە

لەم کارەدا تێکرای دووری گرافی تایبەت بەناوی (complete bipartite graph) واتە گرافی دووبەشکراوی تەواو دەدۆزینەوە کە دەکاتە $\mu(k_{n,m}) = \frac{2n(n-1)+2m(m-1)+2nm}{(n+m)(n+m-1)}$

وە هەروەها تێکرای دووری گرافی دووبەشکراوی تەواو $k_{n,n}$ دەدۆزینەوە وەک حاڵەتێکی تایبەت کە دەکاتە $\mu(k_{n,n}) = \frac{3n-1}{2n-1}$.