Probability Models

Statistics and Informatics Department

Second course

Fourth Stage

Chapter 2

The Poisson Process

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Siméon Denis Poisson

- Born: 6/21/1781- Pithiviers, France
- Died: 4/25/1840-Sceaux, France

"Life is good for only two things: Discovering mathematics and teaching mathematics."

Poisson distribution is widely used in cases where chance of any individual event being success is small and the number of trials tends to be infinite. This distribution is used to describe the rare events. Some examples of Poisson distribution are:

1. The number of blinds born in a town in a particular year.
2. Number of mistakes committed in a typed page.
3. The number of students scoring very high marks in all subjects
4. The number of plane accidents in a particular week.
5. The number of defective screws in a box of 100, manufactured by a reputed company.
6. Number of suicides reported in a particular day.

There are kinds of mathematical equations that have the Poisson distribution

Poisson process , Poisson equation, Poisson kernel, Poisson bracket, Poisson algebra, Poisson regression, Poisson summation formula, Poisson's spot, Poisson's ratio, Poisson zeros

Counting process

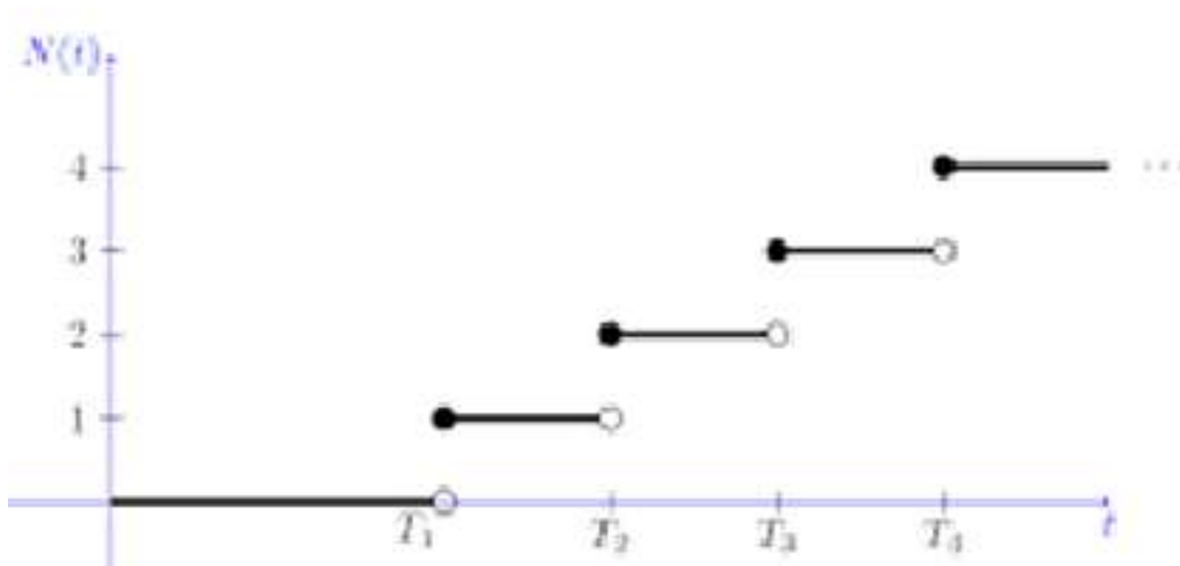
A stochastic process {N(t), t ≥ 0} is a counting process if it represents the number of events that occur up to time t.

Example

If N(t) is the number of customers who have entered McDonald's for service at prior to time t

{N(t), t ≥ 0} is a counting process in which the event corresponds to the number of customers entering McDonald's.

However, if N(t) represents the number of customers in McDonald's at time t, then {N(t), t ≥ 0} is not a counting process.



A counting process $\{N(t), t \geq 0\}$ must satisfy(properties):

1. **$N(t) \geq 0$.**
2. **$N(t)$ is integer-valued.**
3. **$N(s) \leq N(t)$ for any $s < t$, i.e. it must be non-decreasing.**
4. **For $s < t$, $N(t) - N(s)$ is the number of events that have occurred in the interval $(s, t]$.**

The Poisson process

- **The Poisson process is a counting process for the number of events that have occurred up to a particular time.**

Sometimes called a "jump" process because it jumps up to a higher state each time an event occurs. It is also a special case of a continuous Markov process.

Example 1.Number of arrivals to a system at time t
2. Number of customers in queue at time t
3. Number of customers in system at time t
4. Number of busy servers at time t

A Poisson process with parameter or rate λ> 0 is an integer-valued, continuous time stochastic process {N(t),t ≥ 0}

Properties

1. **N(0) = 0;**
2. **It has independent increments**
3. **The number of events in any interval of length t has a Poisson distribution with mean λt, for all s, t ≥ 0,**
4. **The probability of more than one event is negligible. $P((N(t + h) - N(t)) > 1) = o(h)$**
5. **The probability of one event in a small interval is $P((N(t+h) - N(t)) = 1) = \lambda h + o(h)$.**

$$6- \quad P(N(t+h)-N(t)=n)=\frac{e^{-\lambda h}(\lambda h)^{n}}{n!} \quad n=0,1,2\ldots$$

Prove property (4) $P((N(t+h)-N(t))>1) = o(h)$

Prove Property (5) $P((N(t+h)-N(t))=1) = \lambda h + o(h)$.

Example 1:

Consider a 1-hour traffic volume of 120 vehicles, during which the analyst is interested in obtaining the distribution of 1-minute volume counts

- What is the probability of more than 6 cars arriving (in 1-min interval)?

- What is the probability of between 1 and 3 cars arriving (1-min interval)?

H. W

A conversation in a wireless ad-hoc network is severely disturbed by interference signals according to a Poisson process of rate λ =0.1 per minute.

(a) What is the probability that no interference signals occur within the first two minutes of the conversation?

(b) What is the probability at least two interference signals occur within the first two minutes of the conversation?

Notes

Inequality symbols			
$<$	$>$	$\leq$	$\geq$
•less than •fewer than	•greater than •more than •exceeds	•less than or equal to •no more than •at most	•greater than or equal to •no less than •at least

	POISSON	BERNOULLI
Times of Arrival	Continuous	Discrete
Arrival Rate	λ/unit time	p/per trial
PMF of # of Arrivals	Poisson	Binomial
Interarrival Time Distr.	Exponential	Geometric
Time to k-th arrival	Erlang	Pascal

Example 2

The number of customers arriving at a grocery store can be modeled by a Poisson process with intensity λ=10 customers per hour.

Find the probability that there are 2 customers between 10:00 and 10:20.

P[N($t = \frac{1}{3}$) =2] =?

Find the probability that there are 3 customers between 10:00 and 10:20 and 7 customers between 10:00 and 11.

Time between Arrivals

A. Relation of Poisson and exponential distribution:

F(t) = P(T ≤ t) = 1- P(T > t) = 1- P(X=0)

The probability ta we observe the first arrival after time t is the same as the probability that observe no arrival from now until time t. but X is Poisson with parameter λ which has parameter λt over the time interval (0,t). We compute the above using:

To find the pdf pf T we take the derivative of the cdf w.r.t.(with respect to) get:

$$F(t) = 1 - \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = F(t) = 1 - e^{-\lambda t}$$

To find the pdf pf T we take the derivative of the cdf w.r.t t to get:

$$F(t) = F(t)' = \lambda e^{-\lambda t}.$$

We observe that if X ~ Poisson (λ) the time until the first arrival is exponential with parameter λ.

Example3 /Suppose that an average of 20 customers per hour arrive at a shop according to a Poisson process ($\lambda = 1/3$ per minute).

What is the probability that the shopkeeper will wait more than 5 minutes before the first customer arrives?

$P(T_1 \geq 5) = P(N(t) = 0) = e^{-\frac{1}{3}*5} = 0.188$

- H.w/ If two customers

B. Relation of Poisson and gamma distribution

$$F(t) = P(T \leq t) = 1 - P(T > t) = 1 - P(X < k) = 1 - P(X \leq k-1)$$

In words: The probability that we observe the k_{th} arrival after time t is the same as the probability that we observe less that k arrivals from now until time t. But X is Poisson with parameter λ which has parameter λt over the time interval $(0, t)$. We compute the above using:

$$F(t) = 1 - P(X \leq k-1) \;=\; 1 - \sum_{x=0}^{k-1} \frac{(\lambda t)^x e^{-\lambda t}}{x!} = 1 - e^{-\lambda t} \sum_{x=0}^{k-1} \frac{(\lambda t)^x}{x!}$$

To find the pdf pf T we take the derivative of the cdf w.r.t. t to get:

$$f(t) \;=\; F(t)'$$

$$f(t) = e^{-\lambda t} \lambda \frac{(\lambda t)^{k-1}}{(k-1)!} = \frac{t^{k-1} \lambda^k e^{-\lambda t}}{(k-1)!}$$

Example 4

Students ask questions in class according to Poisson process with the average rate of one question per 10 minutes.

Find the probability that the third question is asked during the last 15 minutes of a 50-minute class

Using only the Gamma Distribution's integral definition we have:

Note
λ= average number of events in unit of time.
$1/\lambda$= average time until an event occurs.

$\lambda = \frac{1}{10} = 0.10$

$$P(35<T_3<50) = \int_{35}^{50} \frac{0.10^3}{\Gamma 3} t^{3-1} e^{-0.1t}\, dt$$

$$= \frac{1}{2000} \int_{35}^{50} t^2\, e^{-0.1t}\, dt \quad = 0.196$$

OR solution by relation (Poisson and Gamma)

$$\begin{aligned} P(35 < T_3 < 50) &= P(T_3 > 35) - P(T_3 > 50) \\ &= P(X_{35} \le 2) - P(X_{50} \le 2) \\ &= P\left(Poisson\left(35 \cdot \frac{1}{10}\right) \le 2\right) - P\left(Poisson\left(50 \cdot \frac{1}{10}\right) \le 2\right) \\ &= P(Poisson(3.5) \le 2) - P(Poisson(5.0) \le 2) \\ &= 0.3208 - 0.1247 \\ &= 0.1961 \end{aligned}$$

H.W

You call the IRS hotline and you are told that you are the 56th person in line, excluding the person currently being served. Callers depart according to a Poisson process with a rate of λ = 2 per minute.

A-How long will you have to wait on the average until your service starts

E[T] = 56/λ = 28

B- What is the probability you will have to wait for more than an hour?

P(T>60)

Example 5 **Suppose that people immigrant to particular territory at a Poisson rate of λ=1.5 per day**

1. What is the expected time until the 100th immigrant arrives?
2. What is the probability that the elapsed time between the 100th immigrant and the next immigrant‘s arrival exceeds 2 days?

3. What is the probability that 100th immigrant will arrive after one year? You may approximate this probability with a normal distribution and you may assume there are 365 days in a year.

$T_1, T_2, \ldots, T_{100}$ iid Exp(λ=1.5/day)

1/ $E(T_1+T_2+\ldots+T_{100}) = E(T_1)+\ldots+E(T_2)$

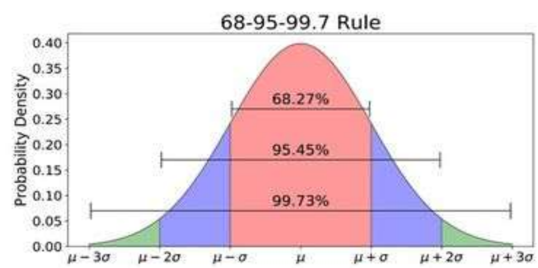
$= 100 * \frac{1}{1.5} = 66.7$ day

2/ $E(T_{101}>2) = e^{-1.5*2} = e^{-3} = 0.04979$

3/ $P(T_1+T_2+\ldots+T_{100} > 365)$ very very small

$T_1+T_2+\ldots+T_{100} \approx \text{Normal}(\frac{200}{3}, \frac{400}{9})$

$= P\left(Z > \frac{365-\frac{200}{3}}{\sqrt{\frac{400}{9}}}\right) = P(Z > 4.475) \approx 0$



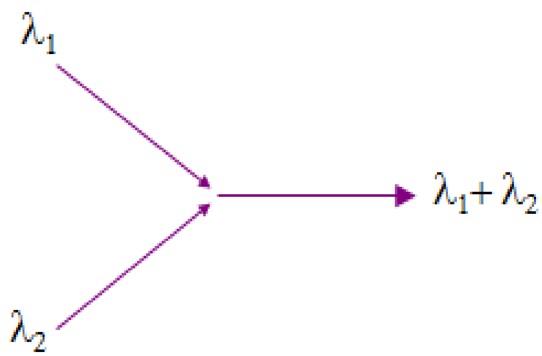
Example6

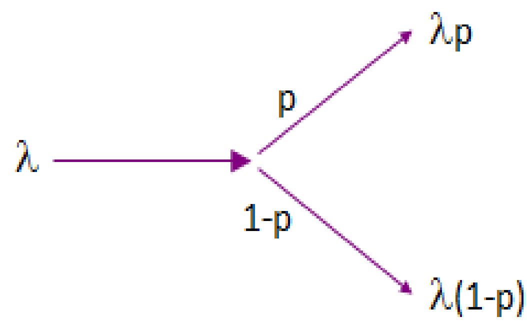
Patients arrive at doctor's office according to a Poisson process with rate $\lambda = \frac{1}{10}$ minute the doctor will not see a patient until at least three patients are in the waiting room.

A. Find the expected waiting time until the patient is admitted to see doctor

B. What is the probability that nobody is admitted to see the doctor in the first hour?

Merging & Splitting Poisson Processes





- $A_1, \ldots, A_k$ independent Poisson processes with rates $\lambda_1, \ldots, \lambda_k$
- Merged in a single process $A = A_1 + \ldots + A_k$
- A is Poisson process with rate $\lambda = \lambda_1 + \ldots + \lambda_k$

- A: Poisson processes with rate λ Split into processes A_1 and A_2 independently, with probabilities p and 1-p respectively
- A_1 is Poisson with rate $\lambda_1 = \lambda p$
- A_2 is Poisson with rate $\lambda_2 = \lambda(1-p)$

Example 7

The arrival of customers wishing to make a deposit at a bank teller window is Poisson with rate 9/hour, and the arrival of customers wishing to make a withdrawal is Poisson with rate 6/hour.

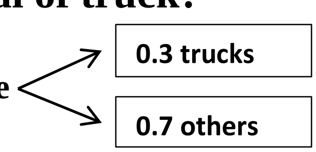
λ_1 =9/hour , λ_2=6/hour it merging Poisson process

λ_1 =9/hour + λ_2=6/hour = 15/hour = λ

Example 8

If the arrival of vehicles at an intersection is Poisson with rate 20/minute, and 30% of the vehicles are trucks, what is rate arrival of truck?

The splitting Poisson Processes for arrival of vehicles λ =20/minute



Then the arrival of trucks is a Poisson process with rate 0.3×20/minute =6/minute.

λ_1 (tricks) =6/minute and λ_2(others)=14/minute

Example 9

The manager of the restaurant notices that customers arrive at the restaurant could males at rate of 15 cust./hr for females at rate 5 cust./hr both types of arrivals are assumed to be Poisson. The restaurant Opens for business at 11:00 am. Find:

a. **Probability of having 10 customers in the restaurant at 11:12 am.**

b. **Probability of having 5 customers in the restaurant at 11:12 am,at least 3 of them are males.**

c. **An arrival arrived at 12:15 to the restaurant, what is the probability that person is a female?**

d. **What is the probability that at most one male arrive between 12:30 and 1:00.**

e. **what is the number of females you expected to see in restaurant at 11:28**

Example 10

Customers enter a store according to a Poisson process of rate $\lambda = 8$ per hour. Each customer independent of others buys something with probability 0.4 and leaves without making a buy's with probability 0.6.

1. **What is the probability of at least 2 buy during a 2 hour period?**

2. **What is the probability wait more 20 minutes until two customers arrives? Gamma**

3. **What is the probability that during the first hour 7 people enter the store with only 4 making buy and 3 do not?**

H.W

You are fishing with a friend. Suppose that you catch fish as a Poisson process of rate 2 per hour, and your friend catch fish as a Poisson process of rate 3 per hour.

1. What is the probability that you catch at least 2 fish in 3 hour? 0.98

2. What is the probability that you time between the first catch with next catch exceed 2 hour?0.018

3. What is the probability that catch 4 fish in 3 hour together (you and friend)

4. What are the expectation and the variance of the time till the 6^{th} catch fish your friend (only friend)?

Exercises

1) You get email according to a Poisson process at a rate of messages $\lambda=0.2$ per hour.
   a. You check your email every hour. What is the probability finding 0 or 1 new messages?
   b. Suppose that you have not checked your email for a whole day. What is the probability of finding 5 new messages?

2) Customers arrive at a store at the rate of 10 per hour. Each is either male or female with probability 0.5 . Assume that you know that exactly 10 women entered within some hour (say, 10 to 11am).
   a. Compute the probability that exactly 10 men also entered.
   b. Compute the probability that at least 2 customers have entered.

3) Births in a hospital occurs randomly at the at an average rate of 1.8 birth per hour. Find the following probability
   a. Exactly 5 births in an hour?
   b. Less than 3 births will occur in a minute?
   c. At least 1 births will occur in 5 minutes?
   d. The mean of births in an hour ?
   e. What is the probability of observing 10 births is male in a given 6 hour at the hospital?
   f. The variance of births in half hour ?

4) The number of passengers arriving at the airport by Poisson 5 passengers per minutes
   a. what is probability more than 3 minutes arrival 8 passengers
   b. what is probability less than 2 minutes arrival 4 passengers

5) The number of failures $N(t)$, which occur in a computer network over the time interval $[0, t)$, can be described by a homogeneous Poisson process $\{N(t), t \geq 0\}$. On an average, there is a failure after every 4 hours, i.e. the intensity of the process is equal to $\lambda = 0.25/h$.
   a. What is the probability of at most 1 failure in $[0, 8)$, at least 2 failures in $[8, 16)$, and at most 1 failure in $[16, 24)$ (time unit: hour)?
   b. What is the probability that the third failure occurs after 8 hours?