Syllabus of Topology

- Basic Concepts in General Topology, Definitions and Examples for Topology.
- Closed sets, Characterization of topological space in term of closed sets.
- Neighborhoods and their properties.
- Bases and Subbases
- First Axiom Space and Second Axiom Space.
- Operators (Interior, Exterior, Boundary, Limit point, and Closure)
- Properties of operators
- Subspaces

- **Connectedness, Path connected,**
- **Continuity in Topological Spaces**
- **Homeomorphism and Homeomorphic.**

  Topological and Hereditary Property.
- **Separation Axioms.**
- **Product Space and Quotient space.**

Useful References

Seymour, L. (1965), Theory and Problem of General Topology, Schaum's Outline series.

Willard S.(2012). General Topology. Courier Corporation.

Sharma J.(1977). Topology, Krishna Prakashan Ltd.

§1 Definition and Examples:

Definition 1.1: Let X be any non-empty set. A family $\mathfrak{J}$ of subsets of X is called a topology on X if it satisfies the following conditions:

(i) $\phi \in \mathfrak{J}$ and $X \in \mathfrak{J}$

(ii) $A, B \in \mathfrak{J} \Rightarrow A \cap B \in \mathfrak{J}$

(iii) $A_\lambda \in \mathfrak{J}$, $\forall\, \lambda \in \Lambda$ (where Λ is any indexing set) $\Rightarrow \bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}$

If $\mathfrak{J}$ is a topology on X, then the ordered pair $\langle X, \mathfrak{J} \rangle$ is called a topological space (or T-space)

Examples 1.2: Throughout X denotes a non-empty set.

1) $\Im = \{\emptyset, X\}$ is a topology on X. This topology is called **indiscrete topology** on X and the T-space $\langle X, \Im \rangle$ is called indiscrete topological space.

2) $\Im = \wp(X)$, ($\wp(X)$ = power set of X is a topology on X and is called **discrete topology** on X and the T-space $\langle X, \Im \rangle$ is called discrete topological space.

Remark: If $|X| = 1$, then discrete and indiscrete topologies on X coincide, otherwise they are different.

3) Let $X = \{a, b, c\}$ then $\Im_1 = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\Im_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ are topologies on X whereas $\Im_3 = \{\emptyset, \{a\}, \{b\}, X\}$ is a not a topology on X.

4) Let X be an infinite set. Define $\mathfrak{I} = \{\emptyset\} \cup \{A \subseteq X \mid X - A \text{ is finite}\}$ then $\mathfrak{I}$ is topology on X.

(i) $\emptyset \in \mathfrak{I}$ ...... (by definition of $\mathfrak{I}$)

As $X - X = \emptyset$, a finite set, $X \in \mathfrak{I}$

(ii) Let $A, B \in \mathfrak{I}$. If either $A = \emptyset$ or $B = \emptyset$, then $A \cap B \in \mathfrak{I}$. Assume that $A \neq \emptyset$ and $B \neq \emptyset$.

Then $X - A$ is finite and $X - B$ is finite. Hence $X - (A \cap B) = (X - A) \cup (X - B)$ is finite set. Therefore $A \cap B \in \mathfrak{J}$. Thus $A, B \in \mathfrak{J} \Rightarrow A \cap B \in \mathfrak{J}$.

(iii) Let $A_\lambda \in \mathfrak{J}$, for each $\lambda \in \Lambda$ (where Λ is any indexing set). If each $A_\lambda = \emptyset$, then

$$\bigcup_{\lambda \in \Lambda} A_\lambda = \emptyset \in \mathfrak{J}.$$

If $\exists\, \lambda_0 \in \Lambda$ such that $A_{\lambda_0} \neq \emptyset$, then $A_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda \Rightarrow X - A_{\lambda_0} \supseteq X - \bigcup_{\lambda \in \Lambda} A_\lambda$.

As $X - A_{\lambda_0}$ is a finite set and subset of finite set being finite we get $X - \bigcup_{\lambda \in \Lambda} A_\lambda$ is finite and hence $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}$. Thus in either case,

$$A_\lambda \in \mathfrak{J}, \;\; \forall\, \lambda \in \Lambda \;\; \Rightarrow \;\; \bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}.$$

From (i), (ii) and (iii) is a topology on X. This topology is called **co-finite topology** on X and the topological space is called co-finite topological space.

Remark: If X is finite set, then co-finite topology on X coincides with the discrete topology on X.

5) Let X be any uncountable set. Define $\mathfrak{I} = \{\emptyset\} \cup \{A \subseteq X \mid X - A \text{ is countable}\}$ Then $\mathfrak{I}$ is a topology on X.

Example1: Let X=R, and $\tau = \{G \subseteq R; -1 \notin G\} \cup \{R\}$, then show that τ is a topology for R.

Example2: Let X=R, and
$\tau = \{G \subseteq R; x \in G \rightarrow -x \in G\} \cup \{\varphi\}$
then show that τ is a topology for R.

Example3: Let X=R, and $\tau = \{G \subseteq R; 3 \in G\} \cup \{\varphi\}$, then show that τ is a topology for R.

Definition 2: Let X be a non-empty set and let $\tau_1\ and \tau_2$ be topologies for X, then

1) We say that τ_1 is weaker than τ_2 or τ_2 is stronger than τ_1 if $\tau_1 \subseteq \tau_2$.

2) If $\tau_1 \subseteq \tau_2$ or $\tau_2 \subseteq \tau_1$ then $\tau_1\ and\ \tau_2$ are comparable otherwise are incomparable.

Example: Let X={a,b,c}, $\tau_1 = \{\varphi, X\}, and\ \tau_2 = P(X),\ \ then$

τ_1 *is weaker than* τ_2 and τ_2 is stronger than τ_1,

τ_1 and τ_2 are comparable.

Example: Let X={a,b,c}, $\tau_1 = \{\varphi, \{a\}, X\}, and\ \tau_2 = \{\varphi, \{b\}, X\}, then$

τ_1 *is not weaker than* τ_2 and τ_2 is not stronger than τ_1,

τ_1 and τ_2 are not comparable.

Union and Intersection of topologies

1) Union of two topologies may not be topology in general.

For example

Example: Let X={a,b,c}, $\tau_1 = \{\varphi, \{a\}, X\}, \, and \; \tau_2 = \{\varphi, \{b\}, X\}, then$

τ_1 and τ_2 are topologies for X, but $\tau_1 \cup \tau_2 = \{\varphi, \{a\}, \{b\}, X\}$,is not

topology for X

Union and Intersection of topologies

2) Intersection of two topologies is also topology.

in general arbitrary intersection of topologies is also topology

for X

Proof: Let τ_1 and τ_2 be two topologies for X, we have to show that $\tau_1 \cap \tau_2$ is also

topology for X.

Closed sets:

Definition: Let (X, τ) be a topological space, A subset F of X is said to be closed set, if its complement is open.

That's mean F is closed set iff F^c is open set.

Example: Let X={a,b,c}, $\tau_1 = \{\varphi, \{a\}, \{a, b\}, X\}$ then closed sets are X, {b,c},{c} and φ.

Remark:

1) Since $\emptyset$ is open, it follows that $\emptyset^c = X$ is closed

2) Since X is open, it follows that $X^c = \emptyset$ is closed

Thus $\emptyset$ and X are open as well as closed sets