Vector Autoregressive (VAR) model.

Vector Autoregressive (VAR) processes possess several key properties that make them valuable in modeling the dynamic interactions among multiple time series variables in economics and other fields:

1. **Multivariate Time Series Analysis: VAR models handle multiple time series variables simultaneously, allowing the analysis of their interdependencies.**
2. **Flexibility: VAR models can accommodate various types of relationships, including contemporaneous and lagged effects among variables.**
3. **No Exogeneity Assumptions: VAR models do not rely on strict exogeneity assumptions, making them suitable for analyzing endogenous relationships without imposing strict causality assumptions.**
4. **Dynamic Modeling: VAR models capture the dynamics of a system by incorporating lagged values of variables, allowing for the examination of short-term and long-term effects.**
5. **Stationarity and Stability: Stationarity of variables is often assumed in VAR models for estimation purposes.**

Examples

There are many other fields in which VAR models like Economy, epidemiology, medicine, and biology…etc.

Example question	Field	Description
How are vital signs in cardiorespiratory patients dynamically related?	Medicine	A VAR system is used to model the past and current relationships between heart rate, respiratory rate, blood pressure and SpO2.
How do risks of COVID-19 infections interact across age groups?	Epidemiology	Count data of past infections across different age groups was used to model the relationships between infection rates across those age groups.
Is there a bi-directional relationship between personal income and personal consumption spending?	Economics	A two-equation VAR system is used to model the relationship between income and consumption over time.

Example question	Field	Description
How can we model the gene expression networks?	Biology	The relationships across large networks of genes are modeled using a sparse structural VAR model.
What is driving inflation more -- monetary policy shocks or external shocks?	Macroeconomics	A structural VAR model is used to compute variance decomposition and impulse response functions following monetary shocks and external system shocks.

- AR(1) Model

$$y_t = \phi y_{t-1} + \varepsilon_t$$

- VAR(1) Model

$$Y_t = AY_{t-1} + \epsilon_t$$

where

$$Y_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \\ \vdots \\ y_{k,t} \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & \dots & A_{1k} \\ A_{21} & \dots & A_{2k} \\ \vdots & & \\ A_{k1} & \dots & A_{kk} \end{pmatrix}, \quad \epsilon_t = \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \vdots \\ \epsilon_{k,t} \end{pmatrix}$$

where k is the number of endogenous variables included in VAR model

- VAR(p) Model

$$Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \dots + A_p Y_{t-p} + \epsilon_t$$

where

$$Y_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \\ \vdots \\ y_{k,t} \end{pmatrix}, \quad A_j = \begin{pmatrix} A_{j,11} & \dots & A_{j,1k} \\ A_{j,21} & \dots & A_{j,2k} \\ \vdots & & \vdots \\ A_{j,k1} & \dots & A_{j,kk} \end{pmatrix}, \quad \epsilon_t = \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \vdots \\ \epsilon_{k,t} \end{pmatrix}$$

- All VAR(p) models can be written as VAR(1) form

$$Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + ... + A_p Y_{t-p} + \epsilon_t$$

$$Z_t = BZ_{t-1} + u_t$$

where

$$Z_t = \begin{pmatrix} Y_t \\ Y_{t-1} \\ \vdots \\ Y_{t-p} \end{pmatrix}, \quad B = \begin{pmatrix} A_1 & A_2 & \dots & A_p & 0 \\ I & 0 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & I & 0 \end{pmatrix}, \quad u_t = \begin{pmatrix} \epsilon_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

- This expression is sometimes useful

- Entering exgoneous variables in VAR(p) model:

$$Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + ... + A_p Y_{t-p} + X_t + \epsilon_t$$

where X_t is exogenously determined (can have its own model for its stochastic process, separated from the main VAR model)

- VARMA(p,q) Model

$$Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + ... + A_p Y_{t-p} + \epsilon_t + B_1 \epsilon_{t-1} + ... + B_q \epsilon_{t-q}$$

Structural SVAR versus reduced form VAR

When we are not confident that the variable in the model is described as exogenous, each variable must be treated homogeneously, an example of which is the time series y_t which is affected by the current variables and the variables preceding x_t Simultaneously, the time series x_t is a series affected by the current value and the previously determined values of the time series y_t. In this case, the simple bivariate model is as follows:

$$y_t = \beta_{10} + \beta_{12}x_t + \gamma_{11}y_{t-1} + \gamma_{12}x_{t-1} + e_{yt} \quad \dots\dots\dots \quad 1$$

$$x_t = \beta_{20} + \beta_{21}y_t + \gamma_{21}y_{t-1} + \gamma_{22}x_{t-1} + e_{xt} \quad \dots\dots\dots \quad 2$$

Where we assume y_t ,x_t are stationary, e_{yt}, e_{xt} is an error term that is not auto correlated, and is characterized as white noise. Equations 1 and 2 form a VAR(1) model because the lag is one. These equations are not equations of reduced form, as y_t has a direct (contemporaneous) effect on x_t, given by the parameter β21, and x_t has a direct (contemporaneous) effect on y_t, given β12. By rewriting the system using matrices, we get the following:

$$\begin{bmatrix} 1 & \beta_{12} \\ \beta_{21} & 1 \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} \beta_{10} \\ \beta_{20} \end{bmatrix} + \begin{bmatrix} \gamma_{11} \gamma_{12} \\ \gamma_{21} \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} u_{yt} \\ u_{xt} \end{bmatrix} \quad \ldots \quad 3$$

Or $\boldsymbol{\beta z_t = \Gamma_0 + \Gamma_1 z_{t-1} + u_{t-1}}$ **.......4**

Where

$$\boldsymbol{B} = \begin{bmatrix} 1 & \beta_{12} \\ \beta_{21} & 1 \end{bmatrix}, \boldsymbol{z_t} = \begin{bmatrix} y_t \\ x_t \end{bmatrix}$$

$$\boldsymbol{\Gamma_0} = \begin{bmatrix} \beta_{10} \\ \beta_{20} \end{bmatrix}, \boldsymbol{\Gamma_1} = \begin{bmatrix} \gamma_{11} \gamma_{12} \\ \gamma_{21} \gamma_{22} \end{bmatrix} \, and \quad \boldsymbol{u_t} = \begin{bmatrix} u_{yt} \\ u_{xt} \end{bmatrix}$$

Multiplying both sides of the equation by $\boldsymbol{B^{-1}}$ we get:

$$\boldsymbol{z_t = A_0 + A_1 z_{t-1} + e_t} \qquad \textbf{.......5}$$

Where $\boldsymbol{A_0 = B^{-1}\Gamma_0}$, $\boldsymbol{A_1 = B^{-1}\Gamma_1}$, $\boldsymbol{e_t = B^{-1}u_t}$

For simplicity, we can use the symbols $\boldsymbol{a_{i0}}$, the i element of the vector A_0 and $\boldsymbol{a_{ij}}$, the element of row i and column j of the matrix A1, and $\boldsymbol{e_{it}}$ representing the i element of vector et. Using these symbols, the VAR model can be written as follows.

$$\boldsymbol{y_t = a_{10} + a_{11} y_{t-1} + \gamma a_{12} x_{t-1} + e_{1t}} \qquad \textbf{..... 6}$$

$$\boldsymbol{x_t = a_{20} + a_{21} y_{t-1} + a_{22} x_{t-1} + e_{2t}} \qquad \textbf{....... 7}$$

To differentiate between the original VAR regression model (1&.2) the system that we obtained in equations (6 &7), the first is called a primitive or structural system while the second is a VAR system in a standard or reduced form. The error term e1t and e2t consist of two shocks uyt and uxt, where et= $\boldsymbol{B^{-1}}$ ut can be obtained.

- Overall Importance: Reduced form is necessary to simplify and organize relationships between variables through statistical or economic reasons, facilitate (easily) analysis and forecasting and provide insight into the nature of these relationships

$$e_{1t} = (u_{yt} + \beta_{12} u_{xt})/(1 - \beta_{12}\beta_{21}) \qquad \text{......8}$$

$$e_{2t} = (u_{xt} + \beta_{21} u_{yt})/(1 - \beta_{12}\beta_{21}) \qquad \text{.....9}$$

Since u_{yt},u_{xt} are white noise processes, it follows that both e_{1t} and e_{2t} are also white noise processes.

Vector Autoregression (VAR) models have faced criticism in various aspects, including concerns about the absence of clear economic theory and issues related to statistical testing.

To overcome these criticisms, proponents of the VAR model estimate what is called the impulse response function. It provides a dynamic framework to observe and quantify how variables respond to unexpected changes, thereby enhancing the understanding and utility of VAR models in analyzing complex systems.

Impulse Response Function (IRF):

Definition: IRF is a statistical technique that measures the response of variables in a VAR model to an exogenous shock or impulse in one of the variables, while holding all other variables constant.

Understanding Dynamics: It shows how a one-time shock or innovation to a particular variable propagates through the system over time, illustrating the impact and persistence of the shock on other variables.

Graphical Representation: Typically displayed as a series of graphs, each depicting the cumulative effects of a shock on the variables over successive time periods.

Interpreting Effects: IRF quantifies the magnitude, direction, and duration of the effects of shocks, providing insights into the transmission channels and dynamic interactions among variables.

Assists in Causal Inference: Helps in inferring potential causal relationships among variables by observing how a shock in one variable affects others over time.

Model Evaluation: Enables assessment of the VAR model's goodness-of-fit and validity by comparing expected responses from the IRF with actual outcomes.

Policy Analysis: Facilitates analysis of the impact of policy changes or external shocks on the economy by forecasting the responses of different variables to these changes.

Causality tests:

Granger causality tests within VAR models enables researchers and analysts to explore and quantify potential directional influences among multiple variables, providing valuable insights into temporal relationships and predictive power among economic or time series.

VAR Granger Causality/Block Exogeneity Wald Tests
Date: 11/23/23 Time: 13:47
Sample: 1947Q1 1994Q4
Included observations: 190

Dependent variable: Y

Excluded	Chi-sq	df	Prob.
X	21.60661	2	0.0000
All	21.60661	2	0.0000

Dependent variable: X

Excluded	Chi-sq	df	Prob.
Y	4.185015	2	0.1234
All	4.185015	2	0.1234

Interpretation:

For 'Y' as the Dependent Variable:

The p-value of 0.0000 indicates strong evidence of Granger causality from 'X' to 'Y' at a very high level of statistical significance ($p < 0.05$). Past values of 'X' significantly predict 'Y'.

For 'X' as the Dependent Variable:

The p-value of 0.1234 suggests that there isn't sufficient evidence to conclude Granger causality from 'Y' to 'X' at a conventional level of significance ($p > 0.05$).