**Using the Poisson queueing model to improve the queue in Periodic Vehicles Inspection PVI company in Erbil city**

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**Abstract**

This research defines the basic concepts of queueing theory and the Poisson queueing situation with multichannel homogeneous parallel servers. All parallel services with the same service rate, a waiting customer is selected from the queue to start service with the first free server.

The birth and death process used to analyze queueing systems, also the queue model derived in terms of steady state.

In this study we have dealt with the number of customers (cars) arrive at Periodic Vehicles Inspection company (PVI) according to Poisson process, the arrival items have been found to follow an exponential distribution and the number of cars that require repair with a certain probability following abnormal distribution.

In order to achive the objectives of this research, the data obtained from traffic directorate. From available data the probability of the number of customers in the system and the probability the number of customers will return to the system are defined also the expected waiting time of customers in the system and queue are determined.

**Keywords:** - compound of Poisson process and binomial distribution, birth and death process, The Poisson queue model with multi servers.

**1. Introduction**

Queueing theory is the mathematical study of waiting lines which are a part of everyday life in different fields, because as a process it has several Important functions.

It is generally considered a branch of operation research because results are often used when making business decisions about the resources needed to provide the service, so queues are an essential way of dealing with flow customers when there are limited resources.

Queueing models are used so that queue length and waiting time can be predicted and make decisions on determining the line queue in order to avoid crowded in the general places.

Most queueing model deals with system performance in steady state and assume that the system has been operating with the same arrival rate, average service time and other characteristics. This research consists of three sections, the first section involves the basic concepts of queueing system, the second section specify application includes describing and analyzing of data and the third section shows the results determined by the practical part.

**2. Methodology**

This section is dedicated for the basic concepts of queueing model, the Poisson and Binomial distribution, the birth and death process in queue theory and the Poisson queue model $M|M|s$.

**2.1 Queueing theory**

The queueing theory means arrival customers at system service channels asking for a certain service, when the system is busy or the service not available the customers will wait in queue until they take their service. The queueing systems are included awaiting line and service channels, for convenience they represented with three characteristics which are arrival distribution, service time distribution and number of servers. (HILLIER & LIEBERMAN, 2015)

**2.1.1 Characteristics (Kendall notation) of queueing models**

The queueing system is represented the following six main characteristics:

1. The arrival (or inter arrival time) distribution.
2. The departure (or service time) distribution.
3. Number of servers.
4. Queue discipline.
5. The maximum number of customers allowed in the System.
6. Size of the calling source.

**2.1.2 Measures of performance of queueing systems**

The most commonly used measures of performance in queueing situation are:

1. The expected number of customers in system $(Ls)$.
2. The expected number of customers in queue $(Lq)$.
3. The expected waiting time in system $(Ws)$.
4. The expected waiting time in queue $(Wq)$.
5. The expected number of busy servers $(ρ)$.
6. The utilization factor for $(s)$ servers $\left(\frac{ρ}{s}\right)$.

Where $λ,μ$ are arrival and departure rates and $ρ=\frac{λ}{μ}$.

The relationship between $Ls$ and $Ws$ (also $Lq$ and $Wq$) is known as little's formula, and is given as:

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| $$Ls=λWs$$ | (1) |
| $$Lq=λWq$$ |
| $$Ws=Wq+\frac{1}{μ}$$ |
| $$Ls=Lq+\frac{λ}{μ}$$ |

When the maximum number of customers allowed in the system has been restricted to $N$, thus the effective arrival rate $λ\_{eff}$ is used instead of $λ$.

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| $$λ\_{eff}=λ\left(1-P\_{N}\right)$$ | (2) |

For large $N$, the $λ\_{eff}$ is equivalent to $λ$. (Shamblin & Stevens, 1974) (Chakravarthy, et al., 2015)

**2.2 Poisson process and binomial distribution**

 Suppose that $N(t)$ be the number $(n)$ of occurrences of an event during $\left(0,t\right]$ is a Poisson process with mean $λt$. Suppose also that each occurrence of that event has a constant probability $(p)$ of being recorded independently of each other. So, the number $(k)$ of occurrences $Y(t)$, which are recorded follows a binomial distribution for a specific number of occurrences of $N(t)$, then (Medhi, 2004) (Haldar & Mahadevan, 2000) (Beichelt & Paul Fatti, 2002)

$$Pr\left(Y\left(t\right)=k\right)=\sum\_{n=k}^{\infty }p\left(Y\left(t\right)=k|N(t)=n\right).p(N(t)=n)$$

$$=\sum\_{n=k}^{\infty }∁\_{k}^{n}p^{k}\left(1-p\right)^{n-k}e^{-λt}\left(λt\right)^{n}/n!$$

$$=\frac{p^{k}e^{-λt}}{k!}\sum\_{n=k}^{\infty }\frac{\left(λt\right)^{n}\left(1-p\right)^{n-k}}{\left(n-k\right)!}$$

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| $$Pr\left(Y\left(t\right)=k\right)=\frac{\left(λpt\right)^{k}e^{-λpt}}{k!} , k=0,1,2,…$$ | (3) |

Which indicates that $Y(t)$ is also Poisson process with parameter $λp$

**2.3 The birth and death process in queue theory**

 It can be shown that the simple birth and death process represented with $(S)$ servers queueing system with Poisson arrivals with parameter $(λ)$ and exponential service time with parameter $(μ)$.

The probability that $\left(k\right)$ events occur between $t$ and $t+h$ given that $(n)$ events occurred at time $t$ can be defined for arrival and departure customers in queue theory as (U. Narayan Bhat, 2002)

$$Pr\left[number of arrivals between t and t+h is k, given that the number of customers in the system at time t is n\right]$$

$$p\left(n,t\right)=\left\{\begin{matrix}1-λ\_{n}h+o(h)&k=0\\λ\_{n}h+o(h)&k=1\\o(h)&k>1\end{matrix}\right.$$

**Pure birth**

The probability of $(n)$ customers in the system during $t $ time

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| $$P\_{n}\left(t\right)=\frac{(λt)^{n}e^{-λt}}{n!} ,n=0,1,2,…$$ | (4) |

Thus, the interarrival distribution is

$$f\left(t\right)=λe^{-λt} ,t>0$$

And

$$pr\left[number of departures between t and t+h is k, given that the number of customers in the system at time t is n\right]$$

$$q\left(n,t\right)=\left\{\begin{matrix}1-μ\_{n}h+o(h)&k=0\\μ\_{n}h+o(h)&k=1\\o(h)&k>1\end{matrix}\right.$$

Where

$$\lim\_{h\to 0}\frac{o(h)}{h}=0$$

**Pure death**

The probability of $n $ customers served after $t$ time is

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| $$p\_{n}\left(t\right)=\frac{\left(μt\right)^{N-n}e^{-μt}}{\left(N-n\right)!} , n=0,1,2,…,N$$ | (5) |

Thus, the service time distribution is (Medhi, 2004)

$$g\left(t\right)=μe^{-μt} ,t>0$$

**2.4 The Poisson queue model** $M|M|S$

 Let the arrivals occur in Poisson process with parameter $(λ)$, and the services times be independent and identically distributed random variable with an exponential distribution with mean $\left(\frac{1}{μ}\right)$. Let there be $(s)$ servers in the system working in parallel independently of each other. Arriving customers queue up in a single line in the order of their arrival. A server who is free takes the customer at the head of the queue for service, it should be noted that in this system all servers offer service approximately at the same rate, thus the number of customers are served per unit time $(μ\_{n})$ is defined as

$$μ\_{n}=\left\{\begin{matrix}nμ if&0\leq n<S\\sμ if&S\leq n\leq N\end{matrix}\right.$$

In order to define the probability of exactly $(n)$ customers in the system $P\_{n}$, the birth and death process assumptions are applied and the following equations must be solved (Taha, 2007)

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| $$P\_{0}\left(t+h\right)=P\_{0}\left(t\right)\left(1-λh\right)+P\_{1}\left(t\right)μh+0\left(h\right) ,n=0 $$ |  |
| $$P\_{0}^{'}\left(t\right)=-λP\_{0}\left(t\right)+μP\_{1}\left(t\right)$$ | (6) |
| $$P\_{n}\left(t+h\right)=P\_{n}\left(t\right)\left(1-\left(λ+nμ\right)h\right)+λhP\_{n-1}\left(t\right)+\left(n+1\right)μhP\_{n+1}\left(t\right)+0\left(h\right) $$$$ 0<n<s$$ |  |
| $$P\_{n}^{'}\left(t+h\right)=-\left(λ+nμ\right)P\_{n}\left(t\right)+λP\_{n-1}\left(t\right)+\left(n+1\right)μP\_{n-1}(t)$$ | (7) |
| $$P\_{n}\left(t+h\right)=P\_{n}\left(t\right)\left(1-\left(λ+sμ\right)h\right)+λhP\_{n-1}\left(t\right)+sμhP\_{n+1}\left(t\right)+0\left(h\right) $$$$ s\leq n<N$$ |  |
| $$P\_{n}^{'}\left(t\right)=-\left(λ+sμ\right)P\_{n}\left(t\right)+λP\_{n-1}\left(t\right)+sμP\_{n+1}(t)$$ | (8) |
| $$P\_{N}\left(t+h\right)=P\_{N}\left(t\right)\left(1-sμh\right)+λhP\_{N-1}\left(t\right)+0\left(h\right) ,n=N$$ |  |
| $$P\_{N}^{'}\left(t\right)=-sμP\_{N}\left(t\right)+λP\_{N-1}\left(t\right)$$ | (9) |

Allowing $t\rightarrow \infty $ and taking the steady state of equations, such that

$$\lim\_{t\to \infty }P\_{n}^{'}\left(t\right)=0 \& \lim\_{t\to \infty }P\_{n}\left(t\right)=P\_{n}$$

Then the equations (6, 7, 8 and 9) are become

$$λP\_{0}=μP\_{1} $$

$$\left(λ+nμ\right)P\_{n}=λP\_{n-1}+\left(n+1\right)μP\_{n+1}$$

$$\left(λ+sμ\right)P\_{n}=λP\_{n-1}+sμP\_{n+1}$$

$$λP\_{N-1}=sμP\_{n} $$

Thus, the limiting distribution of the number of customers in the system $\left(P\_{n}\right)$ is obtained:

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| $$P\_{n}=\left\{\begin{matrix}\frac{\left(\frac{λ}{μ}\right)^{n}}{n!}P\_{0}&0\leq n<s\\\frac{\left(\frac{λ}{μ}\right)^{n}}{s!s^{n-s}}P\_{0}&s\leq n\leq N\end{matrix}\right.$$ | (10) |
| Where  |  |
| $$P\_{0}=\left[\sum\_{n=0}^{s-1}\frac{\left(\frac{λ}{μ}\right)^{n}}{n!}+\sum\_{n=s}^{N}\frac{\left(\frac{λ}{μ}\right)^{n}}{s!s^{n-s}}\right]^{-1}$$ | (11) |

The expected number of customers in the system is

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| $$L\_{s}=\sum\_{n=0}^{N}np\_{n}$$ | (12) |
| and in the queue is |  |
| $$L\_{q}=\sum\_{n=s}^{N}(n-s)p\_{n}$$ | (13) |
| $$Waiting time=\frac{Number of customers in line}{Number of customers served per unit time}$$ | (14) |

For a single server (s=1) and N customers allowed in the system (above) model becomes $M|M|1$, then the probability of the number of customers in the system is given by

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| $$P\_{n}=\frac{1-ρ}{1-ρ^{N+1}}ρ^{n} 0\leq n\leq N$$ | (15) |
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| $$E\left(n\right)=L\_{s}=\frac{ρ[1-\left(N+1\right)ρ^{N}+Nρ^{N+1}]}{(1-ρ)(1-ρ^{N+1})} ,where ρ=\frac{λ}{μ}$$ | (16) |
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# **3. Application**

This section dedicated for application, which includes the computations of the probability $(P) $of the failure car components (Tires, Lights, Brake and Engine), the probability of the number of customers (Cars) arriving at periodic vehicles inspection (PVI) company, the probability of the number of cars that require repair and the measures of performance of queueing system are defined.

## **3-1 Data Description**

The data about the failure cars components as (Tires, Lights, Brake and Engine), which are inspected during 11 years (2011-2020) by PVI company obtained from the traffic directorate of Erbil city.

The PVI company consists of three fields each one has 6-8 parallel channels (servers), They offer equal services independently of each other, with a single special channel is assigned for cars that require repair, whenever a component part fails the customer (car) will return again to the system after their failure parts are repaired, figure (3.1). The customers arrive at the system according to poisson process with mean 55 customer per hour (0.928 customer per minute), and the service time follows an exponential distribution with mean 12 minutes, The mean service rate of any server is 5 per hour, then the mean service rate of the system with 8 servers is 40 per hour (0.67 customer per minute).



Figure 3.1 Showing the PVI System working

So, during 7 hours (8AM – 3PM, but at 2PM the entering customers to the system are no allowable) working daily approximately 300 cars are inspected at each field of the company and the failures are detected, *table (3-1).*

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| **year** | $$T\_{i}$$ | **No. of Cars** | **Tire** | **Light** | **Break** | **Engine** |
| **failure** | $$F(t)$$ | **failure** | $$F(t)$$ | **failure** | $$F(t)$$ | **failure** | $$F(t)$$ |
| **2010** | 1 | 171394 | 685 | 0.004 | 1199 | 0.007 | 1131 | 0.007 | 1268 | 0.007 |
| **2011** | 2 | 199632 | 1597 | 0.008 | 2795 | 0.014 | 2635 | 0.013 | 2954 | 0.015 |
| **2012** | 3 | 222057 | 2665 | 0.012 | 4663 | 0.021 | 4397 | 0.020 | 4930 | 0.022 |
| **2013** | 4 | 243576 | 3897 | 0.016 | 6820 | 0.028 | 6430 | 0.026 | 7210 | 0.030 |
| **2014** | 5 | 289045 | 5781 | 0.020 | 10117 | 0.035 | 9538 | 0.033 | 10695 | 0.037 |
| **2015** | 6 | 288412 | 6922 | 0.024 | 12113 | 0.042 | 11421 | 0.040 | 12805 | 0.044 |
| **2016** | 7 | 304960 | 8539 | 0.028 | 14943 | 0.049 | 14089 | 0.046 | 15797 | 0.052 |
| **2017** | 8 | 272490 | 8720 | 0.032 | 15259 | 0.056 | 14387 | 0.053 | 16131 | 0.059 |
| **2018** | 9 | 278413 | 10023 | 0.036 | 17540 | 0.063 | 16538 | 0.059 | 18542 | 0.067 |
| **2019** | 10 | 347003 | 13880 | 0.040 | 24290 | 0.070 | 22902 | 0.066 | 25675 | 0.074 |
| **2020** | 11 | 250828 | 11477 | 0.046 | 19433 | 0.077 | 18277 | 0.073 | 20700 | 0.083 |
| **Probability of****Failure components** | $$P\_{Tire}=0.024$$ | $$P\_{Light}=0.042$$ | $$P\_{Break}=0.0396$$ | $$P\_{Engine}=0.044$$ |

Table 3. The probability of failure (Tire, Light, Break, Engine) components

Where

$$P=\frac{\sum\_{i=1}^{no. years}F(t\_{i})}{no. years}$$

**3.2** The probability $P\_{n}$ of (n=10, 20, 30, 40, 50, 55) cars being in the system are determined by (10), Where the probability of $P\_{o}=1.62E-11$, founded by (11).

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| --- | --- |
| $$n$$ | $$P\_{n}t$$ |
| 10 | 1.63E-07 |
| 20 | 3.94E-06 |
| 30 | 9.51E-05 |
| 40 | 2.30E-03 |
| 50 | 0.0555 |
| 55 | 0.2727 |

Table 3. The probability of the number of cars being in the system

**3.3** The probability of the number of cars that require repair their failure components (Tire, Light, Break and Engine) are calculated by (3).

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| $$P(Y\_{t}=K)$$ | **0** | **5** | **10** | **15** | **20** | **25** | **30** |
| **Tire** | 0.2671 | 0.0089 | 1.18E-06 | 1.31E-11 | 2.83E-17 | 1.78E-23 | 4.17E-30 |
| **Break** | 0.1133 | 0.0463 | 7.50E-05 | 1.02E-08 | 2.69E-13 | 2.06E-18 | 5.92E-24 |
| **Light** | 0.0993 | 0.0544 | 1.18E-04 | 2.16E-08 | 7.64E-13 | 7.88E-18 | 3.03E-23 |
| **Engine** | 0.0889 | 0.0615 | 1.69E-04 | 3.89E-08 | 1.73E-12 | 2.26E-17 | 1.10E-22 |

Table 3. The probability of the number of cars that require repair their failure components

**3.4** The relationships between $(Ls, Lq, Ws, Wq)$ are founded by (1), and the waiting time in queue of times $(t=1, 2, …,7)$ hours are defined by (14).

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| **Time(hour)** | **No. of customers in queue** | $$Wq$$ | $$Lq$$ | $$Ls$$ | $$Ws$$ |
| 1 | 15 | 0.38 | 20.63 | 31.63 | 0.58 |
| 2 | 30 | 0.75 | 41.25 | 52.25 | 0.95 |
| 3 | 45 | 1.13 | 61.88 | 72.88 | 1.33 |
| 4 | 60 | 1.50 | 82.50 | 93.50 | 1.70 |
| 5 | 75 | 1.88 | 103.13 | 114.13 | 2.08 |
| 6 | 90 | 2.25 | 123.75 | 134.75 | 2.45 |
| 7 | 50 | 1.25 | 68.75 | 79.75 | 1.45 |

Table 3. The relationships between measures of performance of queueing system

# **4. Conclusions**

According to the results of application the following conclusions are found:

1. The number of failure components are increased annually, except in 2020, since the number of arrived customers to the system had lessen, because of covid-19, *table (3.1)*
2. The successive probabilities of the number of customers in the system are increased, *table (3.2)*
3. The successive probabilities of the number of cars that require repair their failures are decreased, *table (3.3)*
4. The number of customers in queue are increased with time, so the measures of performance of the queueing system are increased too, *table (3.4)*

# **Reference**

Beichelt, F. E. & Paul Fatti, L., 2002. ***Stochastic Processes and Their Applications****.* USA: CRC Press.

Chakravarthy, S. R., Kavi, K. M. & Yu, A. J., 2015. ***An Introduction to Queueing Theory Modeling and Analysis in Application****s.* 2nd ed. New York: Springer Science+Business Media.

Haldar, A. & Mahadevan, S., 2000. ***Reliability Assessment Using Stochastic Finite Element Analysis****.* New York: Simultaneously.

HILLIER, F. S. & LIEBERMAN, G. J., 2015. ***Introduction to Operations Research****.* 10th ed. New York: McGraw-Hill Education.

Medhi, J., 2004. ***Stochastic Processes****.* 2nd ed. New Delhi: New Age International.

Shamblin, J. E. & Stevens, G. E., 1974. ***Operation Research a Fundamental Approach****.* USA: McGraw-Hill.

Taha, H., 2007. ***Operation Research****.* s.l.:Prentice Hall.

U. Narayan Bhat, G. K. M., 2002. ***Elements of Applied Stochastic Processes****.* 3rd ed. s.l.:Wiley-Interscience.

**پوختە**

 لەم توێژینەوەیەدا چەمکە بنەڕەتییەکانی تیۆری نۆرەگرتن و دۆخی نۆرەگرتنی پۆیسۆن لەگەڵ فرە کەناڵە هاو تەریبەکان دەناسێنێت. وە هەموو کەنالە هاوتەریبەکان هەمان ئەگەری ڕێژەی خزمەتگوزاریان هەیە کە پێشکەش بە کڕیارەکانی(customers) بکات، نۆرەگرتن و چاوەڕوانی هەر کڕیارێک(customer) کاتێک کۆتای دێت کە لە نزیکی کەنالێکی بەتاڵ هەبێت و هەلی ببژێرێت بۆ پێشکەش کردنی خزمەتگوزاری.

 ئێمە پرۆسەی لەدایکبوون و مردنمان بەکارهێناوە بۆ شیکردنەوەی سیستەمی نۆرەگرتن ، هەروەها مۆدێلی نۆرەگرتن وەرگیراوە لە دۆخی جێگیری.

 لەم لێکۆڵینەوەیەماندا مامەڵەمان لەگەڵ ژمارەی کڕیاران(customers) (ئۆتۆمبێلەکان) کردووە کە دەگەنە کۆمپانیای پشکنینی ئۆتۆمبێلی (PVI) بەپێی پرۆسەی پۆیسۆن، دەرکەوتووە کە گەیشتنی کڕیارەکان(customer) بەدوای دابەشکردنێکی ڕێژەیی، وە ژمارەی ئەو ئۆتۆمبێلانەی کە پێویستیان بە چاککردنەوە هەیە بە دیاریکراوی.

 وە بە مەبەستی گەیشتن بە ئامانجەکانی ئەم توێژینەوەیە و ئەو زانیاریانەی لە بەڕێوەبەرایەتی هاتووچۆوە وەرگیراون. لە زانیارییە بەردەستەکانەوە ئەگەری ژمارەی کڕیاران (customer)لە سیستەمەکەدا و ئەگەری ژمارەی گەڕانەوەی کڕیارەکان(customer) بۆ سیستەمەکە پێناسە دەکات هەروەها کاتی چاوەڕوانی چاوەڕوانکراوی کڕیاران(customer) لە سیستەمەکەدا و نزیکی دیاری دەکات.

**وشەی سەرەکی**: - پێکهاتەی پرۆسەی پۆیسۆن و دابەشکردنی باینۆمیال ، پرۆسەی لەدایکبوون و مردن، مۆدێلی نزیکی پۆیسۆن بە فرەسێرڤەر.

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**الملخص**

 يعرّف هذا البحث المفاهيم الأساسية لنظرية الطابور وحالة الطابور في بواسون مع خوادم متوازية متعددة القنوات ومتجانسة جميع الخدمات الموازية بنفس معدل الخدمة ، يتم اختيار العميل المنتظر من قائمة الانتظار لبدء الخدمة مع أول خادم مجاني عملية الولادة والوفاة المستخدمة لتحليل أنظمة الطابور ، وكذلك نموذج قائمة الانتظار المشتق من حيث الحالة المستقرة.

 في هذه الدراسة تعاملنا مع عدد العملاء (السيارات) الذين يصلون إلى شركة الفحص الدوري للمركبات (PVI) وفقًا لعملية بواسون ، وقد تبين أن عناصر الوصول تتبع توزيعًا أسيًا وعدد السيارات التي تتطلب إصلاحًا بدرجة معينة الاحتمال بعد التوزيع غير الطبيعي.

ولتحقيق أهداف هذا البحث تم الحصول على البيانات من مديرية المرور. من البيانات المتاحة ، يتم تحديد احتمال عدد العملاء في النظام واحتمال عودة عدد العملاء إلى النظام ، كما يتم تحديد وقت الانتظار المتوقع للعملاء في النظام وقائمة الانتظار.

الكلمة الرئيسية: - مركب عملية بواسون والتوزيع ذي الحدين ، عملية الولادة والوفاة ، نموذج قائمة انتظار بواسون بخوادم متعددة.