

Directional Derivatives and Gradient Vectors

DEFINITION The derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \quad (1)$$

provided the limit exists.

The **directional derivative** defined by Equation (1) is also denoted by

$$(D_{\mathbf{u}}f)_{P_0}.$$

“The derivative of f at P_0
in the direction of \mathbf{u} ”

EXAMPLE 1 Using the definition, find the derivative of

$$f(x, y) = x^2 + xy$$

at $P_0(1, 2)$ in the direction of the unit vector $\mathbf{u} = (1/\sqrt{2})\mathbf{i} + (1/\sqrt{2})\mathbf{j}$.

Solution Applying the definition in Equation (1), we obtain

$$\begin{aligned}\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} &= \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} && \text{Eq. (1)} \\ &= \lim_{s \rightarrow 0} \frac{f\left(1 + s \cdot \frac{1}{\sqrt{2}}, 2 + s \cdot \frac{1}{\sqrt{2}}\right) - f(1, 2)}{s} \\ &= \lim_{s \rightarrow 0} \frac{\left(1 + \frac{s}{\sqrt{2}}\right)^2 + \left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right) - (1^2 + 1 \cdot 2)}{s} \\ &= \lim_{s \rightarrow 0} \frac{\left(1 + \frac{2s}{\sqrt{2}} + \frac{s^2}{2}\right) + \left(2 + \frac{3s}{\sqrt{2}} + \frac{s^2}{2}\right) - 3}{s} \\ &= \lim_{s \rightarrow 0} \frac{\frac{5s}{\sqrt{2}} + s^2}{s} = \lim_{s \rightarrow 0} \left(\frac{5}{\sqrt{2}} + s\right) = \frac{5}{\sqrt{2}}.\end{aligned}$$

DEFINITION The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0 .

THEOREM 9—The Directional Derivative Is a Dot Product If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds} \right)_{\mathbf{u}, P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}, \quad (4)$$

the dot product of the gradient ∇f at P_0 and \mathbf{u} .

EXAMPLE 2 Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Solution The direction of \mathbf{v} is the unit vector obtained by dividing \mathbf{v} by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of f are everywhere continuous and at $(2, 0)$ are given

$$f_x(2, 0) = (e^y - y \sin(xy))_{(2,0)} = e^0 - 0 = 1$$

$$f_y(2, 0) = (xe^y - x \sin(xy))_{(2,0)} = 2e^0 - 2 \cdot 0 = 2.$$

The gradient of f at $(2, 0)$ is

$$\nabla f|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

$$(D_{\mathbf{u}}f)|_{(2,0)} = \nabla f|_{(2,0)} \cdot \mathbf{u} \quad \text{Eq. (4)}$$

$$= (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j} \right) = \frac{3}{5} - \frac{8}{5} = -1.$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| |\mathbf{u}| \cos \theta = |\nabla f| \cos \theta,$$

Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos \theta$

1. The function f increases most rapidly when $\cos \theta = 1$ or when $\theta = 0$ and \mathbf{u} is the direction of ∇f . That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P . The derivative in this direction is

$$D_{\mathbf{u}}f = |\nabla f| \cos(0) = |\nabla f|.$$

2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f| \cos(\pi) = -|\nabla f|$.
3. Any direction \mathbf{u} orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and

$$D_{\mathbf{u}}f = |\nabla f| \cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

EXAMPLE 3 Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

- (a) increases most rapidly at the point $(1, 1)$.
- (b) decreases most rapidly at $(1, 1)$.
- (c) What are the directions of zero change in f at $(1, 1)$?

Solution

- (a) The function increases most rapidly in the direction of ∇f at $(1, 1)$. The gradient there is

$$(\nabla f)_{(1,1)} = (xi + yj)_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

(b) The function decreases most rapidly in the direction of $-\nabla f$ at $(1, 1)$, which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

(c) The directions of zero change at $(1, 1)$ are the directions orthogonal to ∇f :

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

Algebra Rules for Gradients

1. *Sum Rule:* $\nabla(f + g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f - g) = \nabla f - \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
4. *Product Rule:* $\nabla(fg) = f\nabla g + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

EXAMPLE 5 We illustrate two of the rules with

$$\begin{aligned} f(x, y) &= x - y & g(x, y) &= 3y \\ \nabla f &= \mathbf{i} - \mathbf{j} & \nabla g &= 3\mathbf{j}. \end{aligned}$$

We have

- $\nabla(f - g) = \nabla(x - 4y) = \mathbf{i} - 4\mathbf{j} = \nabla f - \nabla g$ Rule 2
- $\begin{aligned} \nabla(fg) &= \nabla(3xy - 3y^2) = 3y\mathbf{i} + (3x - 6y)\mathbf{j} \\ &= 3y(\mathbf{i} - \mathbf{j}) + 3y\mathbf{j} + (3x - 6y)\mathbf{j} \\ &= 3y(\mathbf{i} - \mathbf{j}) + (3x - 3y)\mathbf{j} \\ &= 3y(\mathbf{i} - \mathbf{j}) + (x - y)3\mathbf{j} = g\nabla f + f\nabla g \end{aligned}$ Rule 4 ■

EXAMPLE 6

- (a) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.
- (b) In what directions does f change most rapidly at P_0 , and what are the rates of change in these directions?

Solution

- (a) The direction of \mathbf{v} is obtained by dividing \mathbf{v} by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$
$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of f at P_0 are

$$f_x = (3x^2 - y^2)|_{(1,1,0)} = 2, \quad f_y = -2xy|_{(1,1,0)} = -2, \quad f_z = -1|_{(1,1,0)} = -1.$$

The gradient of f at P_0 is

$$\nabla f|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of f at P_0 in the direction of \mathbf{v} is therefore

$$\begin{aligned}(D_{\mathbf{u}}f)|_{(1,1,0)} &= \nabla f|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right) \\ &= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.\end{aligned}$$

(b) The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \quad \text{and} \quad -|\nabla f| = -3. \quad \blacksquare$$