Lecture 2

Time Series

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1. Moving Average Process MA(q)

Definition $\{X_t\}$ *is a* ***moving-average process of order q*** *if*
$X_t = Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}$, (1)
where

$$Z_t \sim WN(0, \sigma^2)$$

and $\theta_1, \ldots, \theta_q$ *are constants.*
Remark. The MA(q) process can also be written in the following equivalent form

$$X_t = \theta(B) Z_t, \quad (2)$$

where the **moving average operator**

$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q \quad (3)$$

Defines: a linear combination of values in the shift operator

$$B^k Z_t = Z_{t-k}.$$

Example. MA(2) process.
This process is written as

$$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} = (1 + \theta_1 B + \theta_2 B^2) Z_t. \quad (4)$$

What are the properties of MA(2)? As it is a combination of a zero mean white noise, it also has zero mean, i.e.,

$$\mathrm{E}\, X_t = \mathrm{E}(Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}) = 0.$$

It is easy to calculate the covariance of X_t and $X_{t+\tau}$. We get

$$\gamma(\tau) = \mathrm{cov}(X_t, X_{t+\tau}) = \begin{cases} (1+\theta_1^2+\theta_2^2)\sigma^2 & \text{for } \tau = 0, \\ (\theta_1 + \theta_1\theta_2)\sigma^2 & \text{for } \tau = \pm 1, \\ \theta_2\sigma^2 & \text{for } \tau = \pm 2, \\ 0 & \text{for } |\tau| > 2, \end{cases} \quad \ldots\ldots(5)$$

which shows that the autocovariances depend on lag, but not on time. Dividing
$\gamma(\tau)$ by $\gamma(0)$ we obtain the autocorrelation function,

$$\rho(\tau) = \begin{cases} 1 & \text{for } \tau = 0, \\ \frac{\theta_1+\theta_1\theta_2}{1+\theta_1^2+\theta_2^2} & \text{for } \tau = \pm 1, \\ \frac{\theta_2}{1+\theta_1^2+\theta_2^2} & \text{for } \tau = \pm 2 \\ 0 & \text{for } |\tau| > 2. \end{cases} \quad \ldots\ldots(6)$$

a weakly stationary, 2-correlated TS.

Proposition. *If* $\{X_t\}$ *is a stationary q-correlated time series with mean zero, then it can be represented as an MA(q) process.*

Figure 1: shows MA(2) processes obtained from the simulated Gaussian

white noise shown in Figure 1 for various values of the parameters (θ_1, θ_2).

The blue series is $x_t = z_t + 0.5z_{t-1} + 0.5z_{t-2}$

While the purple series is $x_t = z_t + 5z_{t-1} + 5z_{t-2}$

As you can see very different processes can be obtained for different sets of the parameters. This is an important property of MA(q) processes, which is a very large family of models. This property is reinforced by the following Proposition.

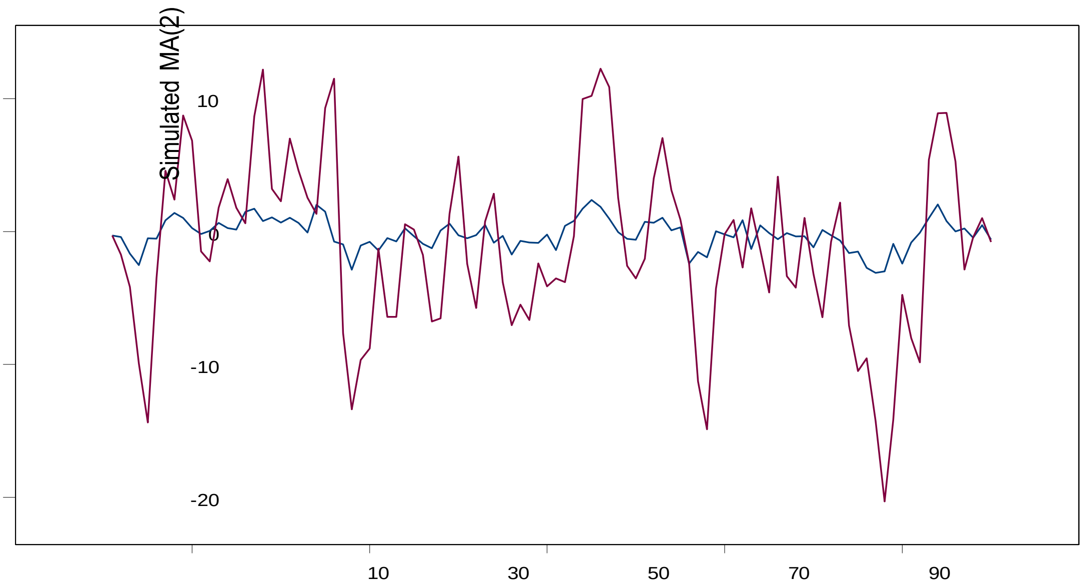


Figure 1: Two simulated MA(2) processes, both from the white noise shown in Figure1, but for different sets of parameters: $(\theta_1, \theta_2) = (0.5, 0.5)$ and $(\theta_1, \theta_2) = (5, 5)$.

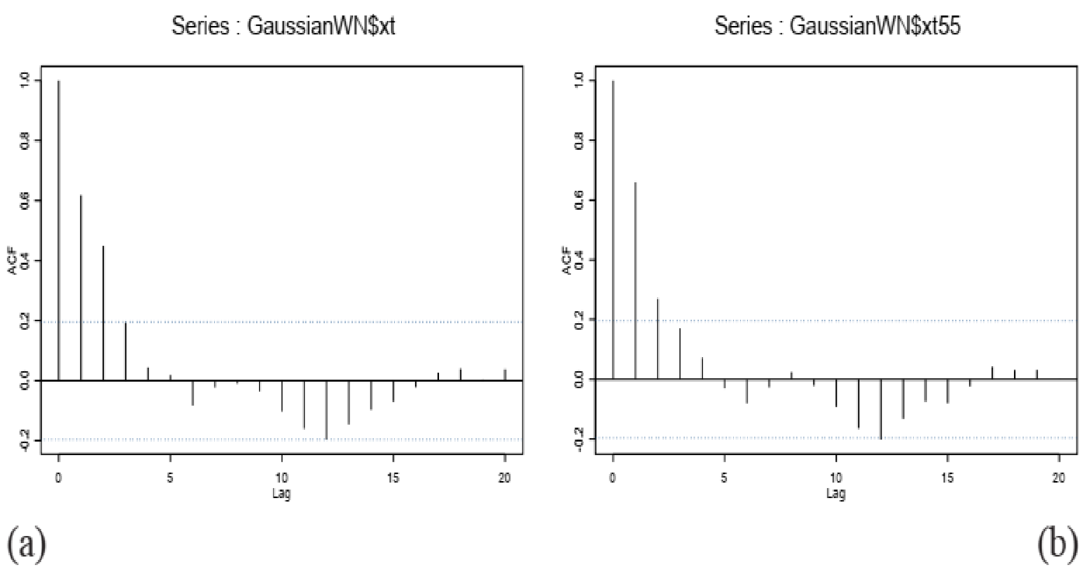


Figure 2 : Series Gaussian WN : ACF

(a) xt = zt + 0.5zt−1 + 0.5zt−2 and (b) xt = zt + 5zt−1 + 5zt−2

2. Invertibility of MA Processes

The MA(1) process can be expressed in terms of lagged values of Xt by substituting repeatedly for lagged values of Zt. We have

$$Z_t = X_t - \theta Z_{t-1}.$$

The substitution yields

$$\begin{aligned} Z_t &= X_t - \theta Z_{t-1} \\ &= X_t - \theta(X_{t-1} - \theta Z_{t-2}) \\ &= X_t - \theta X_{t-1} + \theta^2 Z_{t-2} \\ &= X_t - \theta X_{t-1} + \theta^2(X_{t-2} - \theta Z_{t-3}) \\ &= X_t - \theta X_{t-1} + \theta^2 X_{t-2} - \theta^3 Z_{t-3} \\ &= \ldots \\ &= X_t - \theta X_{t-1} + \theta^2 X_{t-2} - \theta^3 X_{t-3} + \theta^4 X_{t-4} + \ldots + (-\theta)^n Z_{t-n}. \end{aligned}$$

This can be rewritten as

$$(-\theta)^n Z_{t-n} = Z_t - \sum_{j=0}^{n-1} (-\theta)^j X_{t-j}.$$

However, if $|\theta| < 1$, then

$$\mathrm{E}\left(Z_t - \sum_{j=0}^{n-1} (-\theta)^j X_{t-j}\right)^2 = \mathrm{E}\left(\theta^{2n} Z_{t-n}^2\right) \underset{n\to\infty}{\longrightarrow} 0$$

And we say that the sum is convergent in the mean square sense. Hence, we obtain another representation of the model

$$Z_t = \sum_{j=0}^{\infty} (-\theta)^j Z_{t-j} \qquad \ldots\ldots(7)$$

This is a representation of another class of models, called infinite autoregressive (AR) models. So we inverted MA(1) to an infinite AR. It was possible due to the assumption that $|\theta| < 1$. Such a process is called an **invertible process**. This is a desired property of time series.
so in the example we would choose the model with $\sigma^2 = 25$, $\theta = 1/5$.

3. Linear Processes

The class of linear time series models, which includes the class of autoregressive moving-average (ARMA) models, provides a general framework for studying stationary processes. In fact, states that every weakly stationary, purely nondeterministic, stochastic process can be written as a linear combination.This result is known as **Wold's decomposition**

Definition . The TS {Xt} is called a linear process if it has the representation

$$\boldsymbol{x_t - m = Z_t + \psi_1 Z_{t-1} + \psi_2 Z_{t-2} + \cdots . = \sum_{j=0}^{\infty} \psi_j Z_{t-j} , \psi_0 = 1} \quad \ldots .(8)$$

The Zt, t=0,∓1, ∓2, ... are a sequence of uncorrelated random variables often known as innovations
for all t, *where* Zt ~ WN(0, σ2) *and* {ψ_j} *is a sequence of constants such that*

The condition $\sum_{j=1}^{\infty} |\psi j| < \infty$ ensures that the process convergesin the mean square sense, that is

$$\mathrm{E}\,(X_t - \sum_{j=-n}^{n} \psi_j Z_{t-j})^2 \to 0 \quad \text{as } n \to \infty.$$

It is easy to show that the model (8) leads to autocorrelation in x_t. From this equation it follows that:

$$\mathrm{E}(X_t) = 0.$$

$$\gamma(0) = V(x_t) = E(x_t - m)^2$$

$$= E(Z_t + \psi_1 Z_{t-1} + \psi_2 Z_{t-2} + \cdots)^2$$

$$= E(Z_t^2) + \psi_1^2 E(Z_{t-1}) + \psi_2^2 E(Z_{t-1}^2) + \cdots)$$

$$= \sigma^2 + \psi_1^2 \sigma^2 + \psi_2^2 \sigma^2 + \cdots$$

$$= \sigma^2 \sum_{j=0}^{\infty} \boldsymbol{\Psi_j^2} \qquad \ldots\ldots(9)$$

by using the white noise result that $\mathrm{E}(Z_{t-i} Z_{t-j}) = 0$ for $i \neq j$ Now:

$$\gamma(k) = E(x_t - m)(x_{t-k} - m)$$

$$= E(Z_t + \psi_1 Z_{t-1} + \psi_2 Z_{t-2} + \cdots)(Z_{t-k} + \psi_1 Z_{t-k-1} + \psi_2 Z_{t-k-2} + \cdots)$$

$$= \sigma^2 (1.\psi_k + \psi_1 \psi_{k+1} + \psi_2 \psi_{k+2} + \cdots)$$

$$= \sigma^2 \sum_{j=0}^{q} \psi_j \psi_{j+k} \qquad \ldots..(10)$$

And this implies

$$\rho_k = \frac{\sum_{j=0}^{q} \psi_j \psi_{j+k}}{\sum_{j=0}^{q} \psi_j^2} \qquad \text{.......(11)}$$

If the number of ψ-weights in (8) is infinite, the weights must be assumed to be absolutely summable, so that $=\sum_{j=0}^{\infty} \psi_j^q < \infty$ in which case the linear filter representation is said to converge. This condition can be shown to be equivalent to assuming that xt is stationary, and guarantees that all moments exist and are independent of time, in particular that the variance of xt, γ0, is finite.

Example : Find Variances , autocovariance and autocorrelation for MA(1) by Linear Processes

In equation (7) $Z_t = \sum_{j=0}^{\infty} (-\theta)^j Z_{t-j}$

Now consider the model obtained by choosing $\psi_1 =- \theta$ and $\psi_j = 0, j \geq 2$

In equation (9) $\gamma(0)=\sigma^2 \sum_{j=0}^{\infty} \psi_j^2 \;=\; \sigma^2(1+\theta^2)$

In equation (10) $\gamma(1)=\sigma^2 \sum_{j=0}^{q} \psi_j \psi_{j+k}$

$=-\sigma^2\theta \quad , \gamma(0) = 0 \text{ for } k > 1$

$$\rho_1 = -\frac{\theta}{1+\theta^2} \quad \rho_k = 0 \text{ for } k > 1$$

Home Work

1. **Find Variances , autocovariance and autocorrelation by Linear Processes**

   For $Zt= Z_t+0.5Z_{t-1}+0.35Z_{t-2}$

2. **Prove Variances , autocovariance and autocorrelation in Equation(5 and 6) by Linear Processes**

4. Autoregressive Processes AR(p)

The idea behind the autoregressive models is to explain the present value of theseries, X_t, by a function of p past values, $X_{t-1}, X_{t-2}, \ldots, X_{t-p}$.

Definition . *An* ***autoregressive process of order p*** *is written as*

$$X_t = \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \ldots + \varphi_p X_{t-p} + Z_t, \tag{12}$$

where {Zt} is white noise, i.e., {Zt} ~ WN(0, σ2), and Zt is uncorrelated with Xs for each s < t.

Remark . We assume (for simplicity of notation) that the mean of X_t is zero. If the mean is $\mathrm{E}\, Xt = m \neq 0$ $\mathrm{E}\, X_t = \mu$, then we replace X_t by $X_t - \mu$ to obtain

$$X_t - \mu = \varphi_1(X_{t-1} - \mu) + \varphi_2(X_{t-2} - \mu) + \ldots + \varphi_p(X_{t-p} - \mu) + Z_t,$$

what can be written as

$$X_t = \alpha + \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \ldots + \varphi_p X_{t-p} + Z_t,$$

Where $$\alpha = \mu(1 - \varphi_1 - \ldots - \varphi_p).$$

Other ways of writing AR(p) model use:

Vector notation: Denote $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_p)^{\mathrm{T}}$, $\boldsymbol{X}_{t-1} = (X_{t-1}, X_{t-2}, \ldots, X_{t-p})^{\mathrm{T}}$.

Then the formula can be written as

$$X_t = \boldsymbol{\varphi}^{\mathrm{T}} \boldsymbol{X}_{t-1} + Z_t.$$

Backshift operator: Namely, writing the model (7) in the form

$$X_t - \varphi_1 X_{t-1} - \varphi_2 X_{t-2} - \ldots - \varphi_p X_{t-p} = Z_t,$$

and applying $BX_t = X_{t-1}$ we get

$$(1 - \varphi_1 B - \varphi_2 B^2 - \ldots - \varphi_p B^p) X_t = Z_t$$

Or, using the concise notation we write

$$\varphi(B) X_t = Z_t, \qquad (13)$$

where $\varphi(B)$ denotes the **autoregressive operator**

$$\varphi(B) = 1 - \varphi_1 B - \varphi_2 B^2 - \ldots - \varphi_p B^p.$$

Then the AR(p) can be viewed as a solution to the equation (8), i.e.,

$$X_t = \frac{Z_t}{\emptyset(B)} \ \ldots\ldots(14)$$

4.1 AR(1)

According to Definition the autoregressive process of order 1 is given by

$$X_t = \varphi X_{t-1} + Z_t, \qquad (15)$$

where $Z_t \sim WN(0, \sigma^2)$ and φ is a constant.

Is AR(1) a stationary Time Series or not?

Corollary 4.1 says that an infinite combination of white noise variables is a stationary process. Here, due to the recursive form of the TS we can write AR(1) insuch a form. Namely

$$
\begin{aligned}
X_t &= \varphi X_{t-1} + Z_t \\
&= \varphi(\varphi X_{t-2} + Z_{t-1}) + Z_t \\
&= \varphi^2 X_{t-2} + \varphi Z_{t-1} + Z_t \\
&\cdot \\
&\cdot \\
&\cdot \\
&= \varphi^k X_{t-k} + \sum_{j=0}^{k-1} \varphi^j Z_{t-j}
\end{aligned}
$$

This can be rewritten as

$$\varphi^k X_{t-k} = X_t - \sum_{j=0}^{k-1} \varphi^j Z_{t-j} \quad \text{....(16)}$$

What would we obtain if we have continued the backwards operation, i.e., whathappens when $k \to \infty$?

Taking the expectation we obtain

$$\lim_{k\to\infty} \mathrm{E}\left(X_t - \sum_{j=0}^{k-1} \phi^j Z_{t-j}\right)^2 = \lim_{k\to\infty} \phi^{2k}\, \mathrm{E}(X_{t-k}^2) = 0$$

if $|\phi| < 1$ and the variance of X_t is bounded. Hence, we can represent AR(1) as

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$$

in the mean square sense. This is a linear process (4.15) with

$$\psi_j = \begin{cases} \phi^j & \text{for } j \geq 0, \\ 0 & \text{for } j < 0. \end{cases}$$

This technique of iterating backwards works well for AR of order 1 but not for other orders. A more general way to convert the series into a linear process formis the method of matching coefficients.

The AR(1) model is $\varphi(B)X_t = Z_t$,

where $\varphi(B) = 1 - \varphi B$ and $|\varphi| < 1$. We want to write the model as a linear process

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} = \psi(B) Z_t,$$

where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$. It means we want to find the coefficients ψ_j. Substituting Z_t from the AR model into the linear process model we obtain

$$X_t = \psi(B) Z_t = \psi(B)\phi(B) X_t. \tag{4.24}$$

In full, the coefficients of both sides of the equation can be written as

$$\begin{aligned}
1 &= (1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots)(1 - \varphi B) \\
&= 1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots - \varphi B - \psi_1 \varphi B^2 - \psi_2 \varphi B^3 - \psi_3 \varphi B^4 - \dots \\
&= 1 + (\psi_1 - \varphi) B + (\psi_2 - \psi_1 \varphi) B^2 + (\psi_3 - \psi_2 \varphi) B^3 + \dots
\end{aligned}$$

Now, equating coefficients of B^j on the LHS and RHS of this equation we seethat all the coefficients of B^j must be zero, i.e.,

$$\begin{aligned}
\psi_1 &= \phi \\
\psi_2 &= \psi_1 \phi = \phi^2 \\
\psi_3 &= \psi_2 \phi = \phi^3 \\
&\vdots \\
\psi_j &= \psi_{j-1} \phi = \phi^j.
\end{aligned}$$

So, we obtained the linear process form of the AR(1)

Remark. Note, that from the equation (4.24) it follows that $\psi(B)$ is an inverse of $\varphi(B)$, that is

$$\psi(B) = \frac{1}{\phi(B)}.$$

For an AR(1) we have

$$\psi(B) = \frac{1}{1-\phi B} = 1 + \phi B + \phi^2 B^2 + \phi^3 B^3 + \dots$$

As a linear process AR(1) is stationary with mean

$$\mathrm{E}\, X_t = \sum_{j=0}^{\infty} \phi^j\, \mathrm{E}(Z_{t-j}) = 0$$

and autocovariance function given by (4.19), that is

$$\gamma(\tau) = \sigma^2 \sum_{j=0}^{\infty} \phi^j \phi^{j+\tau} = \sigma^2 \phi^\tau \sum_{j=0}^{\infty} \phi^{2j}.$$

However, the infinite sum in this expression is the sum of a geometric progression as $|\phi| < 1$, i.e.,

$$\sum_{j=0}^{\infty} \phi^{2j} = \frac{1}{1-\phi^2}.$$

This gives us the following form for the ACVF of AR(1).

$$\gamma(\tau) = \frac{\sigma^2 \phi^\tau}{1-\phi^2}. \quad (4.28)$$

Then the variance of AR(1) is

$$\gamma(0) = \frac{\sigma^2}{1-\phi^2}.$$

Hence, the autocorrelation function of AR(1) is

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)} = \phi^\tau. \quad (4.29)$$

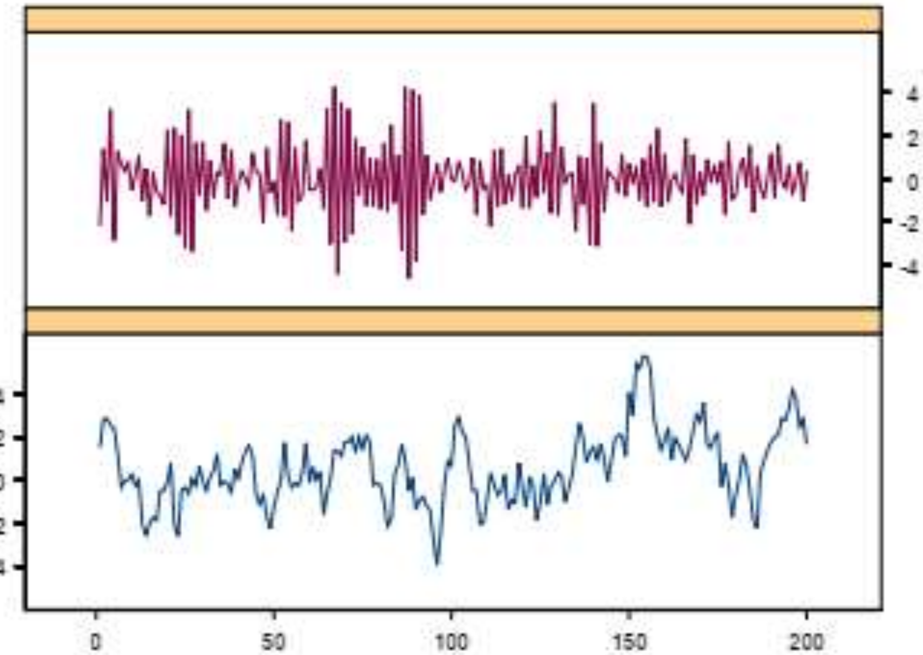


Figure 4.7: Simulated AR(1) processes for $\phi = -0.9$ (top) and for $\phi = 0.9$ (bottom).

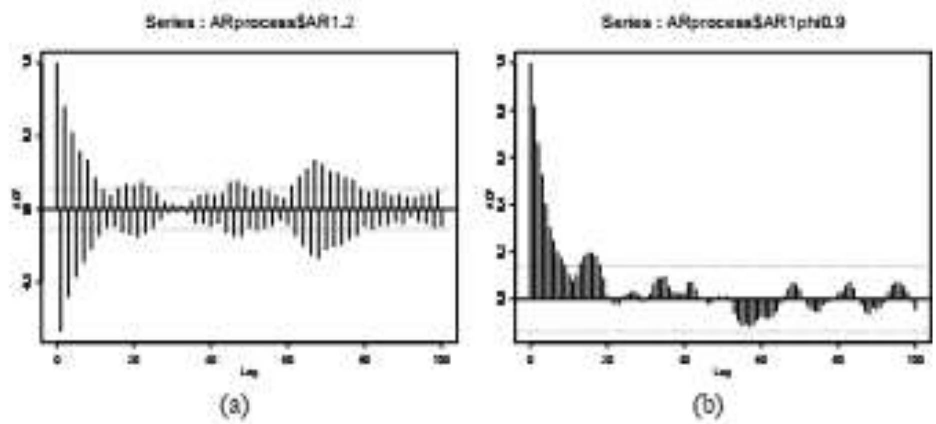


Figure 4.8: Sample ACF for AR(1): (a) $x_t = -0.9x_{t-1} + z_t$ and (b) $x_t = 0.9x_{t-1} + z_t$.

Explosive AR(1) Model and Causality

Random walk, which is AR(1) with $\varphi = 1$ is not a stationary process. So, there is a question if a stationary AR(1) process with $|\varphi| > 1$ exists? Also, what are the properties of AR(1) models for $\varphi > 1$?

Clearly, the sum $\sum_{j=0}^{k-1} \phi^j Z_{t-j}$ will not converge in mean square sense as $k \to \infty$ and we will not get a linear process representation of the AR(1). However, if $|\phi| > 1$ then $\frac{1}{|\phi|} < 1$ and we can express a past value of the TS in terms of a future value rewriting

$$X_{t+1} = \phi X_t + Z_{t+1}$$

as

$$X_t = \phi^{-1} X_{t+1} - \phi^{-1} Z_{t+1}.$$

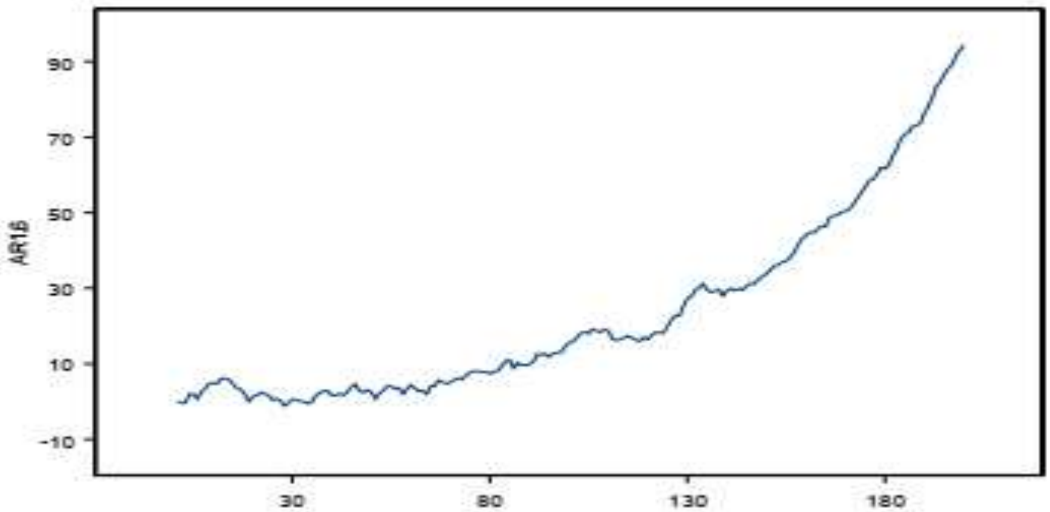


Figure 4.13: Simulated Explosive AR(1): $x_t = 1.02x_{t-1} + z_t$.

Then, substituting for X_{t+j} several times we obtain

$$\begin{aligned} X_t &= \phi^{-1} X_{t+1} - \phi^{-1} Z_{t+1} \\ &= \phi^{-1}(\phi^{-1} X_{t+2} - \phi^{-1} Z_{t+2}) - \phi^{-1} Z_{t+1} \\ &= \phi^{-2} X_{t+2} - \phi^{-2} Z_{t+2} - \phi^{-1} Z_{t+1} \\ &= \dots \\ &= \phi^{-k} X_{t+k} - \sum_{j=1}^{k-1} \phi^{-j} Z_{t+j} \end{aligned}$$

Since $|\phi^{-1}| < 1$ we obtain

$$X_t = -\sum_{j=1}^{\infty} \phi^{-j} Z_{t+j},$$

Which is a future dependent stationary TS. This however, does not have any practical meaning because it requires knowledge of future values to predict the future.
When a process does not depend on the future, such as AR(1) when $|\varphi| < 1$, we say that it is **causal**.
Figure 4.13 shows a simulated series xt = 1.02xt-1+zt. As we can see the values of the time series quickly become large in magnitude, even for φ just slightly above 1. Such process is called **explosive**. This is not a causal TS

3. Autoregressive Moving Average Model ARMA(1,1)

This section is an introduction to a wide class of models ARMA(p,q) which we will consider in more detail later in this course. The special case, ARMA(1,1), is defined by linear difference equations with constant coefficients as follows.

Definition . *A TS $\{X_t\}$ is an* ***ARMA(1,1) process*** *if it is stationary and it satisfies*

$X_t - \varphi X_{t-1} = Z_t + \theta Z_{t-1}$ *for every t, ...(17)*

where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\varphi + \theta \neq 0$.

Such a model may be viewed as a generalization of the two previously introduced models: AR(1) and MA(1). Compare

AR(1): $X_t = \varphi X_{t-1} + Z_t$

MA(1): $X_t = Z_t + \theta Z_{t-1}$

ARMA(1,1): $X_t - \varphi X_{t-1} = Z_t + \theta Z_{t-1}$

Here, as in the MA and AR models, we can use the backshift operator to write the ARMA model more concisely as

$\varphi(B)X_t = \theta(B)Z_t$, ...(18)

where $\varphi(B)$ and $\theta(B)$ are the **linear filters:**

$$\varphi(B) = 1 - \varphi B, \quad \theta(B) = 1 + \theta B.$$

A. Causality and invertibility of ARMA(1,1)

For what values of the parameters φ and θ does the stationary ARMA(1,1) exist and is useful? To answer this question we will look at the two properties of TS, causality and invertibility.

The solution to 17, or to 18, can be written as

$$\boldsymbol{X_t = \frac{\theta(B)\, Z_t}{\varphi(B)}} \qquad \textbf{........(19)}$$

However, for $|\varphi| < 1$ we have

$$\frac{\theta(B)}{\varphi(B)} = (1 + \varphi B + \varphi^2 B^2 + \varphi^3 B^3 + \ldots)(1 + \theta B)$$

$$= 1 + \varphi B + \varphi^2 B^2 + \varphi^3 B^3 + \ldots + \theta B + \varphi\theta B^2 + \varphi^2\theta B^3 + \varphi^3\theta B^4 + \ldots$$

$$= 1 + (\varphi + \theta)B + (\varphi^2 + \varphi\theta)B^2 + (\varphi^3 + \varphi^2\theta)B^3 + \ldots$$

$$= 1 + (\varphi + \theta)B + (\varphi + \theta)\varphi B^2 + (\varphi + \theta)\varphi^2 B^3 + \ldots$$

$$= \sum_{j=0}^{\infty} \psi_j B^j \qquad \ldots\ 5.2$$

where $\psi_0 = 1$ **and** $\psi_j = (\varphi + \theta)\varphi^{j-1}$ for $j = 1, 2, \ldots$.
Thus, we can write the solution to 5.1 in the form of an MA(∞) model,i.e

$$\boldsymbol{X_t = Z_t + (\varphi + \theta)\sum_{j=1}^{\infty} \varphi^{j-1} Z_{t-j}} \qquad \textbf{.... (20)}$$

This is a stationary unique process.

Now, suppose that $|\varphi| > 1$. Then, by similar arguments as in the AR(1) model, it can be shown that

$$X_t = -\theta\phi^{-1} Z_t - (\phi + \theta) \sum_{j=1}^{\infty} \phi^{-j-1} Z_{t+j}.$$

If |φ| = 1 then there is no stationary solution to

While causality means that the process $\{X_t\}$ is expressible in terms of past values of $\{Z_t\}$, the dual property of invertibility means that the process $\{Z_t\}$ is expressible in the past values of $\{X_t\}$. Is ARMA(1,1) invertible?

and so writing the solution for Z_t we have

$$Z_t = \frac{1}{\theta(B)}\phi(B)X_t = \frac{1}{1+\theta B}(1 - \phi B)X_t.$$

what in terms of the backshift operator B can be written as

$$\frac{1}{1+\theta B} = \sum_{j=0}^{\infty}(-\theta)^j B^j.$$

$$Z_t = \sum_{j=0}^{\infty}(-\theta)^j B^j (1 - \phi B) X_t$$

$$= X_t - (\phi + \theta) \sum_{j=1}^{\infty}(-\theta)^{j-1} X_{t-j}.$$

The conclusion is that ARMA(1,1) is invertible if $|\theta| < 1$. Otherwise it is non invertible.

The two properties, causality and invertibility, determine the admissible region for the values of parameters φ and θ, which is the square

$$-1 < \varphi < 1$$
$$-1 < \theta < 1.$$

4.6.1 ACVF and ACF of ARMA(1,1)

The fact that we can express ARMA(1,1) as a linear process of the form

$$= \sum_{j=0}^{\infty} \psi_j Z_{t-j} \quad ....21$$

where Zt is a white noise, is very helpful in deriving the ACVF and ACF of the process.

$$\gamma(\tau) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+\tau}$$

And we can easily derive expressions for $\gamma(0)$ and $\gamma(1)$.

$$\gamma(0) = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2$$

$$= \sigma^2 \left[1 + (\phi + \theta)^2 \sum_{j=1}^{\infty} \phi^{2(j-1)} \right]$$

$$= \sigma^2 \left[1 + (\phi + \theta)^2 \sum_{j=0}^{\infty} \phi^{2j} \right] \qquad \text{......(22)}$$

$$= \sigma^2 \left[1 + \frac{(\phi + \theta)^2}{1 - \phi^2} \right].$$

And

$$\gamma(1) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+1}$$

$$= \sigma^2 \left[1(\phi+\theta) + (\phi+\theta)(\phi+\theta)\phi + (\phi+\theta)\phi(\phi+\theta)\phi^2 + (\phi+\theta)\phi^2(\phi+\theta)\phi^3 + \ldots \right]$$

$$= \sigma^2 \left[(\phi+\theta) + (\phi+\theta)^2 \phi (1 + \phi^2 + \phi^4 + \ldots) \right]$$

$$= \sigma^2 \left[(\phi+\theta) + (\phi+\theta)^2 \phi \sum_{j=0}^{\infty} \phi^{2j} \right] \qquad \text{............(23)}$$

$$= \sigma^2 \left[(\phi+\theta) + \frac{(\phi+\theta)^2 \phi}{1-\phi^2} \right]$$

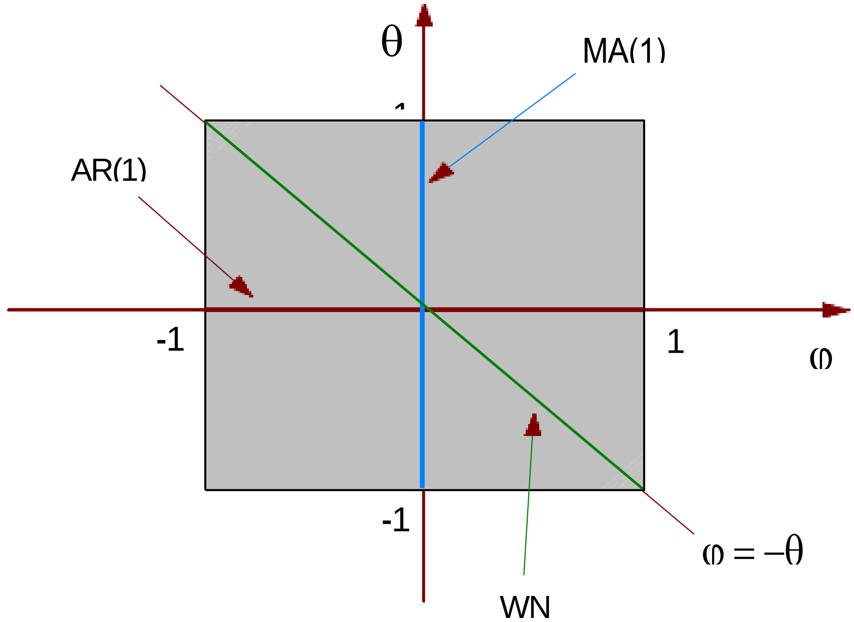
Note : $\sum_{=1}^{\infty} ar^{n-1} = \frac{1}{1-r}$, $\sum_{j=0}^{\infty} \phi^{2j} = \frac{1}{1-\phi^2}$

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{(\phi + \theta)(1 + \phi\theta)}{1 + 2\phi\theta + \theta^2} \qquad \text{-------(24)}$$

H.w: Find Variances, autocovariance and autocorrelation by Linear Processes

For $Zt = 0.6\, Z_{t-1} + Z_t + 0.5 Z_{t-1}$

Graph 4 shows the admissible region for the parameters φ and θ and indicates the regions when we have special cases of ARMA(1,1), which are white noise, AR(1) and MA(1).



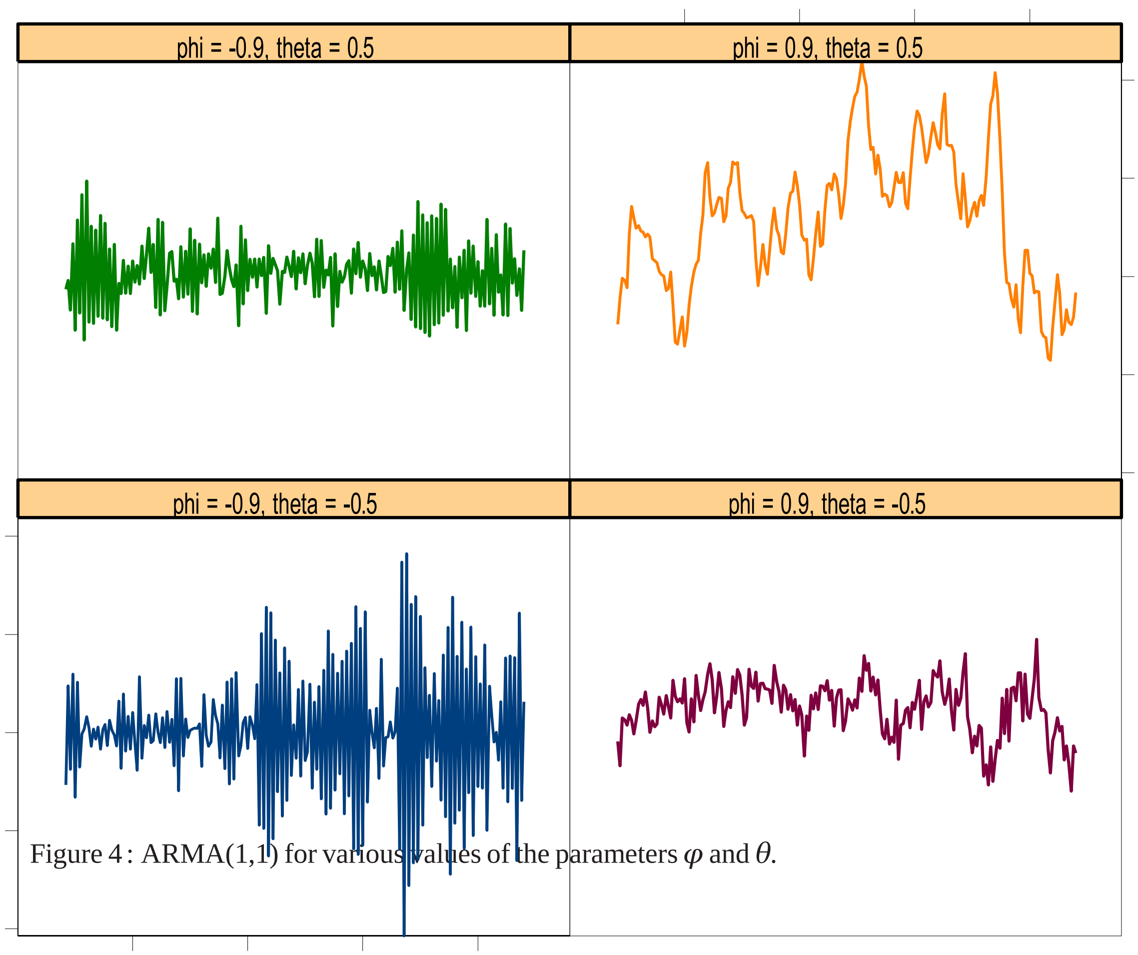


Figure 4 : ARMA(1,1) for various values of the parameters φ and θ.

Causality of ARMA(p,q)

We showed that the condition for stationarity of ARMA(1,1)

Xt - φXt-1 = Zt + θZt-1, for every t,

is that

$$|\phi| \neq 1,$$

that is $1-\phi \neq 0$ or $1+\phi \neq 0$. This is equivalent to say that the polynomial

$$\phi(z) = 1 - \phi z \neq 0 \quad \text{for } |z| = 1.$$

We have also derived the condition for causality of ARMA(1,1), which is

$$|\phi| < 1.$$

This condition can be viewed in terms of the solution to the equation

$$\phi(z) = 1 - \phi z = 0$$

which is $z = \frac{1}{\phi}$ and which should be bigger than 1 or smaller than -1.

Example 6.2. Consider ARMA(2,1)

$$X_t - 0.8X_{t-1} - 0.1X_{t-2} = Z_t + 0.3Z_{t-1}.$$

We can see that the process is causal as the parameters satisfy the conditions. We can also check it by calculating the roots of the *autoregressive polynomial*. These are found by solving the equation

φ(z) = 1 - 0.8z - 0.1z2 = 0.

The discriminant is Δ = 0.82 + 4 · 0.1 = 1.04 and the roots are

$$\left| \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2} \right| > 1.$$

$$z_1 = \frac{0.8 - \sqrt{1.04}}{2(-0.1)} = 1.09902$$
$$z_2 = \frac{0.8 + \sqrt{1.04}}{2(-0.1)} = -9.09902$$

The roots are outside the interval [-1, 1] and so the process is stationary and causal.

Example: Parameter Redundancy, Causality and Invertibility

Consider the process $X_t - 0.4X_{t-1} - 0.45X_{t-2} = Z_t + Z_{t-1} + 0.25Z_{t-2}$.

$(1 - 0.4B - 0.45B^2)X_t = (1 + B + 0.25B^2)Z_t$.

Is this really an ARMA(2,2) process?

We need to check if the polynomials $\varphi(z)$ and $\theta(z)$ have common factors. We have

$\varphi(z) = 1 - 0.4z - 0.45z^2 = (1 + 0.5z)(1 - 0.9z)$,

The model is causal because

$\varphi(z) = 1 - 0.9z = 0$ when **$z = 10/9$,**

which is outside the unit circle. The model is also invertible because

$\theta(z) = 1 + 0.5z = 0$ when **$z = -2$**

This is outside the unit circle too

To obtain a linear process form of the model we need to calculate the coefficients ψ_j. It can be done from the relation

$$\psi_j = \theta_j + \sum_{k=1}^{p} \phi_k \psi_{j-k},$$

where $\theta_0 = 1$, $\theta_j = 0$ for $j > q$, and $\psi_j = 0$ for $j < 0$. This gives

$$\begin{aligned}
&\psi_0 = \theta_0 = 1 \\
&\psi_1 = \theta_1 + \phi_1\psi_0 = \theta_1 + \phi_1 = 0.5 + 0.9 = 1.4 \\
&\psi_2 = \phi_1\psi_1 = \phi_1(\theta_1 + \phi_1) = 0.9 \times 1.4 \\
&\psi_3 = \phi_1\psi_2 = \phi_1^2(\theta_1 + \phi_1) = 0.9^2 \times 1.4 \\
&\ldots \\
&\psi_j = \phi_1\psi_{j-1} = \phi_1^{j-1}(\theta_1 + \phi_1) = 0.9^{j-1} \times 1.4.
\end{aligned}$$

Hence we can write

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} = Z_t + 1.4 \sum_{j=1}^{\infty} 0.9^{j-1} Z_{t-j}.$$

Similarly, we can find the invertible representation of the model which is

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} = X_t - 1.4 \sum_{j=1}^{\infty} (-0.5)^{j-1} X_{t-j}.$$

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