Definition 0.4.1 *Let $\mathcal{U}$ be a non-empty finite set of objects called the universe and R be an equivalence relation on $\mathcal{U}$ named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(\mathcal{U}, R)$ is said to be the approximation space. Let $X \subseteq \mathcal{U}$.*

(i) The lower approximation[57] of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x \in \mathcal{U}} \{R(x) : R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.

(ii) The upper approximation [57] of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x \in \mathcal{U}} \{R(x) : R(x) \cap X \neq \phi\}$

(iii) The boundary region[57] of X with respect to R is the set of all objects, which can be classified neither as X nor as not-X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Remark 0.4.2 [57] *If* $(\mathcal{U}, R)$ *is an approximation space and* $X, Y \subseteq \mathcal{U}$*, then*

(i) $L_R(X) \subseteq X \subseteq U_R(X)$.

(ii) $L_R(\phi) = U_R(\phi) = \phi$ *and* $L_R(\mathcal{U}) = U_R(\mathcal{U}) = \mathcal{U}$

(iii) $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$

(iv) $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$

(v) $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$

(vi) $L_R(X \cap Y) = L_R(X) \cap L_R(Y)$

(vii) $L_R(X) \subseteq L_R(Y)$ *and* $U_R(X) \subseteq U_R(Y)$ *whenever* $X \subseteq Y$

(viii) $U_R(X^C) = [L_R(X)]^C$ *and* $L_R(X^C) = [U_R(X)]^C$

(ix) $U_R U_R(X) = L_R U_R(X) = U_R(X)$

(x) $L_R L_R(X) = U_R L_R(X) = L_R(X)$

Remark 1.1.1 *Let* $\mathcal{U}$ *be an universe of objects and* R *be an equivalence relation on* $\mathcal{U}$. *For* $X \subseteq \mathcal{U}$, *let* $\tau_R(X) = \{\mathcal{U}, \phi, L_R(X), U_R(X), B_R(X)\}$. *We note that* $\mathcal{U}$ *and* $\phi \in \tau_R(X)$. *Since* $L_R(X) \subseteq U_R(X)$, $L_R(X) \cup U_R(X) = U_R(X) \in \tau_R(X)$. *Also,* $U_R(X) \cup B_R(X) = U_R(X) \in \tau_R(X)$ *and* $L_R(X) \cup B_R(X) = U_R(X) \in \tau_R(X)$. *Also,* $L_R(X) \cap U_R(X) = L_R(X) \in \tau_R(X)$; $U_R(X) \cap B_R(X) = B_R(X) \in \tau_R(X)$ *and* $L_R(X) \cap B_R(X) = \phi \in \tau_R(X)$.

Definition 1.1.2 *Let $\mathcal{U}$ be an universe, R be an equivalence relation on $\mathcal{U}$ and $\tau_R(X) = \{\mathcal{U}, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq \mathcal{U}$. $\tau_R(X)$ satisfies the following axioms:*

(i) $\mathcal{U}$ and $\phi \in \tau_R(X)$.

(ii) The union of the elements of any sub-collection of $\tau_R(X)$ is in $\tau_R(X)$.

(iii) The intersection of the elements of any finite sub-collection of $\tau_R(X)$ is in $\tau_R(X)$.

That is, $\tau_R(X)$ *forms a topology on* $\mathcal{U}$ *called the nano topology on* $\mathcal{U}$ *with respect to* X*. We call* $(\mathcal{U}, \tau_R(X))$ *as the nano topological space. The elements of* $\tau_R(X)$ *are called nano-open sets.*

Example 1.1.3 *Let* $\mathcal{U}$ $=\{a,b,c,d,e\}$, $\mathcal{U}/R$ $=\{\{a,b\},\{c,d\},\{e\}\}$, *the family of equivalence classes of* $\mathcal{U}$ *by an equivalence relation* R *and* $X=\{a,c,d\}$*. Then* $U_R(X)=\{a,b,c,d\}$, $L_R(X)=\{c,d\}$ *and* $B_R(X)=\{a,b\}$*. Therefore the nano topology,* $\tau_R(X) = \{\mathcal{U},\phi,\{a,b,c,d\},\{c,d\},\{a,b\}\}$.

Proposition 1.1.5 *If $\tau_R(X)$ is the nano topology on $\mathcal{U}$ with respect to X, then the set $B = \{\mathcal{U}, L_R(X), B_R(X)\}$ is the basis for $\tau_R(X)$.*

Proof:

(i) $\bigcup_{A \in B} \mathrm{A} = \mathcal{U}$.

(ii) Consider $\mathcal{U}$ and $L_R(X)$ from B. Let W $= L_R(X)$. Since U $\cap L_R(X) = L_R(X), W \subset \mathcal{U} \cap L_R(X)$ and every x in $\mathcal{U} \cap L_R(X)$ belongs to W. If we consider $\mathcal{U}$ and B_R(X) from B, taking W $= B_R(X), W \subset \mathcal{U} \cap B_R$(X) and every x in $\mathcal{U} \cap B_R$(X) belongs to W, since $\mathcal{U} \cap B_R(X) = B_R$(X). And when we consider $L_R(X)$ and B_R(X), $L_R(X) \cap B_R(X) = \phi$. Thus, B is a basis for $\tau_R(X)$.

Proposition 1.1.6 *Let $\mathcal{U}$ be a non-empty finite universe and $X \subseteq \mathcal{U}$.*

(i) If $L_R(X) = \phi$ and $U_R(X) = \mathcal{U}$, then $\tau_R(X) = \{\mathcal{U}, \phi\}$, the indiscrete nano topology on $\mathcal{U}$.

(ii) If $L_R(X) = U_R(X) = X$, then the nano topology, $\tau_R(X) = \{\mathcal{U}, \phi, L_R(X)\}$.

(iii) If $L_R(X) = \phi$ and $U_R(X) \neq \mathcal{U}$, then $\tau_R(X) = \{\mathcal{U}, \phi, U_R(X)\}$.

(iv) If $L_R(X) \neq \phi$ and $U_R(X) = \mathcal{U}$, then $\tau_R(X) = \{\mathcal{U}, \phi, L_R(X), B_R(X)\}$.

(v) If $L_R(X) \neq U_R(X)$ where $L_R(X) \neq \phi$ and $U_R(X) \neq \mathcal{U}$, then $\tau_R(X) = \{\mathcal{U}, \phi, L_R(X), U_R(X), B_R(X)\}$ is the discrete nano topology on $\mathcal{U}$.