Chapter Three

Motions in Two Dimension

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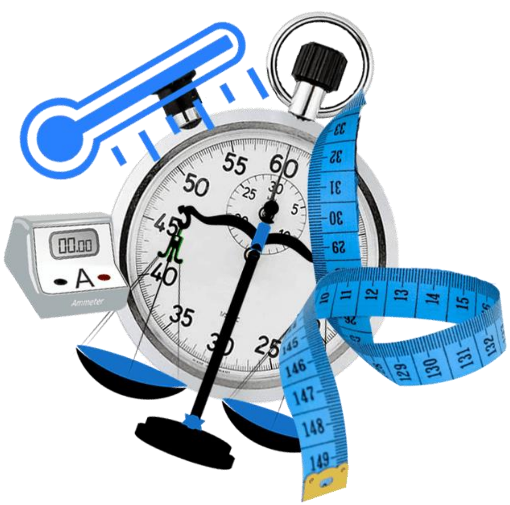

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- In two dimensions, it is necessary to use vector notation to describe physical quantities with both magnitude and direction.
- In this chapter, we will begin by defining displacement, velocity and acceleration as vectors in two dimensions.
- Motion in two dimensions like the motion of projectiles, satellites, and the motion of charged particles in electric fields. Here we shall treat the motion in plane with constant acceleration and uniform circular motion.

Motion in two dimensions with constant acceleration:

Assume that the magnitude and direction of the acceleration remain unchanged during the motion. The position vector for a particle moving in two dimensions (*xy* plane) can be written as:

$$\vec{r} = xi + yj$$

Where x, y, and r change with time as the particle moves. The velocity of the particle is given by

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j$$

$$\vec{v} = \frac{d\vec{r}}{dt} = v_x i + v_y j$$

Since the acceleration is constant then we can substitute

$$v_x = v_{0x} + a_x t \qquad v_y = v_{0y} + a_y t$$

This give

$$\vec{v} = (v_{0x} + a_x t)i + (v_{0y} + a_y t)j$$

$$\vec{v} = (v_{0x} i + v_{0y} j + \left(a_x i + a_y j\right)t$$

Then

$$v = v_0 + at$$

Since our particle moves in two dimension x and y with constant acceleration then

- $x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$
- $y = y_0 + v_{0y} t + \frac{1}{2} a_y t^2$

$$\vec{r} = xi + yj$$

$$\vec{r} = (x_0 + v_{0x}t + \frac{1}{2}a_x t^2)i + (y_0 + v_{0y}t + \frac{1}{2}a_y t^2)j$$

$$\vec{r} = (x_0 i + y_0 \mathrm{j}) + (v_{0x}i + v_{0y}j)t + \frac{1}{2}(a_x i + a_y j)t^2$$

$$r = r_0 + v_0 t + \frac{1}{2}at^2$$

Speed up or slow down

- If the velocity and acceleration components along a given axis have the *same sign* then they are in the same direction. In this case, the object will *speed up*.

- If the acceleration and velocity components have *opposite signs*, then they are in opposite directions. Under these conditions, the object will *slow down*.

Projectile Motion

A particle moves in a vertical plane with some initial velocity but its acceleration is always the free-fall acceleration *g*, which is downward. Such a particle is called a **projectile** and its motion is called **projectile motion**.

- The *horizontal and vertical motions* (at right angles to each other) *are independent*, and the path of such a motion can be found by combining its horizontal and vertical position components.

By Galileo

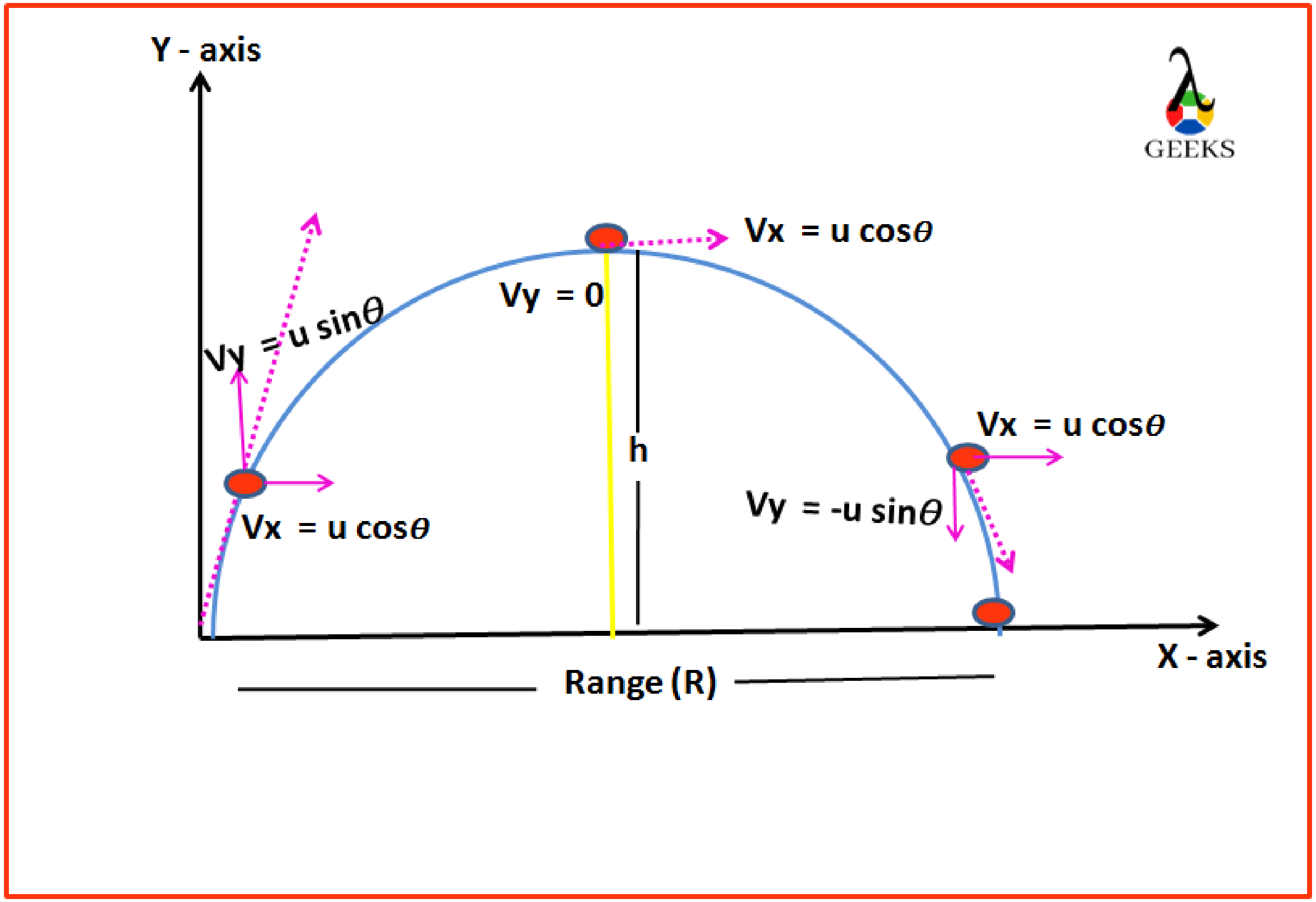
Projectile Motion

- A good example of the motion in two dimensions is the motion of projectile.

we will make the following assumptions:

- The only force present is the force due to gravity.
- The magnitude of the acceleration due to gravity is $|g| = g = 9.8m/s^2$. We choose a coordinate system in which the positive y-axis points up perpendicular to the earth's surface. This definition gives us that $\overrightarrow{a_y} = -g(\hat{j}) = -9.8m/s^2$ and $\overrightarrow{a_x} = 0$.
- The rotation of the earth does not affect the motion.

- To analyze this motion let's assume that at time t=0 the projectile start at the point x_o=y_o=0 with initial velocity v_o which makes an angle q_o, as shown in Figure.



Initial Conditions:

We choose the coordinate system so that the particle leaves the origin ($x_0 = 0, y_0 = 0$) at time $t_i = 0$ with an initial velocity of $\vec{v}_i$. The Procedure for Solving Projectile Motion Problems are as followings:

1. We will separate the motion into the x (horizontal) part and y (vertical) part.

2. Then we will consider each part separately using the appropriate equations. The equations of motion, for each component, become:

   a. x-motion (a_x =0);

$$\vec{v}_x = v_{0x}\,\hat{i}$$

$$\vec{v}_x = v_0\cos(\theta)\,\hat{i} \quad (const.)$$

$$\vec{x}(t) = v_{0x}t\,\hat{i}$$

$$\vec{x}(t) = v_0\cos(\theta)t\,\hat{i}$$

   b. y-motion ($\overrightarrow{a_y} = -g(\hat{j}) = -9.8 m/s^2$);

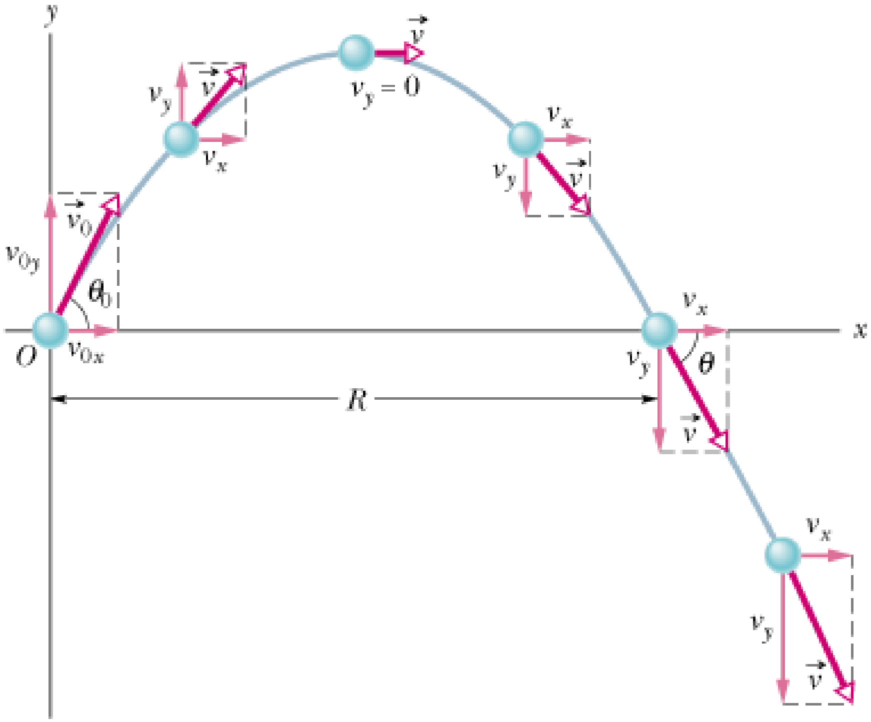
$$\vec{v}_y(t) = v_{0y}\hat{j} - gt\,\hat{j}$$

$$\vec{v}_y(t) = v_0\sin(\theta)\hat{j} - gt\,\hat{j}$$

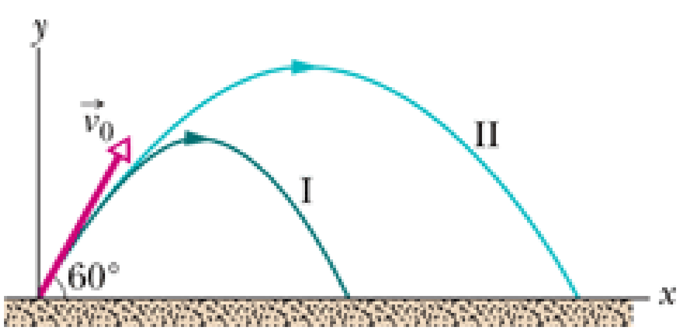
$$\vec{y}(t) = y_0\,\hat{j} + v_{0y}t\;\hat{j} - \frac{1}{2}gt^2\,\hat{j}$$

$$\vec{y}(t) = y_0\,\hat{j} + v_0\sin(\theta)t\;\hat{j} - \frac{1}{2}gt^2\,\hat{j}$$

Properties of *Projectile* Motion



$$v_{0x} = v_0 \cos\theta_0 \quad \text{and} \quad v_{0y} = v_0 \sin\theta_0 .$$



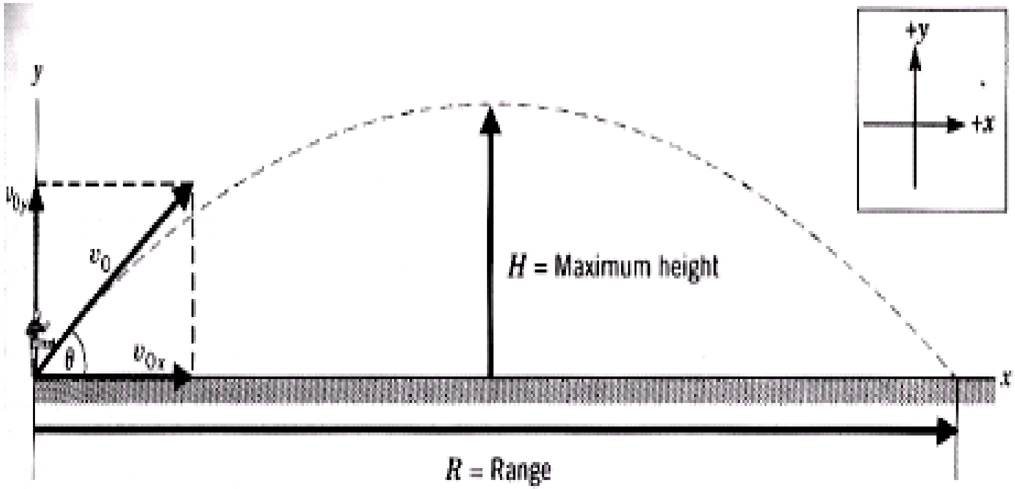
The Horizontal Motion:

- *no acceleration*
- velocity v_x remains unchanged from its initial value throughout the motion
- The horizontal range *R* is maximum for a launch angle of 45°

The vertical Motion:

- *Constant acceleration g*
- velocity v_y=0 at the highest point.

HORIZONTAL RANGE AND MAXIMUM HEIGHT OF A PROJECTILI



object is zero at a time "t" when it reaches at the highest point but the horizontal velocity has some magnitude, then we can write

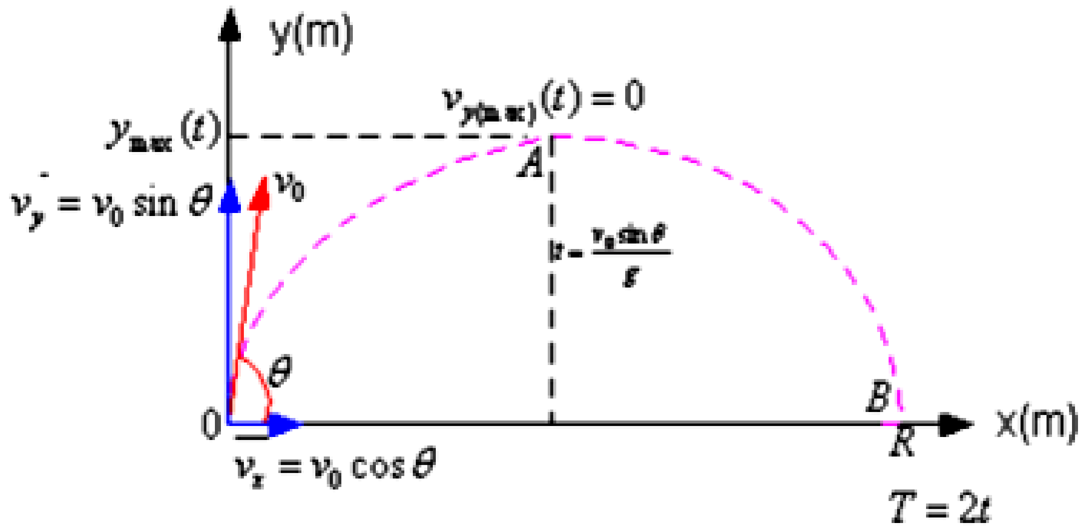
$$\vec{v}_y(t) = v_{0y}\hat{j} - gt\,\hat{j} \quad \Rightarrow \quad 0 = v_0 \sin(\theta)\hat{j} - gt\,\hat{j}$$

then

$$t = \frac{v_0 \sin(\theta)}{g}$$

So, the total time for the projectile to reach from *0* to *B* in a parabolic path is

$$T = 2t = \frac{2v_0 \sin(\theta)}{g}$$



To find the maximum height *h* (y_{max}) we use the equation

$$v_y^2(t) = v_{0y}^2 - 2gy(t)$$

$$R = \frac{v_0^2 \sin(2\theta)}{g}$$

It is obvious that $\theta = 45^\circ$ for the max range, $R_{\max}$.

$0 = v_{0y}^2 - 2gy(t)$, and

$\vec{v}_{0y} = v_0 \sin(\theta)\hat{j}$, then

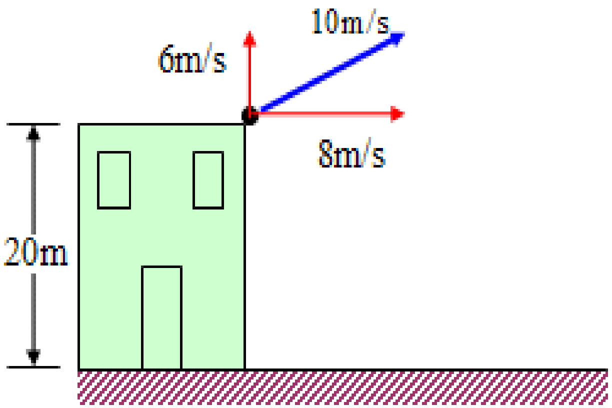
$0 = (v_0 \sin^2(\theta)) - 2gy_{\max}$, finally

$$y_{\max} = \frac{v_0^2 \sin^2(\theta)}{2g}$$

Example: Suppose that in the example above the object had been thrown upward at an angle of 37° to the horizontal with a velocity of 10m/s. Where would it land?

Solution:

Consider the vertical motion



v_{oy} = 6 m/s

a_y = -9.8m/s2

y = 20m

- To find the time of flight we can use

$y = v_{yo}\, t$ - 1/2 $g\, t^2$

- Since we take the top of the building is the origin that we substitute for

y = -20m

-20 = 6 t - 1/2 9.8 t^2

t = 2.73s

- Consider the horizontal motion

$v_x = v_{xo}$ = 8m/s

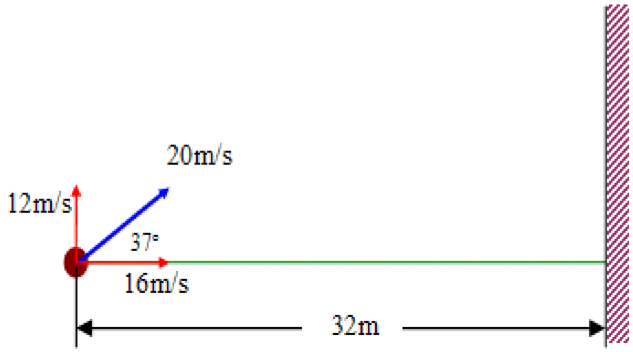
- Then the value of x is given by

$x = v_x\, t$ = 22m

Example:

In the figure shown below **where** will the ball hit the wall

Solution:



$v_x = v_{xo} = 16\text{m/s}$

$x = 32\text{m}$

- Then the time of flight is given by

$x = vt$

$t = 2\text{s}$

- To find the vertical height after 2s we use the relation

$y = v_{yo}\, t - 1/2\, g\, t^2$

Where

$v_{yo} = 12\text{m/s},\ t = 2\text{s}$

$y = 4.4\text{m}$

- Since y is positive value, therefore the ball hit the wall at 4.4m from the ground.
- To determine whether the ball is going up of down we estimate the velocity and from its direction we can know

$v_y = v_{yo} - gt$

$v_y = -7.6\text{m/s}$

- Since the final velocity is negative then the ball must be going down.

Example:

A bombardment aircraft having velocity of 180mi/h leaves its bomb with 30^0 angle downward in horizontal line. The horizontal distance between the point the bomb leaved and the point where it hits the ground is 701m.

a) Find the height at which the aircraft leaves the bomb and
b) Find the flight time of the bomb.

Solution:

We are given:

$v_0 = 180mi/h = 80.5m/s, \quad \theta = 30^0, \quad x_0 = 701m,$

The point where the bomb is leaved is assumed to be "0" point. Then

$$y_0 = 0, \quad y = y_0 + v_0 \sin(\theta)t + \frac{1}{2}at^2 \quad \Rightarrow \quad y = y_0\hat{j} - v_0 \sin(\theta)t\hat{j} - \frac{1}{2}gt^2\hat{j}$$

and

$$x_0 = v_0 \cos(\theta)ti \quad \Rightarrow \quad t = \frac{x_0}{v_0 \cos(\theta)}$$

If we replace "t" in the equation of height, then

$$-y\hat{j} = 0 - v_0 \sin(\theta)t\hat{j} - \frac{1}{2}gt^2\hat{j}, \quad then$$

$$-y\hat{j} = 0 - v_0 \sin(\theta)\left(\frac{x_0}{v_0 \cos(\theta)}\right)\hat{j} - \frac{1}{2}g\left(\frac{x_0}{v_0 \cos(\theta)}\right)^2\hat{j}$$

The solution of the is equation for the unknown, y, gives us that

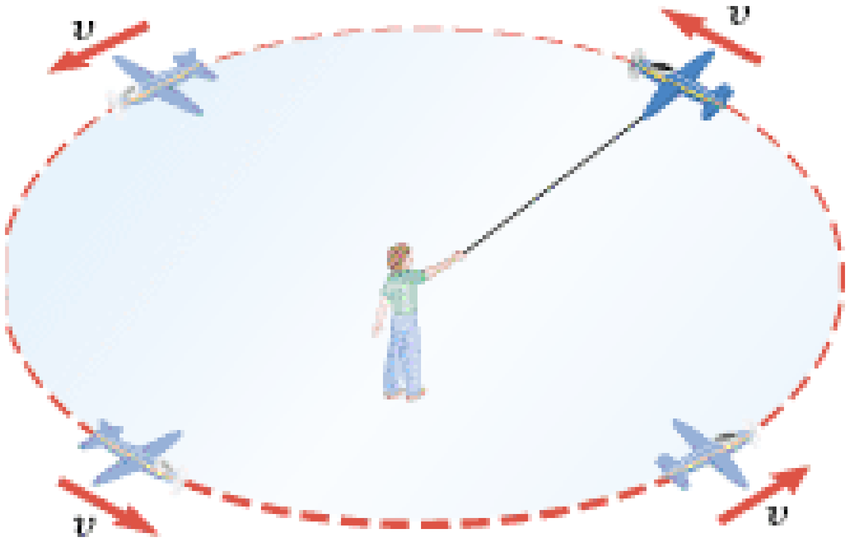
$y = -900\hat{j}$

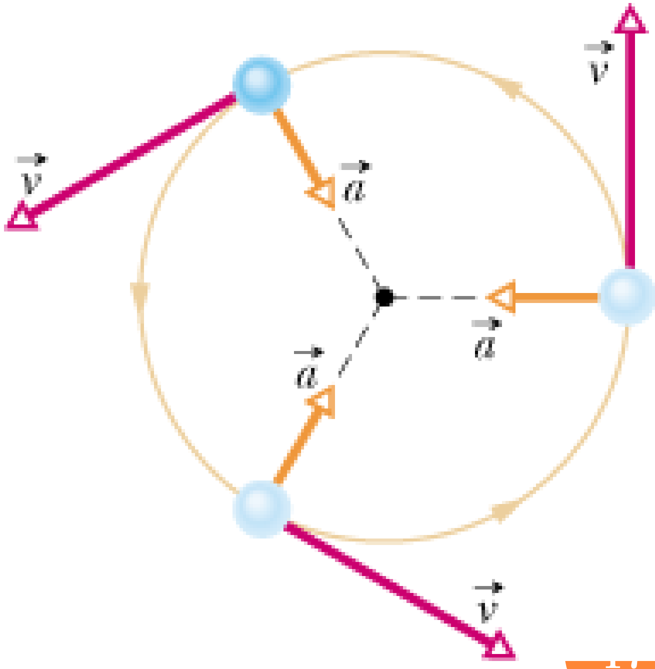
This means that the bomb goes downward vertically. To find the time for the bomb to reach the ground is

$$t = \frac{x_0}{v_0 \cos(\theta)} \quad \Rightarrow \quad t = \frac{701m}{80.5m/s\cos(30)} \quad \Rightarrow \quad t = 10.15s$$

UNIFORM CIRCULAR MOTION

Uniform circular motion is the motion of an object traveling at a constant (uniform) speed on a circular path





- In uniform circular motion an acceleration whose magnitude remains constant but direction changes from one position to another one is called "*centrifugal acceleration*". The difference between centripetal and centrifugal accelerations is quite simple – *centrifugal* acceleration do not exist while *centripetal* accelerations do. let's first examine the words *centripetal* and *centrifugal*.
- centri is derived from the Latin *centr* meaning "center."
- petal is derived from the Latin *petere* meaning "seek."
- fugal is derived from the Latin *fugere* meaning "to flee" as in fugitive.

PROPERTIES OF UNIFORM CIRCULAR MOTION

- **Period** of the motion T: is the time for a particle to go around a closed path exactly once has a special name.

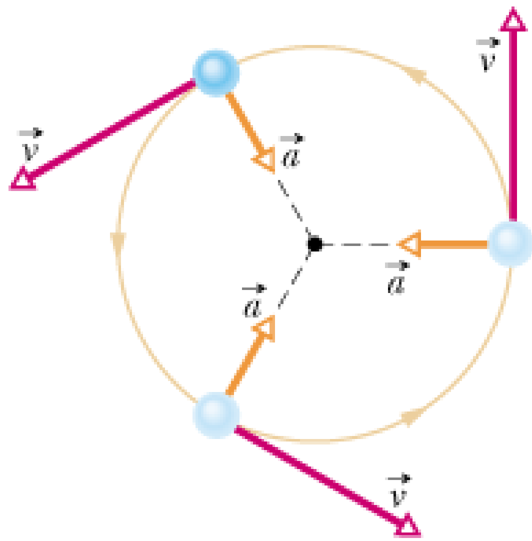
•Average speed is :

$$v = \langle v \rangle = \frac{\text{distance traveled}}{\text{time for the travel}} = \frac{2\pi r}{T} \qquad \text{(average speed)} .$$

•This number of revolutions in a given time is known as the **frequency**, f.

$$f = \frac{1}{T} \qquad \text{(frequency)} .$$

CENTRIPETAL ACCELERATION



The rate of change of this angular displacement with respect to time is given by

$$\vec{w} = \frac{\Delta\theta}{\Delta t}$$

and this change is called "**angular velocity**". For the limit condition;

$$\vec{w} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \quad \Rightarrow \quad \vec{w} = \dot{\theta}\,.$$

Since the angle changes from *0* to 2π in time *T*, then the angular velocity can be written as

$$w = \frac{2\pi}{T} .$$

where the T is the "***period***" of the particle that takes time for "*1 revolution*". So, the revolution for 1 second is written as

$$f = \frac{1}{T} .$$

This is the "***frequency***" of the particle orbiting around a center. Then the angular velocity can be written as

$$w = 2\pi f .$$

The linear velocity of the object is written as,

$$\overrightarrow{\Delta v} = \frac{\overrightarrow{\Delta s}}{\Delta t} \quad \Rightarrow \quad \overrightarrow{\Delta s} = \overrightarrow{\Delta \theta}\ r = \overrightarrow{\Delta w}\ \Delta t\ r \quad \Rightarrow \overrightarrow{\Delta v} = \frac{\overrightarrow{\Delta w}\ \Delta t\ r}{\Delta t} .$$

$$\overrightarrow{\Delta v} = r\ \overrightarrow{\Delta w}$$

and so that the relation between the angular velocity and the linear velocity is given by the equation

$$|v| = |wr| .$$

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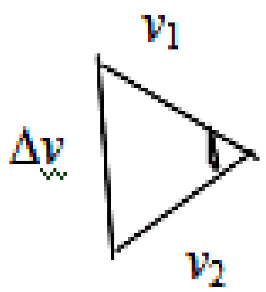

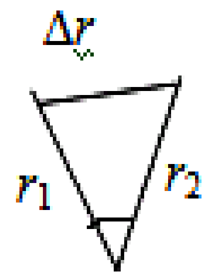

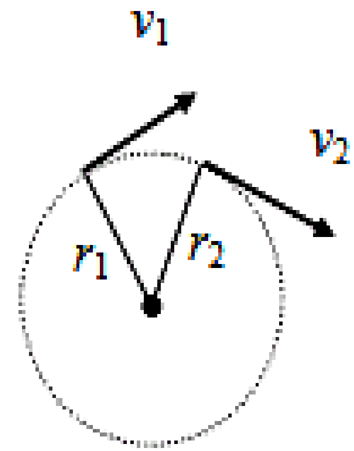

$$\frac{\Delta v}{\Delta r} = \frac{v}{r}$$

$$\Delta v = \frac{v}{r}\Delta r$$

Divide both sides by Δt

$$\frac{\Delta v}{\Delta t} = \frac{v}{r}\frac{\Delta r}{\Delta t}$$

$$a = \frac{v}{r}v = \frac{v^2}{r}$$

$$a_{\perp} = \frac{v^2}{r}$$

$$\vec{r}(t) = r_x\hat{i} + r_y\hat{j} = r\cos(\theta)\hat{i} + r\sin(\theta)\hat{j}$$

using the Equation

$$\vec{r}(t) = r\cos(wt)\hat{i} + r\sin(wt)\hat{j}$$

where i and j, with the little hats, are the unit vectors in the x and y-directions.

The object's velocity is easily found by taking the derivative of its location with respect to time:

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \frac{d}{dt}(r\cos(wt)\hat{i} + r\sin(wt)\hat{j})$$

$$\vec{v}(t) = -rw\sin(wt)\hat{i} + rw\cos(wt)\hat{j}$$

This velocity is always tangent to the circle or equivalently, $\vec{v}(t)$ is always perpendicular to position vector, ***r***, and $\vec{v}(t) \bullet \vec{r} = 0$

The object's acceleration is easily found by taking the derivative of its velocity with respect to time:

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d}{dt}(-rw\sin(wt)\hat{i} + rw\cos(wt)\hat{j})$$

$$\vec{a}(t) = -rw^2\cos(wt)\hat{i} - rw^2\sin(wt)\hat{j}$$

$$\vec{a}(t) = -w^2(\boldsymbol{r}\cos(wt)\hat{i} + r\sin(wt)\hat{j})$$

$$\vec{a}(t) = -w^2\,\vec{r}(t)$$

It is obviously seen that the direction of the object's acceleration $\vec{a}(t)$ is opposite $\vec{r}$, i.e., $\vec{a}(t)$ directed towards the center of motion.

Example: A particle moves in a circular path 0.4m in radius with constant speed. If the particle makes five revolutions in each second of its motion, find:

(a) the speed of the particle and

(b) its acceleration.

Solution:

(a) Since $r = 0.4\text{m}$, the particle travels a distance 0f $2\pi r = 2.51\text{m}$ in each revolution. Therefore, it travels a distance of 12.57m in each second (since it makes 5 rev. in the second).

$$v = 12.57\,m/1sec = 12.6\ m/s$$

$$\text{(b)}\ a_{\perp} = \frac{v^2}{r} = 12.6/0.4 = 395\text{m/s}^2$$

Example:

An object orbiting uniformly around a center, having radius of 1km, spreads 1^0 angle in 0.1s.

a. What is the linear velocity of this object?

b. What is the acceleration of the object?

c. Find the position vector in x and y components of that object for t=1s.

Solution:

We are given:

$$\Delta\theta = 1^0, \quad \Delta t = 0.1s, \quad r = 1km$$

Since the object displaces 1degree in 0.1second, then

$$w = \frac{\Delta\theta}{\Delta t} = \frac{1^0}{0.1} \qquad w = \frac{(1^0 * 2\pi / 360^0)}{0.1} = \frac{2\pi}{36} rad/\sec$$

Then the linear velocity

$$v = wr = 1km * \frac{2\pi}{36} rad/\sec$$

$$v = \frac{2000\pi}{36} = 174.45 m/s$$

The acceleration is

$$a = \frac{v^2}{r} = \frac{174.45^2}{1000} \quad \Rightarrow \quad a = 30.43 m/s^2$$

The position vector is given by

$$\vec{r}(t) = r\cos(wt)\hat{i} + r\sin(wt)\hat{j}$$

then

$$\vec{r}(t) = 1000\cos(\frac{2\pi}{36}t)\hat{i} + 1000\sin(\frac{2\pi}{36}t)\hat{j}.$$

For the 1second, the position vector components are:

$$\vec{r}(t=1) = 1000\cos(\frac{2\pi}{36} * 1)\hat{i} + 1000\sin(\frac{2\pi}{36} * 1)\hat{j}.$$

$$\vec{r}(t=1) = 984.81\hat{i} + 173.65\hat{j}$$

So,

x=984.81m and y=173.65m