Chapter One

Physics and Measurements

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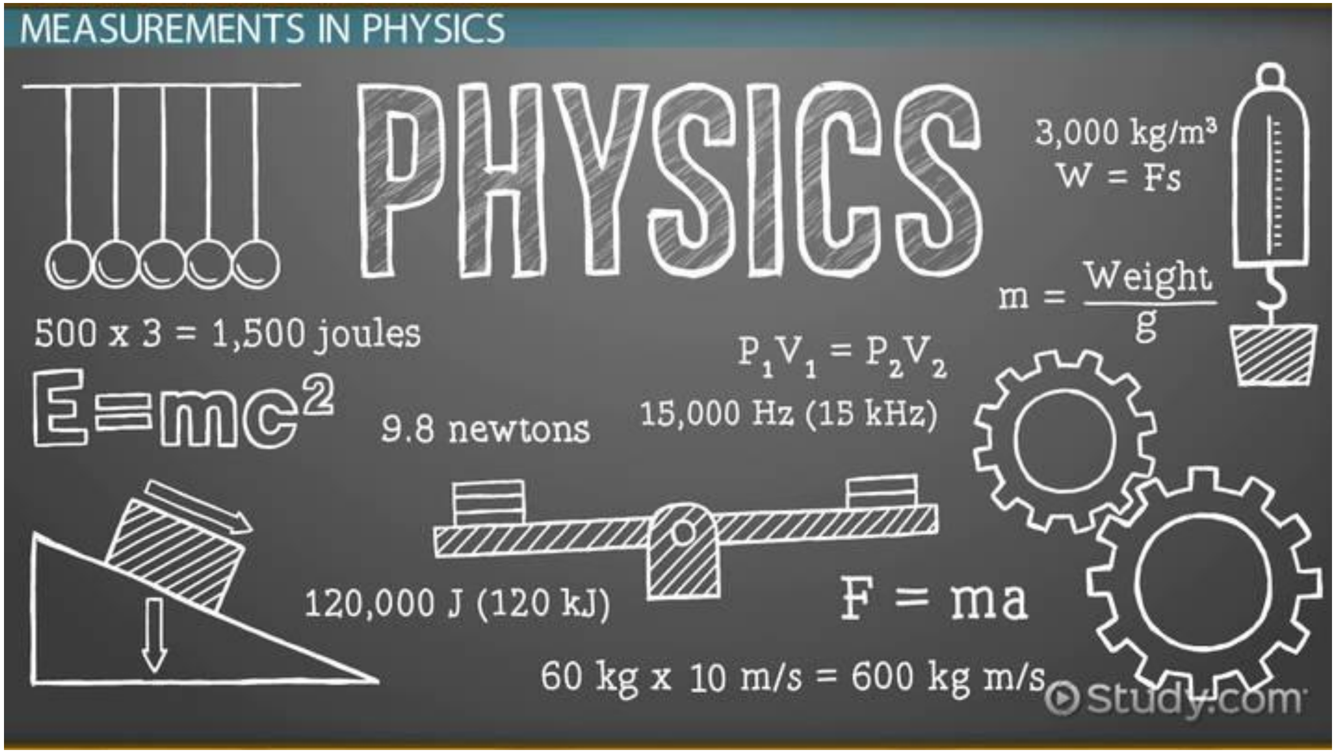
Physics and Measurements

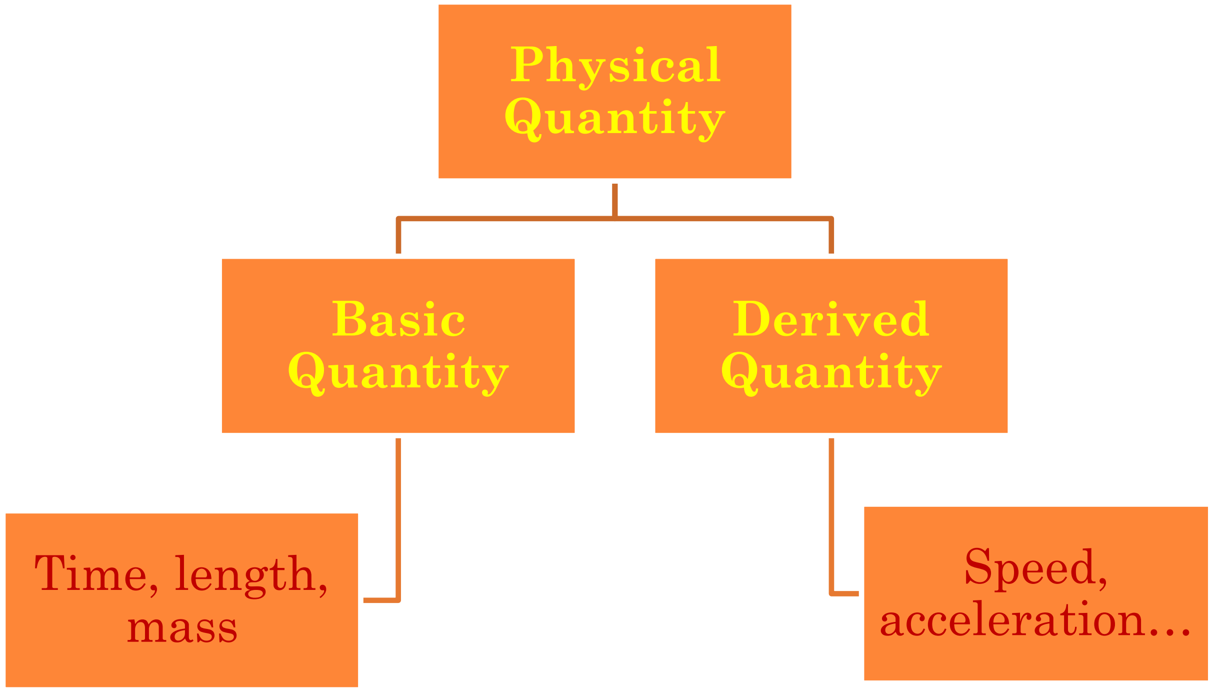
- **Physics is an experimental science that interested to detect the Secrets and mysteries of nature, everything we know about the universe and the laws that governed by it reached through measurements and observations of any natural phenomenon.**



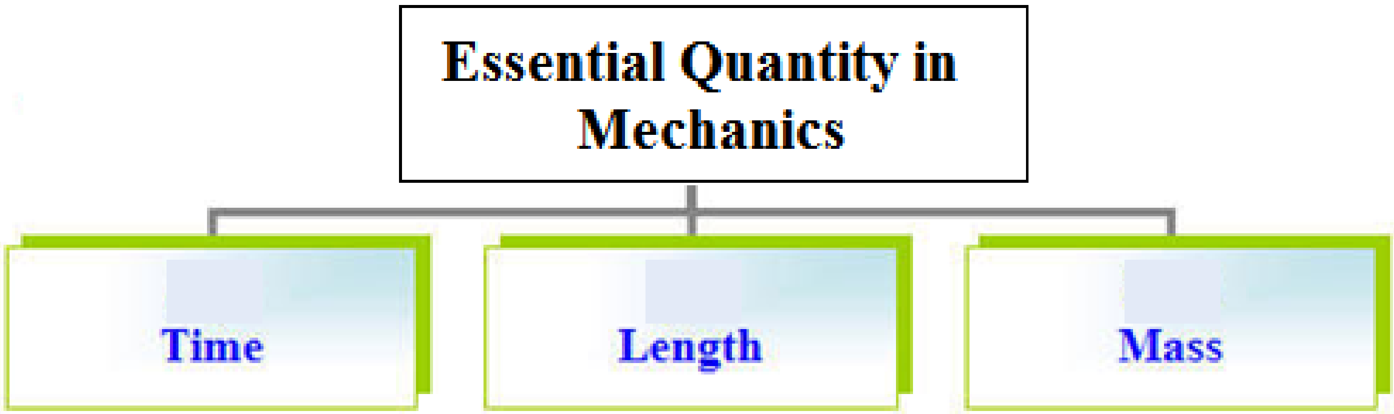
- ***When you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge of it is of a meager and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely, in your thoughts, advanced it to the stage of science.***

Sir William Thompson, Lord Kelvin (1824-1907)

Physical Quantity



- **Derived Physical Quantity**.
- it depends on the description of the method for measuring of any physical quantity.



Mass: The SI unit of mass is the *Kilogram*, which is defined as the mass of a specific platinum-iridium alloy cylinder.

The prototype was made in 1879 by George Matthey (of Johnson Matthey) in the form of a cylinder, 39 mm high and 39 mm in diameter, consisting of an alloy of 90 % platinum and 10 % iridium. The international prototype is kept at Sevres, France

- **Time**: The SI unit of time is the *Second*, which is the time required for a cesium-133 atom to undergo 9192631770 vibrations.
- **Length**: The SI unit of length is Meter, which is the distance traveled by light in vacuum during a time of 1/2999792458 second.

Length

Distance	Length (m)
Radius of Visible Universe	1×10^{26}
To Andromeda Galaxy	2×10^{22}
To nearest star	4×10^{16}
Earth to Sun	1.5×10^{11}
Radius of Earth	6.4×10^{6}
Sears Tower	4.5×10^{2}
Football Field	1×10^{2}
Tall person	2×10^{0}
Thickness of paper	1×10^{-4}
Wavelength of blue light	4×10^{-7}
Diameter of hydrogen atom	1×10^{-10}
Diameter of proton	1×10^{-15}

Time

Interval	Time (s)
Age of Universe	5×10^{17}
Age of Grand Canyon	3×10^{14}
Avg age of college student	6.3×10^{8}
One year	3.2×10^{7}
One hour	3.6×10^{3}
Light travel from Earth to Moon	1.3×10^{0}
One cycle of guitar A string	2×10^{-3}
One cycle of FM radio wave	6×10^{-8}
One cycle of visible light	1×10^{-15}
Time for light to cross a proton	1×10^{-24}

Mass

Object	Mass (kg)
Visible universe	$\sim 10^{52}$
Milky Way galaxy	7×10^{41}
Sun	2×10^{30}
Earth	6×10^{24}
Boeing 747	4×10^{5}
Car	1×10^{3}
You	7×10^{1}
Dust particle	1×10^{-9}
Bacterium	1×10^{-15}
Proton	2×10^{-27}
Electron	9×10^{-31}
Neutrino	$<1 \times 10^{-36}$

H.W.: Compare your age, tall in length and mass with those in the previous table.

International System of Units (SI): It consists of seven base quantities and their corresponding base units

Base Quantity	Base Unit
Length	meter (m)
Mass	kilogram (kg)
Time	second (s)
Electric Current	ampere (A)
Temperature	Kelvin (K)
Amount of Substance	mole (mol)
Luminous Intensity	candela (cd)

dim length = L, dim mass = M, dim time = T

1	Kilometer	km	10^{3} m
1	Decimeter	dm	10^{-1} m
1	Centimeter	cm	10^{-2} m
1	Millimeter	mm	10^{-3} m
1	Micrometer	m	10^{-6} m
1	Nanometer	nm	10^{-9} m
1	Angstrom	A	10^{-10} m
1	Picometer	pm	10^{-12} m
1	Femtomete	fm	10^{-15} m

Example: Astronomical distances are sometimes described in terms of *light-years* (ly). A light-year is the distance that light will travel in one year (yr). How far in meters does light travel in one year?

Solution: Using the relationship

distance= (speed of light) · (time)

we need to know how many seconds are in a year. We know that:

1 year = 365.25 days, 1 day = 24 hours,
1 hour = 60 minutes, 1 minute = 60 seconds.

$$1\ \text{year} = (365.25\ \text{day})\left(\frac{24\ \text{hours}}{1\ \text{day}}\right)\left(\frac{60\ \text{min}}{1\ \text{hour}}\right)\left(\frac{60\ \text{s}}{1\ \text{min}}\right) = 31{,}557{,}600\ \text{s}.$$

So the distance that light travels in a one year is

$$1\ \text{ly} = \left(\frac{299{,}792{,}458\ \text{m}}{1\ \text{s}}\right)\left(\frac{31{,}557{,}600\ \text{s}}{1\ \text{yr}}\right)(1\ \text{yr}) = 9.461 \times 10^{15}\ \text{m}.$$

DIMENSIONAL ANALYSIS

The dimensional analysis is used to check the formula, since the dimension of the left hand side and the right hand side of the formula must be the same.

$$\text{dim velocity} = (\text{length})/(\text{time}) = L \cdot T^{-1}.$$

where $L \equiv \text{length}$, $T \equiv \text{time}$.

Force is also a derived quantity and has dimension

$$\text{dim force} = \frac{(\text{mass})(\text{dim velocity})}{(\text{time})}.$$

where $M \equiv \text{mass}$. We could express force in terms of mass, length, and time by the relationship

$$\text{dim force} = \frac{(\text{mass})(\text{length})}{(\text{time})^2} = M \cdot L \cdot T^{-2}.$$

The derived dimension of kinetic energy is

$$\text{dim kinetic energy} = (\text{mass})(\text{dim velocity})^2,$$

which in terms of mass, length, and time is

$$\text{dim kinetic energy} = \frac{(\text{mass})(\text{length})^2}{(\text{time})^2} = M \cdot L^2 \cdot T^{-2}$$

The derived dimension of work is

$$\text{dim work} = (\text{dim force})(\text{length}),$$

which in terms of our fundamental dimensions is

$$\text{dim work} = \frac{(\text{mass})(\text{length})^2}{(\text{time})^2} = M \cdot L^2 \cdot T^{-2}$$

So work and kinetic energy have the same dimensions.

Power is defined to be the rate of change in time of work so the dimensions are

$$\text{dim power} = \frac{\text{dim work}}{\text{time}} = \frac{(\text{dim force})(\text{length})}{\text{time}} = \frac{(\text{mass})(\text{length})^2}{(\text{time})^3} = M \cdot L^2 \cdot T^{-3}$$

- The word dimension in physics indicates the physical nature of the quantity. For example the distance has a dimension of ***length***, and the speed has a dimension of ***length/time***.

TABLE: DIMENSIONS OF SOME COMMON MECHANICAL QUANTITIES

M ≡ mass, L ≡ length, T ≡ time

Quantity	Dimension	MKS unit
Angle	dimensionless	Dimensionless = radian
Steradian	dimensionless	Dimensionless = radian^2
Area	L^2	m^2
Volume	L^3	m^3
Frequency	T^{-1}	s^{-1} = hertz = Hz
Velocity	$L \cdot T^{-1}$	$m \cdot s^{-1}$
Acceleration	$L \cdot T^{-2}$	$m \cdot s^{-2}$
Angular Velocity	T^{-1}	$\text{rad} \cdot s^{-1}$
Angular Acceleration	T^{-2}	$\text{rad} \cdot s^{-2}$
Density	$M \cdot L^{-3}$	$kg \cdot m^{-3}$
Momentum	$M \cdot L \cdot T^{-1}$	$kg \cdot m \cdot s^{-1}$
Angular Momentum	$M \cdot L^2 \cdot T^{-1}$	$kg \cdot m^2 \cdot s^{-1}$
Force	$M \cdot L \cdot T^{-2}$	$kg \cdot m \cdot s^{-2}$ = newton = N
Work, Energy	$M \cdot L^2 \cdot T^{-2}$	$kg \cdot m^2 \cdot s^{-2}$ = joule = J
Torque	$M \cdot L^2 \cdot T^{-2}$	$kg \cdot m^2 \cdot s^{-2}$
Power	$M \cdot L^2 \cdot T^{-3}$	$kg \cdot m^2 \cdot s^{-3}$ = watt = W
Pressure	$M \cdot L^{-1} \cdot T^{-2}$	$kg \cdot m^{-1} \cdot s^{-2}$ = pascal = Pa

Example: Using the dimensional analysis to check that this equation $x = ½\,at^2$ is correct, where x is the distance, a is the acceleration and t is the time.

Solution:

$$x = \frac{1}{2}\,a\,t^2$$

$$L = \frac{L}{T^2} \times T^2 = L$$

Left hand dimension = Right hand dimension

Example: **Suppose that the acceleration of a particle moving in circle of radius r with uniform velocity v is proportional to the r^n and v^m. Use the dimensional analysis to determine the power n and m.**

Solution: Let **the acceleration is represented by (a); it can be expressed as;**

$$a \propto r^n v^m$$

or

$$a = k\, r^n v^m$$

Where k is the proportionality constant of dimensionless unit.

- The right hand side $a = \frac{L}{T^2}$
- The left hand side $k\, r^n v^m = L^n \left(\frac{L}{T}\right)^m = \frac{L^{n+m}}{T^m}$

The right hand side = The left hand side

$$\frac{L}{T^2} = \frac{L^{n+m}}{T^m}$$

By equating of the power

$1 = n + m$ (1)

$2 = m$ (2), then $n = -1$

$$\boxed{a = k\, r^{-1} v^2}$$

- **H.W.:**
- **1- Show that the equation $d=vt^2$ dimensionally correct or not?**
- **2- Which of the following formulas for F could be correct?**

(a) $F = mvR$

(b) $F = m\left(\frac{v}{R}\right)^2$

(c) $F = \frac{mv^2}{R}$

Coordinate System

A coordinate system consists of four basic elements:

(1) Choice of origin

(2) Choice of axes

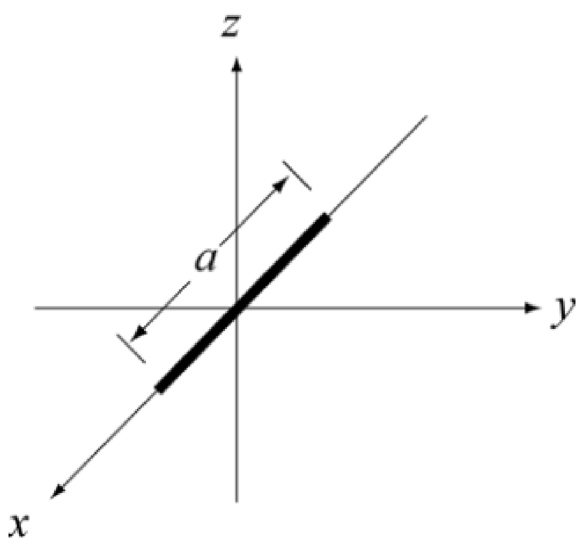
(3) Choice of positive direction for each axis

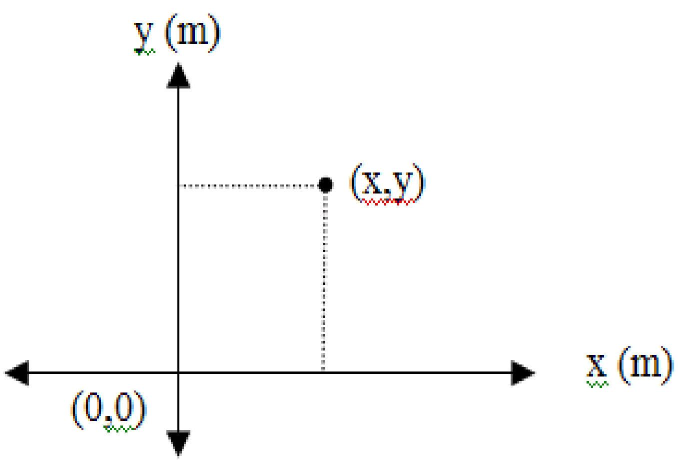
(4) Choice of unit vectors for each axis

There are three commonly used coordinate systems: Cartesian, cylindrical and spherical.

Cartesian Coordinates

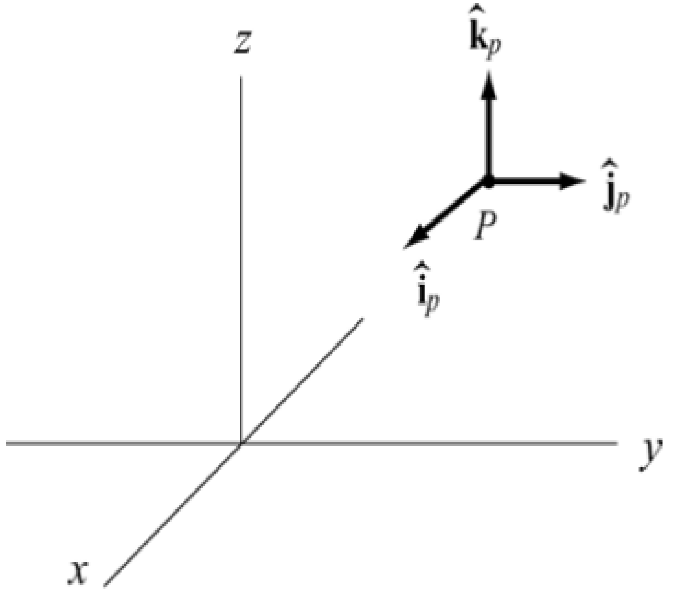
Cartesian coordinates consist of a set of mutually perpendicular axes, which intersect at a common point, the origin *O*. We live in a three-dimensional environment; for that reason, the most common system we will use has three axes, for which we choose the directions of the axes and position of the origin are.





$$-\infty \langle x_P \langle +\infty, -\infty \langle y_P \langle +\infty, -\infty \langle z_P \langle +\infty$$

We now associate to each point P in space, a set of three unit directions vectors $(\hat{\mathbf{i}}_P, \hat{\mathbf{j}}_P, \hat{\mathbf{k}}_P)$. A unit vector has magnitude one: $|\hat{\mathbf{i}}_P| = 1$, $|\hat{\mathbf{j}}_P| = 1$, and $|\hat{\mathbf{k}}_P| = 1$. We assign the direction of $\hat{\mathbf{i}}_P$ to point in the direction of the increasing x-coordinate at the point P. We define the directions for $\hat{\mathbf{j}}_P$ and $\hat{\mathbf{k}}_P$ in the direction of the increasing y-coordinate and z-coordinate respectively.



POLAR COORDINATES

Relation Between Polar and Cartesian Coordinates:

- The relationship between the Cartesian (x, y) and polar coordinates (r, q) shown in the following figure

$$\boldsymbol{x = r\cos\theta}$$

And

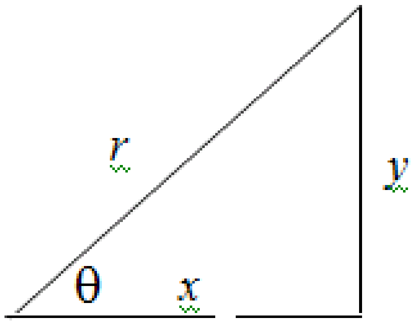
$$\boldsymbol{y = r\sin\theta}$$

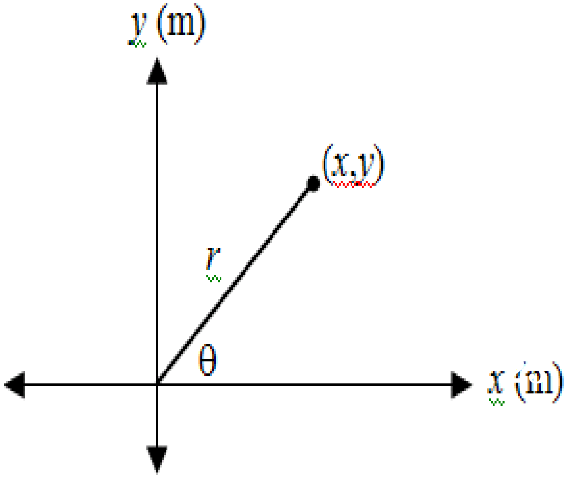
By squaring the two equations then add them we obtain:

$$r = \sqrt{x^2 + y^2}$$

By dividing the tow equations we obtain:

$$\tan\theta = y/x$$





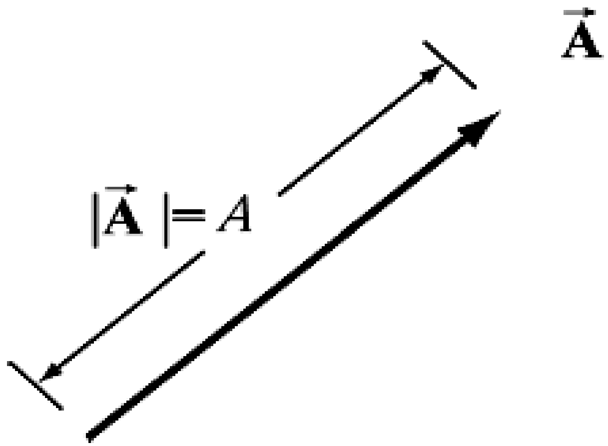
VECTOR

- Physical quantities (essential or derived) can be divided into two types, scalar quantities and vector quantity.

Vector Quantity	Scalar Quantity
Displacement, Position	**Length**
Force, Momentum, Torque	**Mass**
Velocity, Acceleration	**Speed**

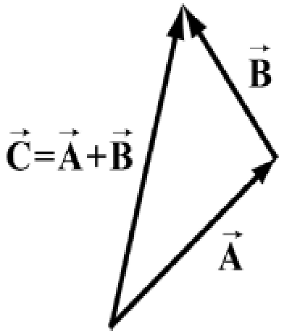
Properties of a Vector

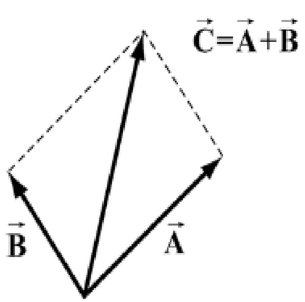
A vector is a quantity that has both direction and magnitude. Let a vector be denoted by the symbol $\vec{\mathbf{A}}$. The magnitude of $\vec{\mathbf{A}}$ is $|\vec{\mathbf{A}}| \equiv A$. We can represent vectors as geometric objects using arrows. The length of the arrow corresponds to the magnitude of the vector. The arrow points in the direction of the vector



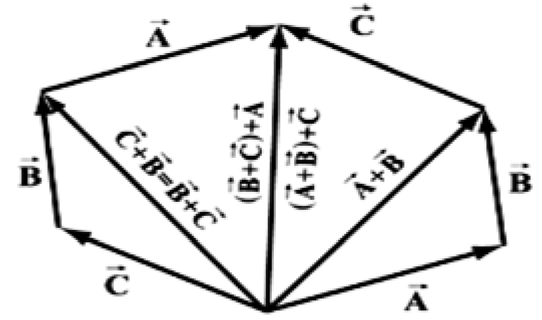
(1) Vector Addition:

Vectors can be added. Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors. We define a new vector, $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$, the "vector addition" of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, by a geometric construction. Draw the arrow that represents $\vec{\mathbf{A}}$. Place the tail of the arrow that represents $\vec{\mathbf{B}}$ at the tip of the arrow for $\vec{\mathbf{A}}$





(i) Commutivity:



The order of adding vectors does not matter;

$$\vec{\mathbf{A}}+\vec{\mathbf{B}}=\vec{\mathbf{B}}+\vec{\mathbf{A}}\,.$$

(ii) Associativity:

When adding three vectors, it doesn't matter which two you start with

$$(\vec{\mathbf{A}}+\vec{\mathbf{B}})+\vec{\mathbf{C}}=\vec{\mathbf{A}}+(\vec{\mathbf{B}}+\vec{\mathbf{C}})$$

(iii) Identity Element for Vector Addition:

There is a unique vector, $\vec{\mathbf{0}}$, that acts as an identity element for vector addition. For all vectors $\vec{\mathbf{A}}$,

$$\vec{\mathbf{A}}+\vec{\mathbf{0}}=\vec{\mathbf{0}}+\vec{\mathbf{A}}=\vec{\mathbf{A}}$$

(iv) Inverse Element for Vector Addition:

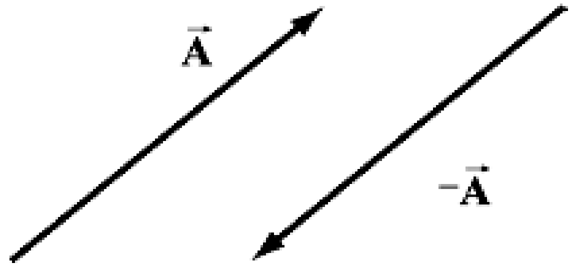
For every vector $\vec{\mathbf{A}}$, there is a unique inverse vector

$$(-1)\vec{\mathbf{A}}\equiv-\vec{\mathbf{A}}$$

such that

$$\vec{\mathbf{A}} + \left(-\vec{\mathbf{A}}\right) = \vec{\mathbf{0}}$$

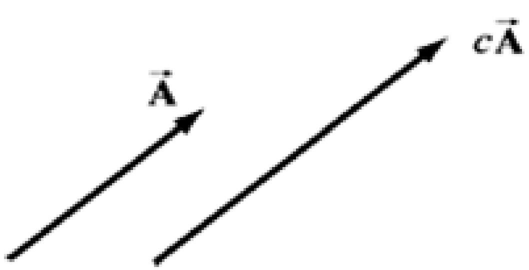
The vector $-\vec{\mathbf{A}}$ has the same magnitude as $\vec{\mathbf{A}}$, $|\vec{\mathbf{A}}| = |-\vec{\mathbf{A}}| = A$, but they point in opposite directions

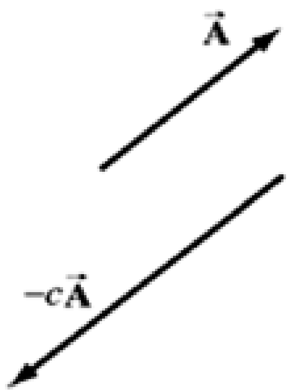


(2) Scalar Multiplication of Vectors:

$$cA = Ac$$

Since $c > 0$, the direction of $c\vec{\mathbf{A}}$ is the same as the direction of $\vec{\mathbf{A}}$. However, the direction of $-c\vec{\mathbf{A}}$ is opposite of $\vec{\mathbf{A}}$





Scalar multiplication of vectors satisfies the following properties:

(i) Associative Law for Scalar Multiplication:

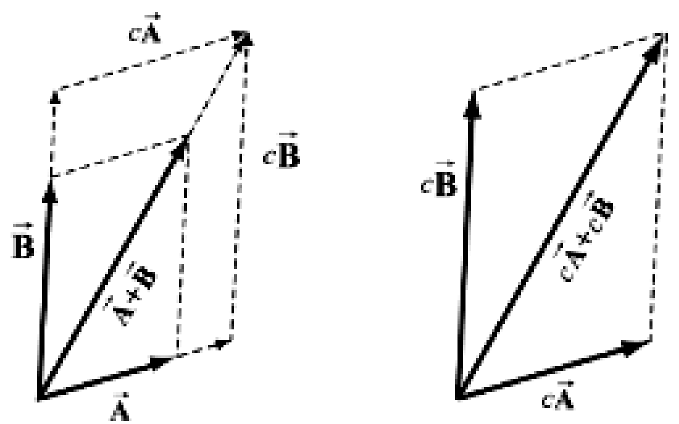
The order of multiplying numbers is doesn't matter. Let b and c be real numbers. Then

$$b(c\vec{\mathbf{A}}) = (bc)\vec{\mathbf{A}} = (cb\,\vec{\mathbf{A}}) = c(b\vec{\mathbf{A}})$$

(ii) Distributive Law for Vector Addition:

Vector addition satisfies a distributive law for multiplication by a number. Let c be a real number. Then

$$c(\vec{\mathbf{A}} + \vec{\mathbf{B}}) = c\,\vec{\mathbf{A}} + c\,\vec{\mathbf{B}}$$

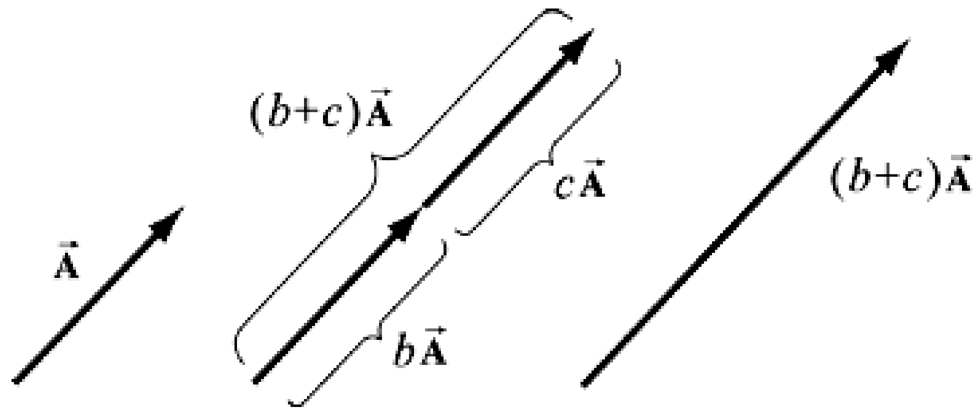


(iii) Distributive Law for Scalar Addition:

The multiplication operation also satisfies a distributive law for the addition of numbers. Let b and c be real numbers. Then

$$(b+c)\vec{\mathbf{A}} = b\vec{\mathbf{A}} + c\vec{\mathbf{A}}$$

Our geometric definition of vector addition satisfies this condition



(iv) Identity Element for Scalar Multiplication:

The number 1 acts as an identity element for multiplication,

$$1\,\vec{\mathbf{A}} = \vec{\mathbf{A}}$$

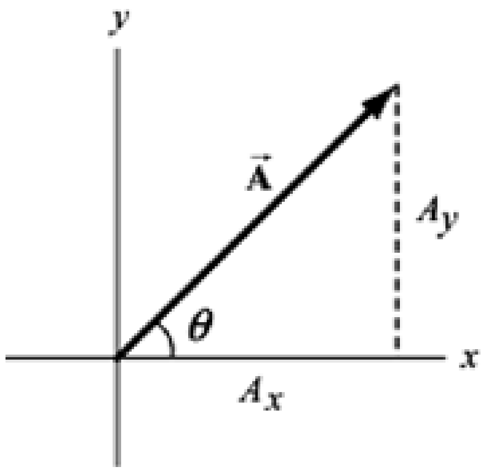
Application of Vectors

(1) Vectors can exist at any point P in space.

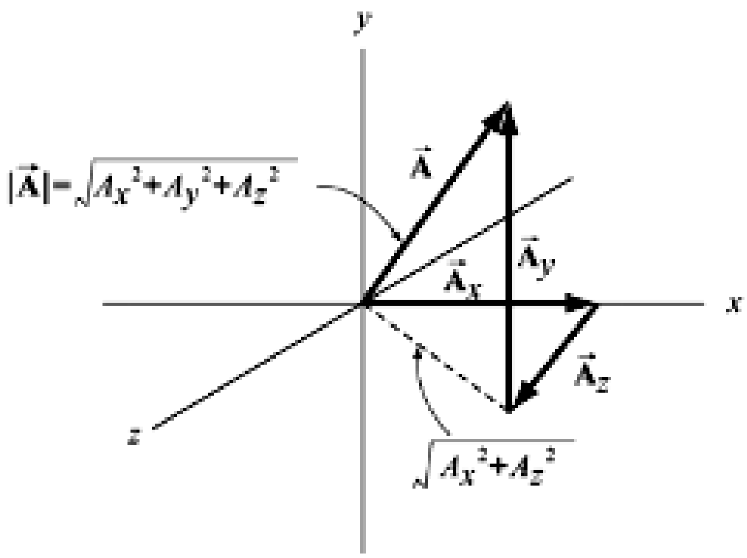
(2) Vectors have direction and magnitude.

(3) Vector Equality: Any two vectors that have the same direction and magnitude are equal no matter where in space they are located.

(4) Vector Decomposition: $\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y$



(5) Unit vectors: The idea of multiplication by real numbers allows us to define a set of unit vectors at each point in space. We associate to each point P in space, a set of three unit vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$. A unit vector means that the magnitude is one: $|\hat{\mathbf{i}}|=1$, $|\hat{\mathbf{j}}|=1$, and $|\hat{\mathbf{k}}|=1$. We assign the direction of $\hat{\mathbf{i}}$ to point in the direction of the increasing x-coordinate at the point P. We call $\hat{\mathbf{i}}$ the unit vector at P pointing in the $+x$-direction. Unit vectors $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ can be defined in a similar manner



(6) Vector Components: Once we have defined unit vectors, we can then define the x-component and y-component of a vector. Recall our vector decomposition, $\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y$. We can write the x-component vector, $\vec{\mathbf{A}}_x$, as

$$\vec{\mathbf{A}}_x = A_x\,\hat{\mathbf{i}}$$

$$\vec{\mathbf{A}}_y = A_y\,\hat{\mathbf{j}}, \quad \vec{\mathbf{A}}_z = A_z\,\hat{\mathbf{k}}$$

A vector $\vec{\mathbf{A}}$ can be represented by its three components $\vec{\mathbf{A}} = (A_x, A_y, A_z)$. We can also write the vector as

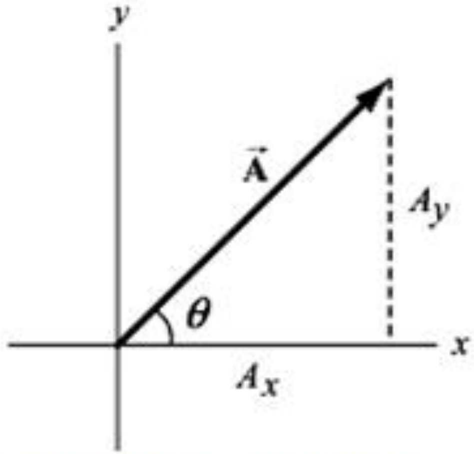
$$\vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}$$

(7) Magnitude:

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

(8) Direction:

$$A_x = A\cos(\theta), \quad A_y = A\sin(\theta)$$



We can now write a vector in the x-y plane as

$$\vec{\mathbf{A}} = A\cos(\theta)\ \hat{\mathbf{i}} + A\sin(\theta)\ \hat{\mathbf{j}}$$

Once the components of a vector are known, the tangent of the angle θ can be determined by

$$\frac{A_y}{A_x} = \frac{A\sin(\theta)}{A\cos(\theta)} = \tan(\theta),$$

that yields

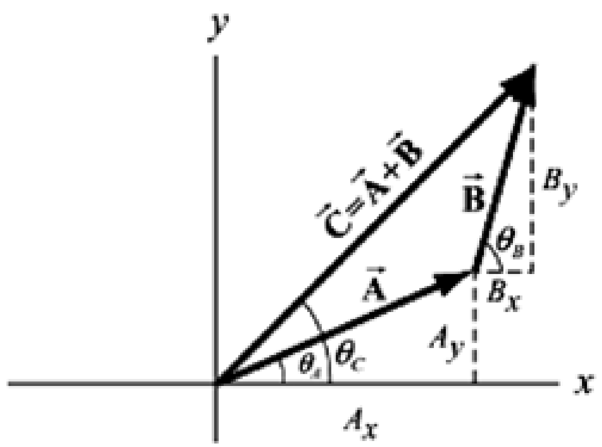
$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right).$$

(9) Vector Addition:

$$\vec{\mathbf{A}} = A\cos(\theta_A)\,\hat{\mathbf{i}} + A\sin(\theta_A)\,\hat{\mathbf{j}}$$

$$\vec{\mathbf{B}} = B\cos(\theta_B)\,\hat{\mathbf{i}} + B\sin(\theta_B)\,\hat{\mathbf{j}}$$

the vector addition $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ is shown. Let θ_C denote the angle that the vector $\vec{\mathbf{C}}$ makes with the positive x-axis.



Then the components of $\vec{\mathbf{C}}$ are

$$C_x = A_x + B_x, \quad C_y = A_y + B_y$$

In terms of magnitudes and angles, we have

$$C_x = C\cos(\theta_C) = A\cos(\theta_A) + B\cos(\theta_B)$$
$$C_y = C\sin(\theta_C) = A\sin(\theta_A) + B\sin(\theta_B)$$

We can write the vector $\vec{\mathbf{C}}$ as

$$\vec{\mathbf{C}} = (A_x + B_x)\,\hat{\mathbf{i}} + (A_y + B_y)\,\hat{\mathbf{j}} = C\cos(\theta_C)\,\hat{\mathbf{i}} + C\sin(\theta_C)\,\hat{\mathbf{j}}$$

Dot Product

We shall now introduce a new vector operation, called the "dot product" or "scalar product" that takes any two vectors and generates a scalar quantity (a number).

***Definition:* Dot Product**

The dot product $\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}$ of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined to be product of the magnitude of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with the cosine of the angle θ between the two vectors:

$$\vec{\mathbf{A}}\cdot\vec{\mathbf{B}} = AB\cos(\theta)$$

Where $A=|\vec{\mathbf{A}}|$ and $B=|\vec{\mathbf{B}}|$ represent the magnitude of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ respectively. The dot product can be positive, zero, or negative, depending on the value of $\cos\theta$. The dot product is always a scalar quantity.

Properties of Dot Product

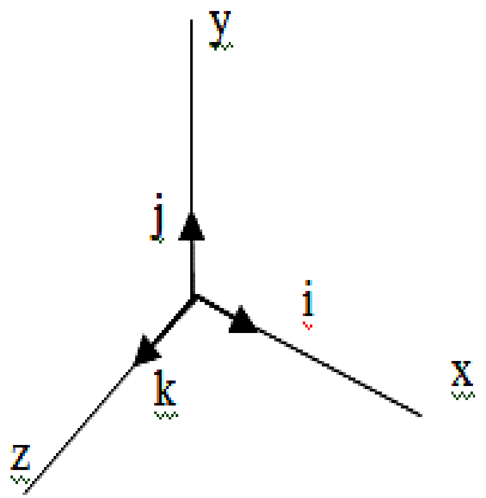
(1a) $$c\vec{\mathbf{A}}\cdot\vec{\mathbf{B}} = c(\vec{\mathbf{A}}\cdot\vec{\mathbf{B}})$$

(2a) $$(\vec{\mathbf{A}}+\vec{\mathbf{B}})\cdot\vec{\mathbf{C}} = \vec{\mathbf{A}}\cdot\vec{\mathbf{C}}+\vec{\mathbf{B}}\cdot\vec{\mathbf{C}}$$

$$\vec{\mathbf{A}}\cdot\vec{\mathbf{B}} = \vec{\mathbf{B}}\cdot\vec{\mathbf{A}}$$

(1b) $$\vec{\mathbf{A}}\cdot c\vec{\mathbf{B}} = c(\vec{\mathbf{A}}\cdot\vec{\mathbf{B}})$$

(2b) $$\vec{\mathbf{C}}\cdot(\vec{\mathbf{A}}+\vec{\mathbf{B}}) = \vec{\mathbf{C}}\cdot\vec{\mathbf{A}}+\vec{\mathbf{C}}\cdot\vec{\mathbf{B}}.$$



$\mathbf{i} \equiv$ a unit vector along the x-axis

$\mathbf{j} \equiv$ a unit vector along the y-axis

$\mathbf{k} \equiv$ a unit vector along the z-axis

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = |\hat{\mathbf{i}}||\hat{\mathbf{i}}| \cos(0) = 1$$

$$\hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = |\hat{\mathbf{i}}||\hat{\mathbf{j}}| \cos(\pi/2) = 0$$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0$$

The result for the dot product can be generalized easily for arbitrary vectors

$$\vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}$$

and

$$\vec{\mathbf{B}} = B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}} + B_z\,\hat{\mathbf{k}}$$

to yield

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_x B_x + A_y B_y + A_z B_z$$

$$\vec{A}.\vec{B} = (A_x i + A_y j + A_z k).(B_x i + B_y j + B_z k)$$

$$\begin{aligned}\vec{A}.\vec{B} = (&A_x i.B_x i + A_x i.B_y j + A_x i.B_z k \\ &+ A_y j B_x i + A_y j B_y j + A_y j.B_z k \\ &+ A_z k.B_x i + A_z k.B_y j + A_z k.B_z k)\end{aligned}$$

Therefore

$$\therefore \vec{A}.\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

The angle between the two vectors is

$$\cos\theta = \frac{\vec{A}.\vec{B}}{|A||B|} = \frac{A_x B_x + A_y B_y + A_z B_z}{|A||B|}$$

Cross Product

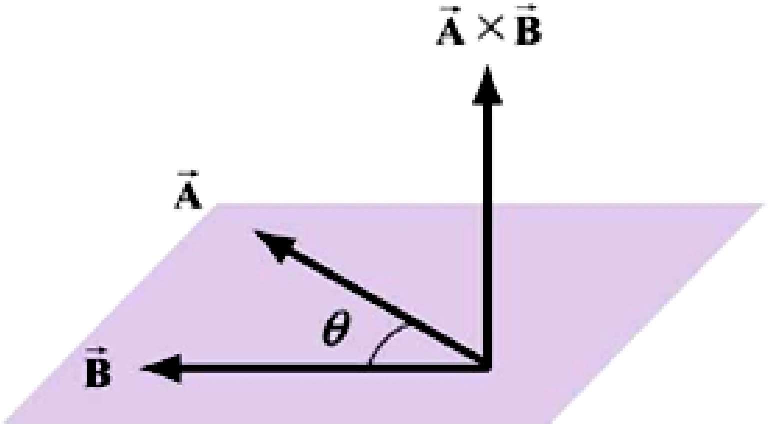
We shall now introduce our second vector operation, called the "cross product" that takes any two vectors and generates a new vector.

Definition: Cross Product

The magnitude of the cross product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined to be product of the magnitude of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with the sine of the angle θ between the two vectors,

$$\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = AB\sin(\theta),$$

where A and B denote the magnitudes of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, respectively. The angle θ between the vectors is limited to the values $0 \le \theta \le \pi$ insuring that $\sin(\theta) \ge 0$.



Properties of the Cross Product

(1) The cross product is anti-commutative since changing the order of the vectors cross product changes the direction of the cross product vector by the right hand rule:

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}}$$

(2) The cross product between a vector $c\vec{\mathbf{A}}$ where c is a scalar and a vector $\vec{\mathbf{B}}$ is

$$c\vec{\mathbf{A}} \times \vec{\mathbf{B}} = c(\vec{\mathbf{A}} \times \vec{\mathbf{B}})$$

Similarly,

$$\vec{\mathbf{A}} \times c\vec{\mathbf{B}} = c(\vec{\mathbf{A}} \times \vec{\mathbf{B}})$$

(3) The cross product between the sum of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with a vector $\vec{\mathbf{C}}$ is

$$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) \times \vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{C}} + \vec{\mathbf{B}} \times \vec{\mathbf{C}}$$

Similarly,

$$\vec{\mathbf{A}} \times (\vec{\mathbf{B}} + \vec{\mathbf{C}}) = \vec{\mathbf{A}} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \vec{\mathbf{C}}$$

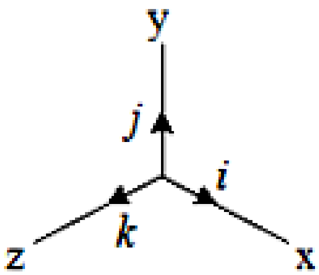
$$\vec{A}\times\vec{B} = AB\sin\theta$$

$$\vec{A}\times\vec{B} = (A_x i + A_y j + A_z k)\times(B_x i + B_y j + B_z k)$$

$$i\times i = 0 \qquad i\times j = k \qquad i\times k = -j$$

$$j\times j = 0 \qquad j\times k = i \qquad j\times i = -k$$

$$k\times k = 0 \qquad k\times i = j \qquad k\times j = -i$$



$$\vec{A}\times\vec{B} = (A_y B_z - A_z B_y)i + (A_z B_x - A_x B_z)j + (A_x B_y - A_y B_x)k$$

If $\vec{C} = \vec{A}\times\vec{B}$, the components of $\vec{C}$ are given by

$$C_x = A_y B_z - A_z B_y$$

$$C_y = A_z B_x - A_x B_z$$

$$C_z = A_x B_y - A_y B_x$$

Example

Two vectors are given by $\vec{A} = 3i - 2j$ and $\vec{B} = -i - 4j$. Calculate (a) $\vec{A} + \vec{B}$, (b) $\vec{A} - \vec{B}$, (c) $|\vec{A} + \vec{B}|$, (d) $|\vec{A} - \vec{B}|$, and (e) the direction of $\vec{A} + \vec{B}$ and $|\vec{A} - \vec{B}|$.

Solution

(a) $\vec{A} + \vec{B} = (3i - 2j) + (-i - 4j) = 2i - 6j$

(b) $\vec{A} - \vec{B} = (3i - 2j) - (-i - 4j) = 4i + 2j$

(c) $|\vec{A} + \vec{B}| = \sqrt{2^2 + (-6)^2} = 6.32$

(d) $|\vec{A} - \vec{B}| = \sqrt{4^2 + 2^2} = 4.47$

(e) For $\vec{A} + \vec{B}$, $\theta = \tan^{-1}(-6/2) = -71.6^\circ = 288^\circ$

For $\vec{A} - \vec{B}$, $\theta = \tan^{-1}(2/4) = 26.6^\circ$

Exercise Find the cross product of the following vectors:

(a) $\mathbf{j} \times \mathbf{k}$ (b) $\mathbf{i} \times 4\mathbf{i}$ (c) $(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \times (3\mathbf{j} + 2\mathbf{k})$

(d) $3\mathbf{j} \times 5\mathbf{i}$ (e) $(\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \times (2\mathbf{i} + \mathbf{j} - \mathbf{k})$

Exercise:

Find (a) the scalar projection of vector $\mathbf{a} = (2, 3, 1)$ in the direction of vector $\mathbf{b} = (5, -2, 2)$.

(b) the angle between $\mathbf{a}$ and $\mathbf{b}$.

(c) the vector projection of $\mathbf{a}$ in the direction of $\mathbf{b}$.

Question

If three vectors are given as

$$\vec{A} = i + j + k, \quad \vec{B} = 2i - j + 3k \quad and \quad \vec{C} = i + 2j - k,$$

then find the results of following equations given below:

a) $(\overrightarrow{A + B}).\vec{C}$ *and* $\vec{C}.(\overrightarrow{A + B})$

b) $\vec{A} \times (\vec{B} + \vec{C})$ *and* $(\vec{B} + \vec{C}) \times \vec{A}$

c) $(\vec{A} \times \vec{B}).\vec{C}$ *and* $\vec{A}.(\vec{B} \times \vec{C})$

d) $\vec{A} \times (\vec{B} \times \vec{C})$

Thank You