Rational numbers and Real numbers

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Archimedes Principle

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Let $S = \{ka : k \in \mathbb{N}\}$. Then S is a nonempty subset of $\mathbb{R}$. Assume that the theorem is not true. Then b become an upper bound of S. Since $\mathbb{R}$ is complete, so S has a supremum in $\mathbb{R}$. Let $\sup(S) = y$.

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- Notice that $y - a < y$, so $y - a$ can not be upper bound for S. Thus, there is $ma \in S$ such that $y - a < ma$. Then $ma + a > y$, so $(m+1)a > y$. But $(m+1)a \in S$ which contradicts $\sup(S) = y$. □

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(The density of Rational numbers) Between any two distinct real numbers there is a rational number.

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S-continuity and L^1-integribility on *finite sets

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Density of Rational numbers

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Let $a, b \in \mathbb{R}$ and $a < b$. Then by Theorem of rational density, there is a rational r_1 between a and b, and there is a rational r_2 between r_1 and b. In general there is a rational r_n between r_{n-1} and b. Thus we get the set $\{r_1, r_2, \ldots, r_n, \ldots\}$ which is a subset of $\mathbb{Q}$. □

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Let $r \in \mathbb{Q}$ and $s \in \mathbb{Q}^c$, where $\mathbb{Q}^c$ is the set of all irrational numbers. Then:
(1) $r + s \in \mathbb{Q}^c$.
(2) $r \cdot s \in \mathbb{Q}^c$, provided that $r \neq 0$.

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Proof.

Let $a, b \in \mathbb{R}$ and $a < b$. assume that there is no irrational number between a and b. This means that all numbers between a and b are rational numbers. By adding $\sqrt{2}$ with a ,b and all numbers between them, we obtain all numbers between $a + \sqrt{2}$ and $b + \sqrt{2}$ are irrational numbers. That is, there is no rational number between $a + \sqrt{2}$ and $b + \sqrt{2}$ which is contradiction with Remark 6. Hence, there is an irrational number between a and b. □

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The density of Irrational numbers

Proof.

Let $a, b \in \mathbb{R}$ and $a < b$. Then by Theorem 7, there is an irrational s_1 between a and b, and there is an irrational s_2 between s_1 and b. In general there is a rational s_n between s_{n-1} and b. Thus we get the set $\{s_1, s_2, \ldots, s_n, \ldots\}$ which is a subset of $\mathbb{Q}^c$. □

(The density of Irrational numbers) Let A and B be two nonempty subsets of $\mathbb{R}$. If $x < y$ for all $x \in A$ and for all B, then prove that $\sup(A) \leq \inf(B)$ Between any two distinct real numbers there is an irrational number.

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The relation between $L^1(\mathbb{T})$ and $L^1(\Omega)$

Proof: Assume that $h \in L^1(\Omega)$, where $\Omega = \{\lfloor -\frac{N}{2} \rfloor + 1, \lfloor -\frac{N}{2} \rfloor + 2, \ldots, 0, \ldots, \lfloor \frac{N}{2} \rfloor\}$ and $N > \mathbb{N}$. Then h is S-integrable and almost S-continuous internal function on Ω. So, there is a rare subset E of Ω such that h is S-continuous on $\Omega - E$. Let $B = \{t \in \mathbb{T} : h(\lfloor \frac{Nt}{2\pi} \rfloor) \notin {}^*\mathbb{C}_{fin}\}$. Since $\int_\Omega |h| dm$ is limited and h is almost S-continuous on Ω, then B is a measurable subset of $\mathbb{T}$ and it has a Lebesgue zero via Loeb Theorem ($B \subseteq \mathbb{T}$ is Lebesgue measurable if and only if $\forall \epsilon > 0$ in $\mathbb{R}$ there are *finite sets A and C such that $A \subseteq \mathrm{st}^{-1} B \subseteq C$ and $\frac{\mathrm{card}(C-A)}{N} < \epsilon$.) In this case $\mathrm{st}(\frac{\mathrm{card} A}{N}) = \mathrm{st}(\frac{\mathrm{card} C}{N})$ and this is the Loeb measure of B.

Bounded and Monotone sequences

Now we define

$$f(t) = \begin{cases} \mathrm{st}(h(\lfloor Nt2\pi \rfloor)) & \text{if } t \in \mathbb{T} - B, \\ 0 & \text{if } t \in B. \end{cases}$$

From the definition of f, we deduce that $|f(t)|$ is limited for all $t \in \mathbb{T}$. So, $\int_{\mathbb{T}} |f(t)|dt$ is limited. That is, $\| f \|_1 = \int_{\mathbb{T}} |f(t)|dt$ is finite.

Cauchy sequences

Moreover, since h is almost S-continuous on Ω, then there exists a rare subset E of Ω such that h is S-continuous on $\Omega - E$. let $a \in \omega - E$ (a nonstandard model of $\mathbb{T} - B$). Then for all $t \in \omega - E$, if $t \approx a$, then $h(t) \approx h(a)$ is true in $\Omega - E$. given $\epsilon > 0$ arbitrary and standard, without loss of generality, let $\epsilon = 1/n$, for some $n \in \mathbb{N}$. We have to find $\delta > 0$ of the form $1/k$, where $k \in \mathbb{N}$ such that

$$\forall t(|t - a| < \delta \Rightarrow |h(t) - h(a)| < \frac{1}{n})$$

is true in $\Omega - E$.

Complete sets

So, for all unlimited $k \in {}^*\mathbb{N}$ we have

$$\forall t(|t - a| < 1/k \Rightarrow |h(t) - h(a)| < \frac{1}{n})$$

is true in $\Omega - E$. Then

$$\forall t(|t - a| < 1/k \Rightarrow |\mathrm{st}(h(t)) - \mathrm{st}(h(a))| < \frac{1}{n})$$

is true in $\mathbb{T} - B$. So,

$$\forall t(|t - a| < 1/k \Rightarrow |f(t) - f(a)| < \frac{1}{n})$$

is true in $\mathbb{T} - B$.

Convergent sequences and Cauchy sequences

Now, let

$$\theta(k) = (k \in {}^*\mathbb{N}) \wedge (k = 0 \vee \neg \forall h(|t-a| < 1/k \Rightarrow |f(t)-f(a)| < \frac{1}{n})).$$

if we couldn't find $\delta = 1/k$, for $k \in \mathbb{N}$, then θ would define $\mathbb{N}$ in ${}^*\mathbb{R}$. Which is contradiction with $\mathbb{N}$ is not definable (not internal) in ${}^*\mathbb{R}$. Therefore, f is a continuous function in on $\mathbb{T} - B$. Hence, f is continuous almost everywhere on $\mathbb{T}$. Since $\mathbb{T}$ is a Lebesgue measurable set, so f is a measurable function on $\mathbb{T}$. Hence, $f \in L^1(\mathbb{T})$ and $f(t) = \mathrm{st}(h(\lfloor \frac{Nt}{2\pi} \rfloor))$ almost everywhere on $\mathbb{T}$.