Multivariate Analysis Statistical

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Chapter one

Principle of multivariate Statistical Analysis

Why Multivariate Analysis:

Multivariate analysis consists of a collection of methods that can be used when several measurements are made on each individual or objects in on or more samples.

We refer to the measurements as variable and to the individuals or objects as units(research units, sampling units, or experimental units)or observations, the below table contains some examples of Multivariate analysis.

پێک هاتووە لە کۆمەڵێک شێوازێک دەتوانین بەکاری بهێنین،کاتێک جەندین قیاساتێکمان هەبێت،ئێمە قیاساتەکان تەنها بۆ یەک کەس وەرناگرین بەڵکو بۆ هەموو کەسێک وەردەگرین.

(measurements) ئەو قیاساتەی کە وەردەگرین پێی دەڵێن (variables).

وە بە (individuals) دەڵێن(units)یاخود پێی دەڵێین(observations)،

لەبەر ئەوەی ئێمە هەندێک شتمان هەیە دەتوانین قیاسی بکەین بە جاو.

قیاسات مەبەستمان لەوەیە کە قیاسی دەم و جاوی کەسێک دەکەین،بۆ (measurements) نموونە قیاسی بەینی جاو و برۆ یان قیاسی نێوان جاو لەگەڵ جاو،یاخود لوت لەگەڵ جاو یان گوێجکە لەگەڵ گوێجکە،ئینجا جەندەها قیاساتی دیکە،ئەمەش تەنها لەسەر مرۆڤ ناکرێت بەڵکوو لەسەر ئاژەڵانیش دەکرێت،یان قیاسات لەسەر بەرد،یان هەرشتێکی دیکە.

Example of Multivariate Data

#	Units (individuals or objects)	Variables (measurements)
1	students	several exam scores in a single course
2	Students	Eyes, ears, nose, mouth, eyebrows, chin, etc
3	people	Grades in mathematics, history. Music. Art. Physics
4	Skulls	Height .weight. percentage of body fat. Risting heart rate
5	Companies	Length. Width. Cranial capacity
6	Manufactured items	Expenditures for advertising, labour, raw ,materials
7	Applicants for bank loans	Various measurements to check on compliance with specifications
8	Segments of literature	Income, education level, length of residence, saving ,account, current debt load
9	Human hairs	Sentence length, frequency of usage of certain words and of style characteristics
10	Birds	Composition of various elements
11	Human face	Lengths of various bones

1- (student) دەبێتە(units)یان دەبێتە(individual)هەردووکیشیان یەک شتە فەرقی نییە.

ئێمە دەتوانین قیاسی(student)بکەین،بۆ نمونە:مەرحەلە سێی ئامار حەوت مادەیان هەیە،کاتێک ئیمتیحان دەکەن کۆرسی یەکەم،ئەو حەوت مادده دەکاتە حەوت دەرەجە،بۆ نمونە ئێمە سەت قوتابیمان هەبێت،کەواتە دەبێتە سەت دەرەجە بۆ قوتابی یەکەم،سەت دەرەجە بۆ قوتابی دووەم،سەت دەرەجە بۆقوتابی سێیەم،سەت دەرەجە بۆ قوتابی جوارەم،سەت دەرەجە بۆ قوتابی پێنجەم،سەت دەرەجە بۆ قوتابی شەشەم،سەت دەرەجە بۆقوتابی حەوتەم،هەتا حەوسەت قوتابی هەریەکە و سەت دەرەجەی هەیە هەیە،ئەگەر سەیر بکەین ئێمە حەوسەت دەرەجەمان وەرگرتییە،بەڵام (variable)ی یەکەم یانی سەت دەرەجە بۆ مادەی(mathematical)،وە(variable)دووەم یانی سەت دەرەجە بۆ مادەی(operation research),وە (variable)ی سییەم یانی سەت دەرەجە بۆ مادەی(regression Analysis)

هەر مادە و سەت دەرەجەی هەیە.

2 - هەر وەکو یەکەم دەتوانین قیاساتی بۆ وەربگرین.

3 - (people)ئینسان،دەتوانین قیاساتی درێژی ئەو کەسە جەندە،یان کێشی جەندە،یان جەند قەڵەوە،یان قیاسی لێدانی دڵی جەندە،جەستەی مرۆڤ جەندین(variable)ی تێدایە یاخود جەندین قیاساتی تێدایە،ئەو قیاساتانە پێی دەڵێن(variable).

4 - (skulls)قیاساتی: درێژی،پانی،کێشی،مێشکی جەندە.

5 - (companies)کاتێک لەسەر کۆمپانیایێک دیراسەی لەسەر دەکەین،کێشەی هەبوو لەسەر راگەیاندن،رێکلام،بۆ نمونە ئەو رێکلامە جەندی تیدەجێت،جەند وەستای هەیە،جەند کرێکاری هەیە،جەند مادەی خاوی هەیە،وە جەندین(variable)ی دیکەی هەیە دەتوانین ئێمە لەناو کۆمپانیا بەکاربهێنین.

6 - (Manufactured items)درووستکردنی کالا،ئێمە ج قیاساتێکی بۆ دەکەین،بۆ نمونە ئاوە تەماتە،ئێمە دەمانەوێت بزانین ئەو ئاوە تەماتەی کە لە دەرەوە دێت،ئێمە جەندین قیاسات لەو ئاوە تەماتەیە وەردەگرین،بۆ نمونە بزانین رێژەی ئاوی ناو ئەو ئاوە تەماتەیە جەندە،یان رێژەی تەماتەی جەندە،یان ئێکسبایەر بووە یاخود نا.

7 - (Applicants for bank loans)ئەو کەسانەی دەجنە بانکی جۆن قەرز بکەن؟ئەی ئەو بانکە جۆن قەرزیان بداتێ،جەندین(variable)ی تێدایە ئەو کەسەی دێت تەقدیمی پێشکەش دەکات بۆ ئەوەی قەرز وەربگرێت.

8 - (Segments of literature)ئەو کەسەی قەرز دەکات ئاستی زانستی جەندە،ئاستی زانستی زۆر گرینگە بۆ بانکەکەی بۆ ئەوەی بزانێت ئاستی ئەو کەسە جەندە کەدەیەوێ قەرز دەکات،بۆ نمونە ئەو کەسەی کە قەرز دەکات ماستەری هەبێت،یان دکتۆرای هەبێت،ئەو کەسە فەرقی هەیە لەگەڵ کەسێک کە هیج شەهادەی نەبێت،یاخود جەند ئیقامەی هەیە خەلکی ئێرەیە یان خەلکی شوێنی دیکەیە،جەند پارە خەزن دەکات،جەند پارە لە دەرەوە دادەنێت،وە جەند پارە دەباتن،جۆری ئەو مەعامەلەی لە بانکی دەکاتن جۆنە.

9 - (hairs Human)باسی نوسینەوە دەکات،بۆ نمونە نوسینی شعرێک لەلایەن شاعیرێک،ئایە ئەو شاعیرە، درێژی رستەکانی جەند بووە؟وشەکانی کە بەکاری هێناون لە ناو شعرەکانی ج وشەیەکی زۆر بەکارهێناوە،ئەمە(frequency)یە،جەند(frequency)بەکارهێناوە؟یان شێوازی وشەکان،یان ئەو هێمایانەی کە بەکاری هێناون جۆنە،بۆ نمونە ئەگەر بێت و شعرەکانی عبداللە پەشێو بێنین،ئێمە دەتوانین(تحلیل احصاء)ی،بۆ بکەین،دەتوانین بەش بەشی بکەین،بۆ نمونە:بیکەینە بەشی سیاسی،یان بەشی خۆشەویستی،یان بەشی نیشتمان،یان بەشی ئابورییە.

10 - (Birds)بۆ نمونە ئەگەر جۆلەکەیەک بێنین،ئایە قاجەکانی جەند درێژە؟یان جەند کورتە؟یان قاجی کۆترێک جەند درێژە؟یان قاجی نعامەیەک جەند درێژە؟ئەوانە هەمووی فەرقی هەیە.

11 - (Human face)ئەوەمان باس کرد لە سەرەتادا.

Multivariate Analysis is concerned generally with two areas, descriptive and inferential Statistics, and the descriptive field, we often obtain optimal linear combinations of variables ,in the inferential area, many multivariate techniques are extensions of univariate procedures.

(Multivariate Analysis)لە دوو بوار ئیش دەکات یەکێکیان(descriptive)یانی بەس وەسفی ئیشکە بکات یاخود بە جۆری دووەم ئیش بکات ئەویش(inferential)یانی بجێتە ناو (test)کان.کە بەکاری

دەهێنین لە(Multivariate)،بۆ نمونە ئێمە پێنج (variables)مان هەیە،ئێمە،دەتوانین(linear combinations)،بۆ دروست بکەین،(linear combinations)پێک هاتووە لە جەندین هاوکێشە،لە بواری(inferential) ئێمە (t-test)مان خوێند،تەنها بۆ یەک(univariate)،واتە،بۆ یەک(variable)بەڵام ئێستا دەیکەینە(Multivariate)،بۆ ئەوەی (t-test)بۆ یەک (variable)وەربگرین،ئێستا ئێمە (t-test)بەکاردەهێنین بۆ جەند (variables)ێک.

جەند جۆرێکمان هەیە لە(داتا)و(تحليل) کردن.***Basic Types of data and Analysis***

The four basic types are as follows. چوار جۆرمان هەیە لە(داتا)و(تحليل)

1- A single sample with several variables measured on each sampling unit (subjects or object)

.یەک سامپلی وەردەگرین،بۆ نمونە لە قۆناغی یەکی ئامار سامپلێک وەردەگرین،لە دە قوتابیی،دە قوتابییەکە بونە (sample)ئینجا دوای ئەوە جەندین(variable)لەو دە کەسە وەردەگرین،بۆ نمونە درێژیان جەندە،کێشیان جەندە،رەنگی جاویان،قەلەوە یان زەحیفە،جەندین(variable) مان،لەو قوتابیانە وەرگرت,بۆ نمونە نمرەکانیان لە (سمستەری) یەکەم،لە مادەی ئامار،(sample)وەردەگرم،لەگەڵ جەندین(variable),یانی شێوازی یەکەم،یەک(sample)وەردەگرم،وە جەندین(variable)لەخۆ دەگرێت.

2- A single samples with two sets of variables measured on each unit.

(sample)ێکمان وەرگرتییە،بەڵام دوو جۆر(set)لە(variable)وەردەگرین،بۆ نمونە(variable)ێک بۆ کجەکان،وە(variable)ێک بۆ کورەکان،یانی خاڵی یەکەم یەک(set)بوو،وە خاڵی دووەم دوو جۆر(set)وەردەگرین.

3- Two samples with several variables measured on each unit.

دوو(sample)وەردەگرین،لەگەڵ جەندین(variable).جیاوازی لەگەڵ خالی دووەم ئەوەیە،خاڵی دووەم یەک(sample)وەردەگرین،بەڵام لەخاڵی سێیەم دوو(sample)وەردەگرین.

4- three or more samples with several variables measured on each unit.

بۆ سێ(sample)یان زیاتر،وە جەندین(variable)وەرگرتییە کە قیاسمان کردییە لەسەر ئەو(unit)تانە.

وە هەروەها دەتوانین ئەو چوار جۆرە تێکەڵی یەکیان بکەین،یانی جۆری یەکەم لەگەڵ جۆری سێیەم،یان دووەم لەگەڵ جۆری یەکەم،ئەوانی دیکەش بەهەمان شێوە،دەتوانین تێکەڵی یەکیان بکەین.

ئینجا دوای ئەوە نمونە بۆ هەر یەک لەو جوار حالەتەی کە باسمان کرد.

1- A single sample with several variables measured on each sampling unit (subjects or object).

a- test the hypothesis that the means of the variables have specified values.

لێرەدا یەک (sample)مان وەرگرتییە،مەعەدەڵی دەرەجەی قوتابیان لە مادەی بیرکاری،ئێمە دوو(hypothesis)مان هەیە،ئێمە(test)ی،ئەو(hypothesis)دەکەین،کە دەڵێت،

$H_0: m=50$ $H_1: \neq 50$ مەعەدەڵی قوتابیان=50

بۆ نمونە لە مادەی ئامار لە قۆناغی یەک،دەڵێین مەعەدەڵی ئامار=بە مەعەدەڵی ئابوری=بە مەعەدەڵی ئیدارە

ئینجا ئێمە دەڵێین مەعەدەلیان وەکو یەکە،ئینجا ئایە ئەوە تەواوە یان تەواو نییە؟

یەک(sample)وەرگرتییە جەندین(variable)بە خۆوە وەردەگریتن..

B- test the hypothesis that the variable are uncorrelated and have a common variance.

لێرەدا (test)ی (hypothesis)دەکەین،یەک (sample)مان هەیە،بەڵام جەندین (variable)لە خۆوە دەگرێت،ئەو(variable)مانە، ئایە(correlation)لە نێوانیاندا هەیە،یاخود(correlation)لە نێوانیاندا نییە؟

c- find a small set of linear combination of the original variables that summarizes most of the variation in the data (principal components)

دەڵێت مەجموعەیەکی بجوک لە(combination)دروست بکە،لێرەدا جەندین(variable)مان هەیە،دەتوانین بڵێین(variable)ی،یەکەم لەگەڵ(variable)دووەم،یان لەگەڵ(variable)سێیەم،پێکیان وەدەنێم دەیکەمە یەک هاوکێشە،ئەوانە پێی دەوترێت(principal components)دەیکەمە،کۆمبۆنێنتی یەکەم،ڤاریەبڵی یەکەم،لەگەڵ پێنجەم ،لەگەڵ حەوتەم ،ئەوا دەبێتە(components leaner)ئەوکاتە دەبێتە(principal components)ی،دووەم.

d- Express the original variables as linear functions of a smaller set of under_lying variables that accounts for the original variables and intercorrelations (factor Analysis).

باسی (factor Analysis) دەکەین کە جەندین(variable)مان هەیە،ئەوجارە دەڵێین جەندین(function linear)مان هەیە،ئەو(function linear)نانە،لەو (variable)انە،دروست بووە،ئەوا جوینە ناو(factor Analysis).

ئەو چوار نمونانە بۆ خاڵی یەکەم ئەگەر هاتوو یەک(sample)وەربگرین.

2- A single samples with two sets of variables measured on each unit.

له خـــــاڵی دووەم ئەگەر یەک(sample)وەربگـــــرین بەڵام هەمـــــوو(variable)کـــــان)بەیەکەوە نەبن،یـــانی بیکەیـــنە دوو مەجمـــوعە گروپ،یـــانی مەجمـــوعە(variable)لەو دەســـتی جەب،وە مەجموعە(variable)لە دەستی راست.

A- Determine the number ,the size, and the nature of relationships between the two sets of variables (canonical correlation),for example, you may wish to relate a set of interest variables to a set of achievement variables, How much overall correlation is there between there two sets.

لێرەدا بابەتێکمان هەیە پێی دەڵێن(canonical correlation)ئێمە یەک (sample)هەیە،بەڵام ژمارە و قەبارە و جـۆری ئەو عـیلاقەیەی کە هەیە لە نێـوان ئەو دوو مەجمـوعە(variable)ە،دوو مەجمــوعە(variable)مــان هەیە لەیەک(sample)ی،ئەوە پــێی دەڵــێن(canonical correlation)،بــــــــۆ نمــــــــونە دوو جــــــــــۆرە(variable)م هەیە لەســــــــەر یەک(sample)ی،مەجمــوعە(variable)ک پــێی دەوترێــت(inters variables)،ئەو (variable)ی کە لە لامان گرینگە،کە ئێمە زۆر بایەخی پێ دەدەین،وە مەجموعە(variable)دیکە، پــێی دەوترێــت(achievement variable)یــانی ڤــاریەبلی دەســـتکەوتەکان،ئایە جەند(correlation)هەیە لە بەینی ئەو دوو مەجموعەیە؟

B- find a model to predict one set of variables from the other set(multivariate multiple regression).

(regression)پێـک هـاتووە لەیەک(Y)وە لەجەنـدین(X)یـانی یەک(depended)هەیە،لەگەل جەنـدین(independed variable)،باشـەئەدی ئەگەر هـاتوو جەنـدین(depended variable)هەبوو،یانی جەندین(Y)هەبوو،وە جەندین(X)هەبوو؟ئەو کاتە پێی ناڵێین(multiple regression),بەڵکو پێی دەڵێین(multivariable multiple regression)،بەڵام هەر(single sample)ە.

3- Two samples with several variables measured on each unit.

کە دوو(sample)ە،کە هەریەکەو جەندین(variable)ی هەیە

a- compare the means of the variables across the tow samples(Hotellings T^2-Test)

دوو(sample)مـــــــــان هەیە،هەریەکەو جەنـــــــــــدین(variable)ی،لە خــــــــــــۆوە گرتووە،یـــانی(sample)ی،یەکەم،جەنـــد(variable)لەخـــۆوە گرتـــووە،وە(sample)ی،دووەم جەنــــــــــدین(variable)لە خــــــــــۆوە گرتووە.ئەوکــــــــاتە ئــــــــێمە نــــــــاتوانین(T-test)بەکاربهێنین،بەڵکو(Hotellings T^2-Test)بەکاردەهێنین.

B- Find a linear combination of the variable that best separates the two samples(discriminant analysis)

مەوزوعێکمان هەیە،پێی دەڵێن(discriminant analysis)ئێمە دوو(sample)وەردەگرین،بـۆ نمــــــونە ئەوانەی نەخۆشــــــن،لەگەڵ ئەوانەی نەخـــــــۆش نــــــین, ئەو نەخۆشـــــــانە مەجمـوعە(variable)هەیە,وە ئەوانەی کە نەخـۆش نـین مەجمـوعە(variable)هەیە,ئـایە جـۆن

جیاوازی نیوان ئەو دوو(sample)دەکەم؟ئەمە(discriminant analysis)ی،پێ دەڵێن،کە دوو(sample)لەخۆ دەگرێت،وە جەندین(variable)لەخۆ دەگرێت.

C- Find a function of the variables that accurately allocates the units into the two groups (classification analysis)

لێرەشـدا بـابەتێکی دکەمـان هەیە پـێی دەڵـێن(classification analysis)ئەگەر بێـت و دوو(sample)مان، هەبێت،وە جەنـدین(variable)لەخـۆ دەگرێـت،جۆن دەتوانین،بیـان کەینە دوو(groups)جۆن بیان کەینە دوو بەش؟

4- three or more samples with several variables measured on each unit.

لەسێ (sample)زیاتر کە هەر (sample)ێک جەندین(variable)لە خۆوە بگرێت.

A- Compare the means of the variables across the groups(multivariable analysis of variance)

بـــــۆ نمـــــونە پێـــــنج گروپمـــــان هەیە،یـــــانی پێـــــنج(sample)هەیە،هەر (sample)ێک،جەنـــــــــدین(variables)لەخـــــــــۆوە دەگرێـــــــــت،ئەوانە هەرهەمویـــان(mean)هەیە،ئـــێمە(ANOVA)مـــان هەیە،(analysis of variance)یەک(Y)هەیە,وە جەنـــــــــــــــدین(X)هەیە,بەڵام لـــــــــــــــێرەدا جەنـدین(Y)مان،دەبێـت،دەبێتە(multivariable analysis of variance)کە پـێی دەڵێن(MANOVA)

B- Extension of 3(b)to more than two groups.

c- Extension of 3(c)to more than two groups.

یـــــانی ئەگەر بمانەوێـــــت زیـــــاتر لە دوو(sample)وەربگرین،لەجیـــــاتی ئەوەی ئـــــێمە دوو(sample)وەرگرین.

Data Organization

We will use the notation to indicate the particular value of the K^{-th} variable that is observed on the j^{th} item, or trial. That is,

ئێمە جەندین کەسمان هەیە،وە جەندین(variable)بۆ ئەو کەسانە وەرگرتییە،بۆ نمونە دە کەسم وەرگـرتییە،هەر کەسـێک دە(variable)م،لێ وەرگرتییە،یـانی دە قیاسـاتم لێ وەرگرتییە،جـۆن بـزانم قیـاسی دووەمی ئی کەسی دووەمە،وە قیـاسی سـێیەمی ئی کەسی سـێیەمە،جۆن دەنوسرێـت؟بۆ نمـونە درێـژی دە قوتـابی وە کێشی۔ دە قوتابی،ئێستـا دوو(variable)مـان هەیە،دەکـات جەنـد قیاسـمان وەرگرتییە؟دەکاتە (٢٠)قیاسم وەرگرتییە،جونکە ئەو دە کەسە،بۆ درێژی دە قیاسی وەردەگرم،وە بۆ کێشەکەشی دە قیاسم وەرگرتییە،کەواتە دەکاتە(٢٠)قیاسم وەرگرتییە،وە بە ئیحصائی بەم شێوەیە دەنوسرێت:

X_{yk}=measurement of the K^{-th} variable on the j^{th} item.

درێژی=K قیاسی جۆری (ج)بۆ جۆری(ج)=X،

Consequently, N measurement on p variables can be displayed follows.

یانی(N=10)وە(P=2)

=20

Q) there are 4 types of data , what are they?

1- A single sample with several variables measured on each sampling unit (subject or object.

2- A single sample with two sets of variables measured on each unit.

3- two sample with several variables measured on each unit.

4- three or more samples with several variables measured on each unit.

Chapter two

Matrix Algebra

Def: matrix (or rectangular matrix, or {m, n}matrix)is a rectangular array of numbers, that are written M rows and N of column ,the matrix A is accepted in from:

واته پێک هاتووە لە چەند صەفێک و چەند عامودێک،ژمارەی صەفەکان(m)وە ژمارەی عمود(n).

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_1 & m_2 & m_3 & \cdots & m_n \end{bmatrix}$$

Or in abbreviated from A=A{m,n}=(a_{ij})$_{mn,}$i=1,2,……,m, j=1,2,……..,n.

(matrix algebra) هەندێک پرۆسە لە . Some Important operation in Matrix Algebra

1. Transpose

If <u>A</u> is Square matrix then.

I. $(A')' = A$ A= $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}' = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}' = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

II. $(A+B)' = A' + B'$

A=$\begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 8 & 1 \\ 3 & 6 \end{bmatrix}$

$(A+B)'$=$\begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix} + \begin{bmatrix} 8 & 1 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 1+8 & 5+1 \\ 3+3 & 7+6 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 6 & 13 \end{bmatrix}' = \begin{bmatrix} 9 & 6 \\ 6 & 13 \end{bmatrix}$

A'=$\begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}' = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$ B'=$\begin{bmatrix} 8 & 1 \\ 3 & 6 \end{bmatrix}' = \begin{bmatrix} 8 & 3 \\ 1 & 6 \end{bmatrix}$

A'=$\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} + B'$=$\begin{bmatrix} 8 & 3 \\ 1 & 6 \end{bmatrix}$=$\begin{bmatrix} 1+8 & 3+3 \\ 5+1 & 7+6 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 6 & 13 \end{bmatrix}$

III. IF $A'A$ =$A\,A'$ $\rightarrow$ then A is a symmetric matrix, for instance : واتە سێگۆشەکان یەکسانن

A= $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix}' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix}$

ئەگەر بێت و (Transpose)ی بۆ وەربگرین
هەمان ئەنجام بۆمان دەردەچێتەوە

IV. if $A\,A' = 0$ $\rightarrow$ A =0 A=$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$واتە هەموو قیمەتەکانی سفر بێت

A=$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = A = \begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix}$

V. $(AB\ )'$ = $B'\,A'$ A=$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$

$(AB\ )'$=$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} . \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1*3+2*2 & 1*4+2*1 \\ 3*3+4*2 & 3*4+4*1 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 17 & 16 \end{bmatrix} = \begin{bmatrix} 7 & 17 \\ 6 & 16 \end{bmatrix}$

B'=$\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ $A' = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$B'\,A'$=$\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3*1+2*2 & 3*3+2*4 \\ 4*1+1*2 & 4*3+1*4 \end{bmatrix} = \begin{bmatrix} 7 & 17 \\ 6 & 16 \end{bmatrix}$

2. **Multiplication** جاران کردنی دوو ریزکراوە

لە جاران کردندا مەرجە لە ریزکراوەی یەکەم ستونەکان یەکسان بێت،بە ریزەکانی ماتریکسی دووەم.

$A_{3*\underline{5}} . B_{\underline{5}*5} = C_{3*5}$

$I.$ $AI = IA = A$,what is I Matrix?

$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} . I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution:

$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} . I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2*1+3*0 & 2*0+3*1 \\ 4*1+6*0 & 4*0+6*1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 1*2+0*4 & 1*3+0*6 \\ 0*2+1*4 & 0*3+1*6 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

هیج فەرقی نییە هەردووکیان ج(A I)بکەی ج(I A)بکەی.

ئەگەر(I_3)بوو ئەوا سێ یەکی تێدایە $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ وە ئەگەر(I_4)بوو ئەوا چوار یەکی تێدایە

$II.$ $A0 = 0A = 0$ A= $\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}(0) = 0$

$(0)\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix} = 0$

$III.$ In general, AB$\neq BA$

$A = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix}$ $B = \begin{bmatrix} -3 & -3 \\ -1 & 1 \end{bmatrix} \to AB \to A = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix} . B = \begin{bmatrix} -3 & -3 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -4*-3+2*-1 & -4*-3+2*1 \\ 2*-3+(-4*-2) & 2*-3+(4*1) \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ -2 & -10 \end{bmatrix}$

حالەتی یەکەم (AB)

ئینجا دێینە سەر حاڵەتی دووەم(BA)

$B = \begin{bmatrix} -3 & -3 \\ -1 & 1 \end{bmatrix} . A = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix} =$

$\begin{bmatrix} -3*-4+(-3*2) & -3*2+(-3*-4) \\ -1*-4+1+2 & -1*2+1*-4 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & -6 \end{bmatrix}$

کەواتە یەکسان نەبوون(AB$\neq BA$)

$But\ AB = BA\ IF\ \ A = B\ OR\ B = A$

$A = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} . \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1*5+(-2*-2) & 1+(-2+(-2*1)) \\ 2*5+5*(-2) & 2*-2+5*1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} . \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 5*1+(-2*2) & 5*-2+(-2*5) \\ -2*1+1*2 & -2*-2+1*5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

یانی ئەگەر هاتوو(AیانB)ی(Identity matrix)بوو ئەوکاتە دەڵێین(AB = BA)

حاڵەتی دووەم ئەگەر هاتوو(AیانB)ی(Identity matrix)بوو ،ئەوکاتە دەڵێین(AB = BA)

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1*1+0*3 & 1*2+0*4 \\ 0*1+1*3 & 0*2+1*4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} . \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1*1+2*0 & 1*0+2*1 \\ 3*1+4*0 & 3*0+4*1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

حاڵەتی سێیەم ئەگەر هاتوو(AیانB)(Zero matrix)بوو،ئەوکاتە دەڵێین(AB = BA)

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} . \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} . \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

حاڵەتی چوارەم ئەگەر هاتوو($A=B^{-1}$)یان($B=A^{-1}$)

$A.B=B.A=I$, $B.B^{-1}= B^{-1}.B=I$, $A.A^{-1}= A^{-1}.A=I$ $\qquad A=B^{-1}$, $B=A^{-1}$

A=$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ B=$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$= I

حاڵەتێکی دیکە ئەگەر هاتوو(if A or B is diagonal matrix, for instance)

واتە ج(A)یان(B)سێگۆشەکانیان (0)بێت،وە(diagonal)هەر ژمارەیەک بێت،ئەوکاتە(AB = BA).

A=$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ B=$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ ڕێگای یەکەم

AB=$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} . \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1*3+0*0+0*0 & 1*0+0*1+0*0 & 1*0+0*8+0*2 \\ 0*3+4*0+0*0 & 0*0+4*1+0*0 & 0*0+4*0+0*2 \\ 0*3+0*0+5*0 & 0*0+0*1+5*0 & 0*0+0*0+5*2 \end{bmatrix}$

=$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

BA=$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 3*1+0*0+0*0 & 3*0+0*4+0*0 & 3*0+0*0+0*5 \\ 0*1+1*0+0*0 & 0*0+1*4+0*0 & 0*0+1*0+0*5 \\ 0*1+0*0+2*0 & 0*0+0*4+2*0 & 0*0+0*0+2*5 \end{bmatrix}$

BA=$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

(1*4*5)*(3*1*2)=120 ڕێگای دووەم ئاسانترە , (3*1*2). (1*4*5)=120

AB=BA ڕێگای دووەم

IV. If D_1 and D_2 are two diagonal matrices of all the same order then $D_1D_2=D_2D_1$.

$D_1=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$ $D_2=\begin{bmatrix}3&0&0\\0&6&0\\0&0&8\end{bmatrix}$ ڕێگای یەکەم

$D_1D_2\begin{Bmatrix}1&0&0\\0&2&0\\0&0&3\end{Bmatrix}.\begin{bmatrix}3&0&0\\0&6&0\\0&0&8\end{bmatrix}=\begin{bmatrix}1*3+0*0+0*0&1*0+0*6+0*0&1*0+0*0+0*8\\0*3+2*0+0*0&0*0+2*6+0*0&0*0+2*0+0*8\\0*3+0*6+3*0&0*0+0*6+3*0&0*0+0*6+3*8\end{bmatrix}$

$=\begin{bmatrix}3&0&0\\0&12&0\\0&0&24\end{bmatrix}$=3*12*24=864

$D_2D_1=\begin{bmatrix}3&0&0\\0&6&0\\0&0&8\end{bmatrix}.\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}=\begin{bmatrix}3*1+0*0+0*0&3*0+0*2+0*0&3*0+0*0+0*3\\0*1+6*0+0*0&0*0+6*2+0*0&0*6*0+0*3\\0*1+0*0+8*0&0*0+0*2+8*0&0*0+0*0+8*3\end{bmatrix}$

$\begin{bmatrix}3&0&0\\0&12&0\\0&0&24\end{bmatrix}$=3*12*24=864

ڕێگای دووەم:

$D_1=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$=1*2*3=6 , $D_2=\begin{bmatrix}3&0&0\\0&6&0\\0&0&8\end{bmatrix}$=3*6*8=144

6*144=864

$D_2=\begin{bmatrix}3 & 0 & 0\\0 & 6 & 0\\0 & 0 & 8\end{bmatrix}$=3*6*8=144 , $D_1=\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}$=1*2*3=6 ,144*6=864

3- Determinants:

Def[n]: for any square matrix An.n , then the determinant of A (|A|) is defined by:

واتە دەبێت ژمارەی(ریزەکان=ژمارەی ستونەکان)یانی دەبێت(square matrix)بێت.

$|\mathbf{A}| = \sum a_{ij} A_{ij}$ a = نرخەکانی ریزکراوە, j=ژمارەی سونەکان، i=ژمارەی ریزەکان

Where Aij is the cofactor aij which is equal to Aij $=(-1)^{i+j} \times$ minor the minor of element aij is the determinant of the sub matrix A obtained by deleting the i^{th} row amd the j^{th} column of A.

$$\mathbf{A}=\begin{bmatrix}\boldsymbol{a_{11}} & \boldsymbol{a_{12}} & \boldsymbol{a_{13}}\\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} & \boldsymbol{a_{23}}\\ \boldsymbol{a_{31}} & \boldsymbol{a_{32}} & \boldsymbol{a_{33}}\end{bmatrix}$$

Arrow method to find determinant:

$$|\mathbf{A}|=\begin{bmatrix}\boldsymbol{a_{11}} & \boldsymbol{a_{12}} & \boldsymbol{a_{13}}\\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} & \boldsymbol{a_{23}}\\ \boldsymbol{a_{31}} & \boldsymbol{a_{32}} & \boldsymbol{a_{33}}\end{bmatrix}\begin{matrix}\boldsymbol{a_{12}} & \boldsymbol{a_{13}}\\ \boldsymbol{a_{22}} & \boldsymbol{a_{23}}\\ \boldsymbol{a_{32}} & \boldsymbol{a_{33}}\end{matrix}$$

$$|\boldsymbol{A}| = (\boldsymbol{a_{11}}.\boldsymbol{a_{22}}.\boldsymbol{a_{33}}) + (\boldsymbol{a_{12}}.\boldsymbol{a_{23}}.\boldsymbol{a_{32}}) + (\boldsymbol{a_{13}}.\boldsymbol{a_{22}}.\boldsymbol{a_{33}}) - (\boldsymbol{a_{31}}.\boldsymbol{a_{22}}.\boldsymbol{a_{13}}) - (\boldsymbol{a_{32}}.\boldsymbol{a_{23}}.\boldsymbol{a_{12}}) - (\boldsymbol{a_{33}}.\boldsymbol{a_{22}}.\boldsymbol{a_{13}})$$

For example:

A= $\begin{vmatrix} -7 & -10 & 4 \\ 3 & -9 & 2 \\ 7 & 1 & 2 \end{vmatrix}$,solution→|A|= $\begin{vmatrix} -7 & -10 & 4 \\ 3 & -9 & 2 \\ 7 & 1 & 2 \end{vmatrix}\begin{matrix} -7 & -10 \\ 3 & -9 \\ 7 & 1 \end{matrix}$

$$|A| = (-7\times -9\times 2) + (-10\times 2\times 7) + (4\times 3\times 1) - (7\times -9\times 4) - (1\times 2\times -7) - (2\times 3\times -10)$$
$$= 126 - 140 + 12 + 252 + 14 + 60 = 322$$

Theorems about the <u>properties</u> of determinants:

1- the determinant of a diagonal matrix of identity matrix is the product of diagonal elements.

جیاوازی نێوان(diagonal)و(identity)بەم شێوەیە:

$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 24 \end{bmatrix}$ → **diagonal** →سێگۆشەکانیان سفرە، وە ئەو ژمارانەی سەهمی بەسەردا هاتووە نابێت،(١)بێت

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ → **identity**→سێگۆشەکانیان سفرە وە ئەو ژمارانەی سەهمی بەسەردا هاتووە دەبێت(١)بێت

For example: into diagonal:

$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{bmatrix}$,solution→ 3×5×8=120

For example: into identity:

A= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ solution→1× 1 × 1=1

2. let A be (n*n) matrix, then B obtained from A by multiply row(or column) of A by a scalar C ,then|B|=C|A|.

Example// A= $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ **C=5**

S0lution// 5× $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ **=B=** $\begin{bmatrix} 10 & 15 \\ 5 & 20 \end{bmatrix}$

بەهۆی نرخی(C)ی، و مەصفوفەی(B)یم،دەست کەوت،دێینە سەر بەشی دووەم.

|B|= $\begin{bmatrix} 10 & 15 \\ 5 & 20 \end{bmatrix}$ **=10** $\times 20 - 5 \times 15$ **=200-75= 125**

C|A|=5× $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ = $\begin{bmatrix} 10 & 15 \\ 5 & 20 \end{bmatrix}$ **=200-75= 125**

کەواتە هەمان نرخ دەرهات.

3- if B obtained from A by interchanging two rows or columns, then |B|= -|A|

$|B| = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = -|A| = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ (واتە گۆڕینی دوو ریز)

$-|A| = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = |B| = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

4- if a row or column of a square matrix is zero, then the determinant zero.

A= $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \\ 2 & 4 & 1 \end{bmatrix}$ **=0, solution→**(سفر)واتە یەکێک لە کۆلۆمەکان هەموو ژمارەکانی (سفر)ئەوکاتە نەتیجەی کۆتایی دەکاتە

ئینجا ئێمە ئەو کۆلۆمە هەلدەبژێرین کە(سفرە)

0 det $\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} + 0\, det \begin{vmatrix} 3 & 4 \\ 4 & 1 \end{vmatrix} - 0$ **det** $\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix}$

$0(1*4-2*3)+0(3*1-4*4)-0(1*1-2*4)$

$=0(-2)+0(-5)-0(-7)=0$

بەڵام پێویست ناکات هەموو ختواتەکان بکەین مادەم یەکێک لە عامووەدەم یان یەکێک لە سەفەکان سفر بوو ئەوا یەکسەر دەنوسین دەکاتە سفر، وەکوو ئەو نمونەی سەرەوە، یەکێک لەسەفەکان هەمووی سفرە.

5- if two rows or columns in A are similar, then |A|=0

$A=\begin{bmatrix} 4 & 1 & 2 \\ 3 & 2 & 5 \\ 4 & 1 & 2 \end{bmatrix}=$ یانی ئەگەر هاتوو دوو سەف یان دوو ریز وەکو یەک بن ئەکاتە دەکاتە(0)

لێرەدا ریزی یەکەم و سێەم وەکو یەکە بۆیە دەکاتە (سفر)

$|A|=\begin{bmatrix} 4 & 1 & 2 \\ 3 & 2 & 5 \\ 4 & 1 & 2 \end{bmatrix}=$ریزی یەکەم هەڵدەبژێرین

$|A|=\begin{bmatrix} 4 & 1 & 2 \\ 3 & 2 & 5 \\ 4 & 1 & 2 \end{bmatrix}=4+\begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}-1\begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}+2\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$

$|A|=4(2*2-1*5)-1(3*2-4*5)+2(3*1-4*2)=4(4-5)-(6-20)+(3-8)$

$|A|=4(-1)-1(-14)+2(-5)=-4+14-10=0$ کەواتە سفر دەرچوو

بەڵام پێویست ناکات شیکاری بۆ بکەین،من تەنها بۆ سەلماندن شیکارم کردووە.

6- IF A has an inverse, then $|A^{-1}| = |A|^{-1} = \frac{1}{|A|}$

$|A^{-1}| = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$

$|A^{-1}| = \frac{1}{|A|}$adj A $\qquad |A| = (4*6)-(2*7) = 10$,adj= $\begin{bmatrix} 6 & -7 \\ -2 & 2 \end{bmatrix}$

$|A^{-1}| = \frac{1}{10}\begin{bmatrix} 6 & -7 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{6}{10} & \frac{-7}{10} \\ \frac{-2}{10} & \frac{2}{10} \end{bmatrix} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$

$|A|^{-1} = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} = (4*6)-(2*7) = 10$, adj= $\begin{bmatrix} 6 & -7 \\ -2 & 2 \end{bmatrix}$

$|A|^{-1} = \frac{1}{10}\begin{bmatrix} 6 & -7 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{6}{10} & \frac{-7}{10} \\ \frac{-2}{10} & \frac{2}{10} \end{bmatrix} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$, $|A^{-1}| = |A|^{-1}$

7. If A and B have <u>determinants</u> and in the same order, then |AB| =|A||B|.

For example:

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix}$$

$|A| = \begin{vmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{vmatrix}$ =ریزی یەکەم هەڵدەبژێرین

$$|A| = (1)\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 0 & 2 \\ 1 & O \end{vmatrix} + 2\begin{vmatrix} 0 & 3 \\ 1 & O \end{vmatrix}$$
$$= 1(3*1-0*2) + 2(0*0-1*2) + 3(0*0-1*\ 2) = 7$$

B= $\begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix}$ =ریزی یەکەم هەڵدەبژێرین

$|B| = 2\begin{vmatrix} -1 & -2 \\ 1 & -2 \end{vmatrix} + 0\begin{vmatrix} 0 & -2 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ 3 & 1 \end{vmatrix}$ =2(-1*-2)-(1*-2)+0(0*-2)-(3*-2)+(0*1)-(3*-1)=11

|A||B|=7*11=77

|AB|=$\begin{bmatrix}1 & -2 & 2\\0 & 3 & 2\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\0 & -1 & -2\\3 & 1 & -2\end{bmatrix}$=

$$\begin{bmatrix}1*2+(-2)*0+2*3 & 1*0+(-2)(-1)+ & 1*1+(-2)(-2)+2*(-2)\\0*2+3*0+2*3 & 0*0+3*-1+2*1 & 0*1+3*-2*+2*-2\\1*2+0*0+1*3 & 1*0+0*-1+1*1 & 1*1+0*-2+1*-2\end{bmatrix}$$

$\begin{bmatrix}8 & 4 & 1\\6 & -1 & -10\\5 & 1 & -1\end{bmatrix}$=ریزی یەکەم هەڵدەبژێرین

$$8\begin{bmatrix}-1 & -10\\1 & -1\end{bmatrix}-4\begin{bmatrix}6 & -10\\5 & -1\end{bmatrix}+\begin{bmatrix}6 & -1\\5 & 1\end{bmatrix}$$

8(-1*-1)-(1*-10)+4(6*-1)+ (5*-10)+1(6*1)-(5*-1)= 8(1+10)+4(-6+50)+(6+5)=77

$|A||B|=|AB|$

77=77

4- Matrix Inverse:

Def^n :the inverse of matrix $A_{n\times n}$= $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{31} & a_{32} & \cdots & a_{3n} \end{bmatrix}$

Could be found as follows.

$A^{-1}=\frac{adj(a)}{|A|}$

Example: $A=\begin{bmatrix} 2 & 2 \\ 4 & 5 \end{bmatrix}$

Solution: $A^{-1}=\frac{adj(a)}{|A|}$

یانی ژماره(2 و 5 ،)شوێنیان دەگۆرین،وە ژمارە (2 و 4)بەس نیشانەیان دەگۆرین

$Adj=\begin{bmatrix} 2 & 2 \\ 4 & 5 \end{bmatrix}=\begin{bmatrix} 5 & -2 \\ -4 & 2 \end{bmatrix}$

$|A|=\begin{bmatrix} 2 & 2 \\ 4 & 5 \end{bmatrix}=(2*5)-(4*2)=2$

$A^{-1}=\frac{\begin{bmatrix} 5 & -2 \\ -4 & 2 \end{bmatrix}}{2}=\begin{bmatrix} 2.5 & -1 \\ -2 & 1 \end{bmatrix}$ یانی هەموو ژمارەکان دابەشی دوو دەکەین

Properties matrix inverse:

1- $(A^{-1})^{-1}=A$ یانی دوو جار **(inverse)** بۆ وەربگرین

Example: $A=\begin{bmatrix} 3 & 5 \\ 8 & 6 \end{bmatrix}$

$(A^{-1})=\begin{bmatrix} 3 & 5 \\ 8 & 6 \end{bmatrix}=\begin{bmatrix} 6 & -5 \\ -8 & 3 \end{bmatrix}$ بۆ جاری دووەم **(inverse)**بۆ وەردەگرین

$=\begin{bmatrix} 6 & -5 \\ -8 & 3 \end{bmatrix}=\begin{bmatrix} 3 & 5 \\ 8 & 6 \end{bmatrix}$کەواتە وەکو خۆی لێهاتەوە

$Example//\ Adj(A) = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$

Solution// عامودی یەکەم هەڵدەبژێرین

$(A) = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$

A_{ij} =$(-1)^{i+j}$ =$(-1)^{1+1}$= 1|M_{11}|=$\begin{bmatrix} 1 & 6 \\ 4 & 0 \end{bmatrix}$ =(1*0) – (4*6) = -24

A_{ij} =$(-1)^{i+j}$=$(-1)^{1+2}$=-|M_{12}|=$\begin{bmatrix} 0 & 5 \\ 4 & 0 \end{bmatrix}$ = -(0*0) – (4*5) = 20

A_{ij} =$(-1)^{i+j}$=$(-1)^{1+3}$=|M_{13}|=$\begin{bmatrix} 0 & 5 \\ 1 & 6 \end{bmatrix}$ = - (0*6) - (1*5) =- 5

ئینجا عامودی دووەم هەڵدەبژێرین

$(A) = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$

A_{ij} =$(-1)^{i+j}$=$(-1)^{2+1}$=|M_{21}|=$\begin{bmatrix} 2 & 6 \\ 3 & 0 \end{bmatrix}$ = - (2*0) - (3*6) = 18

A_{ij} =$(-1)^{i+j}$=$(-1)^{2+2}$=|M_{22}|=$\begin{bmatrix} 1 & 5 \\ 3 & 0 \end{bmatrix}$ = (1*0) - (3*5) = -15

A_{ij} =$(-1)^{i+j}$=$(-1)^{2+3}$=|M_{23}|=$\begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix}$ = - (1*6) - (2*5) = -4

ئەوجارە عامودی سێیەم هەڵدەبژێرین

$(A) = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$

A_{ij} =$(-1)^{i+j}$=$(-1)^{3+1}$=|M_{31}|=$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ = (2*4) - (3*1) = 5

A_{ij} =$(-1)^{i+j}$=$(-1)^{3+2}$=|M_{32}|=$\begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$ = - (1*4) - (3*0)=- 4

A_{ij} =$(-1)^{i+j}$=$(-1)^{3+3}$=$|M_{33}|$=$\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}$ = (1*1) - (2*0)= 1

Example: $\begin{bmatrix}\mathbf{2} & \mathbf{0} & \mathbf{1}\\\mathbf{0} & \mathbf{2} & \mathbf{-1}\\\mathbf{1} & \mathbf{-1} & \mathbf{3}\end{bmatrix} = \begin{bmatrix}\mathbf{5} & \mathbf{-1} & \mathbf{-2}\\\mathbf{-1} & \mathbf{5} & \mathbf{-2}\\\mathbf{-2} & \mathbf{-2} & \mathbf{4}\end{bmatrix}$

A_{ij} =$(-1)^{i+j}$ =$(-1)^{1+1}$= 1$|M_{11}|$=$\begin{bmatrix}2 & -1\\-1 & 3\end{bmatrix}$ =(2*3) – (-1*-1) = **5**

A_{ij} =$(-1)^{i+j}$=$(-1)^{1+2}$=-$|M_{12}|$=$\begin{bmatrix}0 & 1\\-1 & 3\end{bmatrix}$ = -(0*3) – (-1*1) = **-1**

A_{ij} =$(-1)^{i+j}$=$(-1)^{1+3}$=$|M_{13}|$=$\begin{bmatrix}0 & 1\\2 & -1\end{bmatrix}$ = (0*-1) - (2*1) = **-2**

ئینجا عامودی دووەم هەڵدەبژێرین

A_{ij} =$(-1)^{i+j}$=$(-1)^{2+1}$=$|M_{21}|$=$\begin{bmatrix}0 & -1\\1 & 3\end{bmatrix}$ = - (0*3) - (1*-1) = **-1**

A_{ij} =$(-1)^{i+j}$=$(-1)^{2+2}$=$|M_{22}|$=$\begin{bmatrix}2 & 1\\1 & 3\end{bmatrix}$ = (2*3) - (1*1) = **5**

A_{ij} =$(-1)^{i+j}$=$(-1)^{2+3}$=$|M_{23}|$=$\begin{bmatrix}2 & 1\\0 & -1\end{bmatrix}$ = - (2*-1) - (0*1) = **2**

ئەوجارە عامودی سێیەم هەڵدەبژێرین

A_{ij} =$(-1)^{i+j}$=$(-1)^{3+1}$=$|M_{31}|$=$\begin{bmatrix}0 & 2\\1 & -1\end{bmatrix}$ = (0*-1) - (1*2) = **-2**

A_{ij} =$(-1)^{i+j}$=$(-1)^{3+2}$=$|M_{32}|$=$\begin{bmatrix}2 & 0\\1 & -1\end{bmatrix}$ = - (2*-1) - (1*0)= **2**

A_{ij} =$(-1)^{i+j}$=$(-1)^{3+3}$=$|M_{33}|$=$\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$ = (2*2) - (0*0)= **4**

2- $(AB)^{-1} = B^{-1} A^{-1}$

Example: $A=\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ $B=\begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}$

$(AB)^{-1}=\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.\begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}=\begin{bmatrix} 2*4+3*5 & 2*2+3*3 \\ 4*4+5*5 & 4*2+5*3 \end{bmatrix}=\begin{bmatrix} 13 & 13 \\ 41 & 23 \end{bmatrix}$

کاتێک(**AB**)پێکەوە جارانمان کرد ئینجا دێین(**inverse**)وەردەگرین.

$(AB)^{-1}=\begin{bmatrix} 13 & 13 \\ 41 & 23 \end{bmatrix}$

$Adj=\frac{adj(ab)}{|AB|}$,$adj=\begin{bmatrix} 23 & -13 \\ -41 & 13 \end{bmatrix}$,$|AB|=\begin{bmatrix} 13 & 13 \\ 41 & 23 \end{bmatrix}$=(13*23)-(41 *13)=529 -533=-4

$(AB)^{-1}=\frac{\begin{bmatrix} 23 & -13 \\ -41 & 13 \end{bmatrix}}{-4}=\begin{bmatrix} \frac{-23}{4} & \frac{13}{4} \\ \frac{41}{4} & \frac{-13}{4} \end{bmatrix}$

ئینجا دێین (**inverse**)ی(**A**)(**B**) بەجیاواز وەردەگرین ئینجا جارانی یەکتریان دەکەین.

$A^{-1}=\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix}$

|A|=5*2-12

=-2

$,\frac{\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix}}{-2}=\begin{bmatrix} \frac{-5}{2} & \frac{3}{2} \\ \frac{4}{2} & 1 \end{bmatrix}$

$|B|=\begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}$=(3*4)-(2 *5)=2 , $Adj=\frac{adjA}{|A|}=\frac{\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}}{2}$

$B^{-1}=\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$

$\begin{bmatrix} \frac{3}{2} & -1 \\ \frac{-5}{2} & 2 \end{bmatrix}$

$$B^{-1}A^{-1}=\begin{bmatrix} \frac{3}{2} & -1 \\ \frac{-5}{2} & 2 \end{bmatrix}\begin{bmatrix} \frac{-5}{2} & \frac{3}{2} \\ \frac{4}{2} & 1 \end{bmatrix}=\begin{bmatrix} \frac{3}{2}*\frac{-5}{2}+(-1*\frac{4}{3}) & \frac{3}{2}*\frac{3}{2}+(-1*1) \\ \frac{-5}{2}*\frac{-5}{2}+2*\frac{4}{2} & \frac{-5}{2}*\frac{3}{2}+2*1 \end{bmatrix}=\begin{bmatrix} \frac{-23}{4} & 13 \\ \frac{41}{4} & \frac{-13}{4} \end{bmatrix}$$

if K is non zero scalar and A has in inverse then.

$$(KA)^{-1}=\frac{1}{K}A^{-1}$$

Example: K=5 ,A= $\begin{bmatrix} 7 & 5 \\ 3 & 2 \end{bmatrix}$

Solution//: det= (7*2) -(3*5)=1 ,adj= $\begin{bmatrix} 2 & -5 \\ -3 & 7 \end{bmatrix}$

$$(5A)^{-1}=5^{-1}A^{-1}=\frac{1}{5(1)}\begin{bmatrix} 2 & -5 \\ -3 & 7 \end{bmatrix}=\frac{\begin{bmatrix} 2 & -5 \\ -3 & 7 \end{bmatrix}}{5}=\begin{bmatrix} \frac{2}{5} & -1 \\ \frac{-3}{5} & \frac{7}{5} \end{bmatrix}$$

5- ***Orthogonal Matrix***

def[th] =the square matrix A is an Orthogonal if: $A^{-1}=A$. An example of this kind of matrices is as follows:

$$A=\begin{bmatrix} 0 & 1` & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}=A^{-1}=A^{T}=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

مەرجی یەکەم دەبێت(square)بێت،وە کاتێک پێی دەڵێن(orthogonal)ئەگەر(inverse)و(transpose
)مان بۆ وەرگرت،هەمان نرخ دەرهات.

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}=A'\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A^{-1}=\frac{adj}{|A|}$$

دۆزینەوەیان تۆزەک قورسە لە جوار بای جواردا.

Example//

$$A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

Solution//

$$A' =\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$= A^{-1}=\frac{adj}{|A|}$$

$$\text{Det } |A|=(\cos\theta * \cos\theta) - (\sin\theta * -\sin\theta)$$

$$\text{Det } |A|=\cos2\theta + \sin2\theta = 1$$

$$\text{Adj} =\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$A^{-1} =\frac{\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}}{1}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

کەواتە هەم(transpose)وە هەم(inverse)یەکسان بوون،کەواتە دەبێتە(orthogonal).

If A is an orthogonal matrix then A exists and is orthogonal:

Proof:

$$A^{-1} = A'$$

بە وەرگرتنی(INVERSE)بۆ هەردوولا

$$(A^{-1})^{-1} = (A')^{-1}$$

کەواتە(inverse)لەگەل(inverse)دەروات،بەس(A)دەمێنێتەوە،ئەوە لای راست،وە لای جەب،(A′)
بەخۆی دەڵێت یەکسانە بە(A^{-1})

یانی بەم شێوەیە لێدێت.

$(A^{-1})^{-1} = (A^{-1})^{-1}$

A= A

IF A and B are two orthogonal matrix and have the same order, then (A,B)is an orthogonal.

Proof:

واتە (orthogonal)وە هەم(A)هەم(B)بوون،وە هەمان (Order)بوون.

$A^{-1} = (A)'$, $B^{-1} = (B)'$

$(AB)^{-1} = (AB)$

$(AB)^{-1} = B^{-1}\ A^{-1}$ ∴ A ,B are orthogonal.

$(AB)^{-1'} = B^{-1}\ A^{-1}$

$\therefore B^{-1}\ A^{-1} = (B)'\ (A)'$

The determinant of an orthogonal matrix is either (+1)or(-1).

مەبەستی لەوەیە کەوا(orthogonal)یان سالب یەک یان موجەب یەک دەردەچێت.

6- ***Idempotent Matrix***

If A square matrix of order n then is called Idempotent matrix if :

مەبەستی لەوەیە ئەگەر هاتوو مەصفوفەیەک جانی خۆی کرا هەمان نەتیجە دەرهات ئەوا پێی دەڵێن(Idempotent matrix)

$$A^2 = A.A = A$$

$$Example// A = \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$$

$$S0lution// A^2 = A.A = A$$

A = $\begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$. A = $\begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}\begin{bmatrix} \frac{16}{25}+\frac{4}{25} & -\frac{8}{25}-\frac{2}{25} \\ -\frac{8}{25}-\frac{2}{25} & \frac{4}{25}+\frac{1}{25} \end{bmatrix}$=$\begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$

یانی هەمان دالە جارانی خۆی دەکەین،دەبێت،وە دەبێت،هەمان نرخی پێشو دەربجێت.

وە هەروەها مەرج نییە،تەنها یەک جار جارانی خۆی بکەینەوە دەکرێت بەم شێوەیەش بێت(A^3)یانی دووجار جارانی خۆی بکەینەوە،یان(A^4)جارانی خۆی بکەینەوە.

$$A^3 = A.A.A$$

Example/

$$A^3 = \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$$

$$A.A.A = \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}.\begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}.\begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$$

$$.AA = \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}.\begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}\begin{bmatrix} \frac{16}{25}+\frac{4}{25} & -\frac{8}{25}-\frac{2}{25} \\ -\frac{8}{25}-\frac{2}{25} & \frac{4}{25}+\frac{1}{25} \end{bmatrix}=\begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$$

$$A^2 . A = \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}.\begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix} =$$
$$\begin{bmatrix} \frac{16}{25}+\frac{4}{25} & -\frac{8}{25}-\frac{2}{25} \\ -\frac{8}{25}-\frac{2}{25} & \frac{4}{25}+\frac{1}{25} \end{bmatrix}=\begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$$

کەواتە هەمان نەتیجە دەردێت.بەلام لە تاقیکردنەوە پێویست ناکات،شیکاری بکەین،راست،نەتیجەکە دادەنێن،جونکە مامۆستا لە لەسەر ئەسیلە پێمان دەلێت(***Idempotent Matrix***)دە.

Example// let X = $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, show that $\left(\frac{XX'}{X'X}\right)^2$ is an **Idempotent** matrix.

Solution:

دەبێت یەکەم جار بەهۆی(X)(transpose)بدۆزینەوە ئینجا دەبێت جاران و دابەشەکە جێ بەجێ بکەین.

$$X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = X' = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$$

ئنجا جارانی یەکتریان دەکەین.

$$X.X' = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.\begin{bmatrix} 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1*1 & 1*2 & 1*-1 \\ 2*1 & 2*2 & 2*-1 \\ -1*1 & -1*2 & -1*-1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}$$

ئێستا نرخی کەرتی سەرەوەمان دەرهێنا ئینجا دێینە سەر نرخی کەرتی خوارەوە.

$$X.X' = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}.\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 1*1 + 2*2 + (-*-1) = 6$$

ئینجا دێین دابەشەکە دەکەین.

$$=\left(\frac{XX'}{X'X}\right)^2 = \frac{\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}}{6} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{-1}{6} \\ \frac{1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{6} & \frac{-1}{3} & \frac{1}{6} \end{bmatrix}$$

ئینجا دوای ئەوە لەبەر ئەوەی توان دووە دەبێت جارانی خۆی بکەینەوە ،وە لە هەمام کاتدا دەبێت،هەمان نرخ دەربچێتەوە،جونکە(Idempotent)ە.پێویست ناکات،لە تاقیکردنەوە جارانی بکەین،من تەنها بۆ ساغکردنەوە دەیکەم.

$$\mathrm{x.x} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{-1}{6} \\ \frac{1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{6} & \frac{-1}{3} & \frac{1}{6} \end{bmatrix}.\begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{-1}{6} \\ \frac{1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{6} & \frac{-1}{3} & \frac{1}{6} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{6}*\frac{1}{6}+\frac{1}{3}*\frac{1}{3}+(\frac{-1}{6}*\frac{-1}{6}) & \frac{1}{6}*\frac{1}{3}+\frac{1}{3}*\frac{2}{3}+(\frac{-1}{6}*\frac{-1}{3}) & \frac{1}{6}*\frac{-1}{6}+\frac{1}{3}*\frac{-1}{3}+\frac{-1}{6}*\frac{1}{6} \\ \frac{1}{3}*\frac{1}{6}+\frac{2}{3}*\frac{1}{3}+\frac{-1}{3}*\frac{-1}{6} & \frac{1}{3}*\frac{1}{3}+\frac{2}{3}*\frac{2}{3}+\frac{-1}{3}*\frac{-1}{3} & \frac{1}{3}*\frac{-1}{6}+\frac{2}{3}*\frac{-1}{3}+\frac{-1}{3}*\frac{1}{6} \\ \frac{-1}{6}*\frac{1}{6}+\frac{-1}{3}*\frac{1}{3}+\frac{1}{6}*\frac{-1}{6} & \frac{-1}{6}*\frac{1}{3}+\frac{-1}{3}*\frac{2}{3}+\frac{1}{6}*\frac{-1}{6} & \frac{-1}{6}*\frac{-1}{6}+\frac{-1}{3}*\frac{-1}{3}+\frac{1}{6}*\frac{1}{6} \end{bmatrix}=$$

$$\begin{bmatrix} \frac{1}{36}+\frac{1}{9}+\frac{1}{36} & \frac{1}{18}+\frac{2}{9}+\frac{1}{18} & \frac{-1}{36}+\frac{-1}{9}+\frac{-1}{36} \\ \frac{1}{18}+\frac{2}{9}+\frac{1}{18} & \frac{1}{9}+\frac{4}{9}+\frac{1}{9} & \frac{-1}{18}+\frac{-2}{9}+\frac{-1}{18} \\ \frac{-1}{36}+\frac{-1}{9}+\frac{-1}{36} & \frac{-1}{18}+\frac{-2}{9}+\frac{-1}{36} & \frac{1}{36}+\frac{1}{9}+\frac{1}{36} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{-1}{6} \\ \frac{1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{6} & \frac{-1}{3} & \frac{1}{36} \end{bmatrix}$$

کەواتە هەمان نەتیجەیە.

7-***Trace of matrix***

If $A = (a_{ij})$ is a square matrix of order n, then the trace of A is:

Tr(A)=$\sum_{i=1}^{n} a_{ij} = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$

A=$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$; tr(A)=$\sum_{i=1}^{3} a_{11} + a_{22} + a_{33}$

وەکوا(**diagonal**)ە،تەنها جیاوازیان ئەوەیە،(**diagonal**)جاران دەکرێت،بەڵام(trace)کۆ دەکرێتەوە.وە مەرجیش نییە سێگۆشەکانیان یەکسان بن.

$$Example//A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$Solution//\ tr = 2 + (-6) + 4 = 0$

Diagonal=2*-6*4=-48

Example//

Solution: A=$\begin{bmatrix} 9 & -4 \\ 8 & -2 \end{bmatrix}$ = 9+(-2)=7

$$Example// \ A = \begin{bmatrix} 3 & 8 & 5 \\ 6 & -2 & 7 \\ 3 & 4 & 1 \end{bmatrix} \Rightarrow solution = 3 + (-2) + 1 = 2$$

$$Example// \ A = \begin{bmatrix} 16 & 0 & 16 & 0 \\ 20 & 8 & 4 & 0 \\ 6 & 0 & 6 & 20 \\ 10 & 0 & 10 & 12 \end{bmatrix} \Rightarrow solution = 16 + 8 + 6 + 12 = 42$$

If A and B are two matrix of order n such that (AB) is defined a square matrix,then

$$Tr(AB) = tr\,(BA)$$

$$.A = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} . \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1*5+(-2*-2) & 1+(-2+(-2*1)) \\ 2*5+5*(-2) & 2*-2+5*1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = Tr(1+1) = 2$$

$$BA = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} . \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 5*1+(-2*2) & 5*-2+(-2*5) \\ -2*1+1*2 & -2*-2+1*5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = tr(1+1) = 2$$

If A and B are two matrices of order n and let C_1 and C_2 be two Scalar, then:

$Tr(C_1A + C_2B) = C_1tr(A) + C_2tr(B)$

Example// C1=3 ,C2=2 ,A=$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$,B=$\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$Tr(C_1A + C_2B)$=3 $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ +2$\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ = جارانی ناو مەصفوفەیان دەکەین

A = $\begin{bmatrix} 3 & 9 \\ 6 & 12 \end{bmatrix}$+B=$\begin{bmatrix} 6 & 4 \\ 2 & 2 \end{bmatrix}$ئینجا دێین (ترێیس)وەردەگرین

{3+12=15}+{6+2=8}=15+8=23

$C_1tr(A) + C_2tr(B)$=3A=$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$+2B=$\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

=3(1+4)+2 (3+1)=15+8=23 کەواتە یەکسان بوون

Let A a square matrix of order n and let C is non-singular matrix (|C|)≠0, then

$$tr\ (C^{-1}\ AC)\ =\ tr\ (A)$$

Proof//

$$tr\ (C^{-1}\ AC) = tr(AC^{-1}\ C)\ =\ tr\ (A)$$

جونكە (C) لەگەل (C⁻¹) دەروات

And is C is an Orthogonal matrix then

$$Tr\ (C'\ AC)\ =\ tr\ (AC'C)$$

$$=tr\ (AC^{-1}C)$$

$$= tr\ (A\ I)\ =\ tr\ (A)$$

8- *Vectors*

$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad \underline{X} = [x_1 \quad x_2 \quad x_3]$ پێی دەڵێن (vector) جونكە يەک عامود سێ سەفی تێدايە

Defth1: let x_1,x_2,….x_k be K*1 vector and c_1,c_2,….c_k be scalars. Then $\sum_{i=1}^{k} c_i\ x_i$ is said to be a linear combination of X.

Def^{n} 2= A set of vectors X_1,X_2,….,X_K is linearly dependent if there exist

a_1,a_2,….,a_k not all zero, such that $\sum_{i=1}^{k} a_i\ x_i\ = 0$ at least one of $a_i \neq 0$.the set linearly independent if the only scalars which satisfy:

$\sum_{i=1}^{k} a_i\ x_i\ = 0 \rightarrow a_1$ =a_2=……=a_k=0.

Linear dependent :

یانی نرخەکانی (a)ی،دەست نیشانی دەکات کەوا ئەو(vector)ە،(dependent)ە،یان(independent)ە.

لە(dependent)نابێت(scalar)کانی هەموو (vector)کان سفر بن،یانی دەبێت بەلایەنی کەم یەکێکیان نابێت سفر بێت،مەبەستمان لە(scalar)یانی.هاوکۆلکەی پێش(vector),یانی ئەگەر هاتوو هەر جواریان(0)بوون،ئەو کاتە دەبووە:***Linear independent***

Linear independent :

کاتێک(**independent**) دەبێت ئەگەر هاتوو هەموو (a)کان سفر بوون، واتە(vector)(0)دەردەجن.

Example : $\underline{X}_1 = [1 \quad 1 \quad 0 \quad 1]'$ $\underline{X}_2 = [0 \quad -2 \quad 1 \quad 1]'$ $\underline{X}_3 = [-2 \quad 0 \quad -1 \quad 1]'$

Show that $\underline{X}_1$, $\underline{X}_2$ and $\underline{X}_3$ are linear independent.

Solution\\

Let $a_1, \; a_2, \; a_3$ are three constants.

$\sum_{i=1}^{k} \; a_i \, x_j = 0 \rightarrow \sum_{i=1}^{3} a_i \, \underline{x}_i = 0$, $a_1\underline{X}_1 + a_2 X_2 + a_3 = 0$

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

ئینجا (a_1)جارانی ناو (vector)دەکەین، بۆ هەر سێ(vector)ەکان.

$a_1 + a_2(0) - 2a_3 = 0 \rightarrow a_1 - 2a_3 = 0 \rightarrow a_1 = 2a_3$ eq.1

$a_1 - 2a_2 + a_3(0) = 0 \rightarrow a_1 - 2a_2 = 0 \rightarrow a_1 = 2a_2$ eq.2

$a_1(0) + a_2 - a_3 = 0 \rightarrow a_2 = a_3$ eq.3

$a_1 + a_2 + a_3 = 0$ eq.4

From eq.4

$a_1 + a_2 + a_3 = 0$

ئینجا لە شوێنی هەر(a)کان کە دۆزیمانەوە لە شوێنیان دادەنێین.

$$2a_3 + a_3 + a_3 = 0 \to 4\,a_3 = 0 \quad \to a_3 = \frac{4}{0} = 0 \;,\; a_3 = 0$$

$$\therefore\; a_1 = 2a_3 = 2(0) = 0 \;,\; a_2 = a_3 = 0$$

$$a_1 = a_2 = a_3 = 0$$

$$\boldsymbol{a_1} + \boldsymbol{a_2} + \boldsymbol{a_3} = 0$$

$\therefore \sum_{i=1}^{3} a_i\, \underline{x}_i = 0$

$\therefore \underline{X}_1$, $\underline{X}_2$ and $\underline{X}_3$ are linear independent.

کەواتە هەموو نەتیجەکانی کۆتایی (0)دەرچوو بۆیە (linear independent.)ە،بەلام ئەگەر یەکێکیان،(0)نەبوا،ئەوکاتە دەبووە(linear dependent.)

Example // into dependent:

$$\boldsymbol{a_1}\begin{bmatrix} \mathbf{1} \\ \mathbf{-2} \\ \mathbf{0} \end{bmatrix} + \boldsymbol{a_2}\begin{bmatrix} \mathbf{4} \\ \mathbf{0} \\ \mathbf{8} \end{bmatrix} + \boldsymbol{a_3}\begin{bmatrix} \mathbf{3} \\ \mathbf{-1} \\ \mathbf{5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution// $\sum_{i=1}^{k} a_i\, x_j = 0 \to \sum_{i=1}^{3} a_i\, \underline{x}_i = 0$, $a_1\underline{X}_1 + a_2 X_2 + a_3 = 0$

$\boldsymbol{a_1} + \mathbf{4}\boldsymbol{a_2} + \mathbf{3}\boldsymbol{a_3} = \mathbf{0}$ eq.1

$-\mathbf{2}\boldsymbol{a_1} + (\mathbf{0})\boldsymbol{a_2} - \boldsymbol{a_3} = \mathbf{0} \to -\mathbf{2}\boldsymbol{a_1} - \boldsymbol{a_3} \to \quad -\mathbf{2}\boldsymbol{a_1} = \boldsymbol{a_3}, \quad \boldsymbol{a_1} = \frac{-1}{2}\boldsymbol{a_3}$ eq.2

$\boldsymbol{a_1}(\mathbf{0}) + \mathbf{8}\boldsymbol{a_2} + \mathbf{5}\boldsymbol{a_3} = 0 \to \mathbf{8}\boldsymbol{a_2} = -\mathbf{5}\boldsymbol{a_3} = \quad \boldsymbol{a_2} = \frac{-5}{8}\boldsymbol{a_3}$ eq.3

$a_1 + a_2 + a_3 = 0$ eq.4

From eq.4

$$a_1 + a_2 + a_3 = 0$$

ئێستا دێین نرخەکانی(a_1)و(a_2)لە (eq.1)دانەنێین،بۆ دۆزینەوەی نرخی(a_3) .

$a_1 + a_2 + a_3 = 0$

$\frac{-1}{2} a_3 - 4\frac{5}{8} a_2 + 3a_3 = 0$, $a_3 = 0$

کەواتە لە سێ(vector)دوو(vector)نرخەکانیان (0)دەرنەجوو،کەواتە(dependent)ئەگەر هەرسێک(0)بان،ئەوکاتە دەبوو بە(independent) .

Example:

Let $\underline{X} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ $find\ (3 \times 1) vectors\ \underline{Y}\ and\ \underline{Z}\ such\ that\ \underline{X}, \underline{Y}\ and\ \underline{Z}$ are **orthogonal**?

Solution: $\underline{X} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \underline{X}' = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$

$\underline{X}'\underline{X} = 1$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \left[\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + 0 \times 0 \right] = \frac{1}{2} + \frac{1}{2} + 0 = 1$$

$\underline{X}'\ \underline{Y} = 0$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0 \rightarrow \left[\frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{2}} y_2 + 0y_3 = 0 \right] \times \sqrt{2}$$

$$\frac{\sqrt{2}}{\sqrt{2}} y_1 + \frac{\sqrt{2}}{\sqrt{2}} y_2 = y_1 + y_2 = 0 \rightarrow y_1 = -y_2$$

$\underline{Y}'\ \underline{Y} = 1$

$$[\, y_1 \quad y_2 \quad y_3 \,] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 1$$

$$\left[{y_1}^2 + \quad {y_2}^2 + \quad {y_3}^2 \right] = 1$$

$$-2{y_2}^2 \quad + \quad {y_3}^2 = 1$$

$${y_3}^2 = 1 - \ 2{y_1}^2$$

Let $y_1 = \frac{1}{\sqrt{2}}$

$$y_2 = -\frac{1}{\sqrt{2}}$$

$$y_3{}^2 = 1 - 2y_2{}^2 = 1 - \left(-\frac{1}{\sqrt{2}}\right)^2$$

$$= 1 - 2\left(\frac{1}{2}\right) = 1 - 1 = 0$$

$$\underline{Y} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix}, \underline{Y}' = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$\underline{X}'\underline{Y} = 0$$

$$\underline{Y}'\underline{Y} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} . \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix} = \frac{1}{2} - \frac{1}{2} + 0 = 0$$

$$\underline{Y}'\underline{Y} = 1$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix} . \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix} = \frac{1}{2} + \frac{1}{2} + 0 = 1$$

$$\underline{X}'\,\underline{Z} = 0$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} . \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = 0$$

$$\frac{1}{\sqrt{2}}Z_1 + \frac{1}{\sqrt{2}}Z_2 + 0Z_3 = 0) * \sqrt{2}$$

$$Z_1 + Z_2 = \rightarrow Z_1 = -Z_2$$

$$\underline{Z}' \, \underline{Z} = 1$$

$$[\, Z_1 \quad Z_2 \quad Z_3]. \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = 1$$

$$Z_1{}^2 + Z_2{}^2 + Z_3{}^2 = 1$$

$$(-Z_2{}^2)^2 + Z_3{}^2 = 1$$

$$-2Z_2{}^2 + Z_3{}^2 = 1 \rightarrow Z_3{}^2 = 1 - 2Z_2{}^2$$

$$\underline{X}' \, \underline{Z} = 0 \qquad \underline{Z}' \, \underline{Z} = 1$$

Let $Z_1 = \frac{-1}{\sqrt{2}} \quad Z_2 = \frac{1}{\sqrt{2}}$

$$Z_3{}^2 = 1 - 2Z_2{}^2 = 1 - 2(\frac{1}{\sqrt{2}})^2$$

$$Z_3{}^2 = 1 - 2\left(\frac{1}{2}\right)^2 = 1 - 1 = 0$$

$$\underline{Z} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \underline{Z}' = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$\underline{X}' \, \underline{Z} = 0$$

$$\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \frac{-1}{2} + \frac{1}{2} + 0 = 1$$

$\underline{Z}' \, \underline{Z} = 0$

$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \frac{1}{2} + \frac{1}{2} + 0 = 1$ is proved

هەر باش بوو شیکار بوو ئەو هەموو کاتە،خەریک بوو سیقەم بە بیرکاری نەمێنێت.

Q1// find determinant

A= $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

$solution = (2 \times 5) - (4 \times 3) = -2$

B= $\begin{bmatrix} 1 & 100 & 50 \\ 0 & -2 & 2 \\ 0 & 0 & 5 \end{bmatrix}$

Solution//

$$\begin{vmatrix} 1 & 100 & 50 \\ 0 & -2 & 2 \\ 0 & 0 & 5 \end{vmatrix} \begin{matrix} 1 & 100 \\ 0 & -2 \\ 0 & 0 \end{matrix}$$

$$(1 \times -2 \times 5) + (100 \times 2 \times 0) + (50 \times 0 \times 0) - (0 \times -2 \times 50) - (0 \times 2 \times 1) - (5 \times 0 \times 100) = -14$$

C) $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

SOlution// ماده‌م (diagonal)ه، ته‌نها ناوه‌راستیان جاران ده‌که‌ین

-2*-2*5=20

D) $\begin{bmatrix} 3 & 5 & 6 \\ 1 & 4 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

Solution//لەبەر ئەوەی ستونی سێیەم هەمووی سفرە کەواتە دەکاتە سفر

$$\begin{vmatrix} 3 & 5 & 6 \\ 1 & 4 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Q2) find the inverse of the following matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \\ 0 & 2 & 5 \end{bmatrix}$

SOLUTION//

A^{-1}=$\frac{adj(a)}{|A|}$

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \\ 0 & 2 & 5 \end{vmatrix} \begin{matrix} 1 & 2 \\ 0 & -2 \\ 0 & 2 \end{matrix}$$

$=(1 \times -2 \times 5) + (2 \times 2 \times 0) + (1 \times 0 \times 2) - (0 \times -2 \times 1) - (2 \times 2 \times 1) - (5 \times 0 \times 2) = -14$

A_{ij} =$(-1)^{i+j}$ =$(-1)^{1+1}$= 1|M_{11}|=$\begin{bmatrix} -2 & 2 \\ 2 & 5 \end{bmatrix}$ =(-2*5) – (2*2) = -14

A_{ij} =$(-1)^{i+j}$=$(-1)^{1+2}$=-|M_{12}|=$\begin{bmatrix} 2 & 1 \\ 2 & 5 \end{bmatrix}$ = -(2*5) – (2*1) = -12

A_{ij} =$(-1)^{i+j}$=$(-1)^{1+3}$=|M_{13}|=$\begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix}$ = (2*2) - (-2*1) = 6

A_{ij} =$(-1)^{i+j}$=$(-1)^{2+1}$=|M_{21}|=$\begin{bmatrix} 0 & 2 \\ 0 & 5 \end{bmatrix}$ = - (0*5) - (0*2) = 0

A_{ij} =$(-1)^{i+j}$=$(-1)^{2+2}$=|M_{22}|=$\begin{bmatrix} 1 & 1 \\ 0 & 5 \end{bmatrix}$ = (1*5) - (0*1) = 5

$A_{ij} = (-1)^{i+j} = (-1)^{2+3} = |M_{23}| = \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix}$ = - (1*2) - (0*-2) = -2

$A_{ij} = (-1)^{i+j} = (-1)^{3+1} = |M_{31}| = \begin{bmatrix} 0 & -2 \\ 0 & 2 \end{bmatrix}$ = (0*2) - (0*-2) = 0

$A_{ij} = (-1)^{i+j} = (-1)^{3+2} = |M_{32}| = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ = - (1*2) - (0*2)= -2

$A_{ij} = (-1)^{i+j} = (-1)^{3+3} = |M_{33}| = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ = (1*-2) - (0*2)= -2

$$A' = \begin{bmatrix} -14 & -12 & 6 \\ 0 & 5 & -2 \\ 0 & -2 & -4 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} -14 & -12 & 6 \\ 0 & 5 & -2 \\ 0 & -2 & -2 \end{bmatrix}}{-14} = \begin{bmatrix} 1 & 4/7 & -3/7 \\ 0 & -5/7 & 1/7 \\ 0 & 1/7 & 1/7 \end{bmatrix}$$

Q3/ If $X = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, Y = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & 1 \end{bmatrix}, z = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ find the following:

$$X + Y, \qquad Y - Z', \qquad 5Y - 5Z', \; ZZ', \qquad YZ'$$

Solution:

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & 1 \end{bmatrix}$$

$X + Y \to not\ applicable$

جونکە یاسای کۆکردنەوەی لەسەر جێ بەجێ نابێت.

$Y - Z'$

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

ZZ'

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} . \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}' = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} . \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 2 + 3 \times 3 & 2 \times 0 + 3 \times 1 & 2 \times 1 + 3 \times 1 \\ 0 \times 2 + 1 \times 3 & 0 \times 0 + 1 \times 1 & 1 \times 1 + 1 \times 1 \end{bmatrix} = \begin{bmatrix} 13 & 3 & 5 \\ 3 & 1 & 2 \end{bmatrix}$$

YZ'

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & 1 \end{bmatrix} . \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}' = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & 1 \end{bmatrix} . \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$not\ applicable$

جونکە یاسای جارانی لەسەر جێ بەچی نابێت.

Q4/ let $\underline{P} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \underline{N} = \begin{bmatrix}0\\1\\2\end{bmatrix}, \underline{M} = \begin{bmatrix}0\\1\\2\end{bmatrix}$

Find the following: $2\underline{P} + \underline{M} - 2\underline{N}$

Solution:

$$2\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\1\\2\end{bmatrix} - 2\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}2\\4\\6\end{bmatrix} + \begin{bmatrix}0\\1\\2\end{bmatrix} - \begin{bmatrix}0\\2\\4\end{bmatrix} = \begin{bmatrix}2\\5\\8\end{bmatrix} - \begin{bmatrix}0\\2\\4\end{bmatrix} = \begin{bmatrix}2\\3\\4\end{bmatrix}$$

Q5/ show how to write each of the following vectors as a linear combination of constant vectors with scalar coefficients

$x, y, or\ z: \begin{bmatrix}3x+2y\\-z\\x+y+5z\end{bmatrix}$

Solution:

$3x + 2y$ لە هاوکێشەی یەکەم

$x = 3, \quad y = 2, \quad z = 0$

$-z$ لە هاوکێشەی دووەم

$x = 0, \quad y = 0, \quad z = -1$

$x + y + 5z$ لە هاوکێشەی یەکەم

$x = 1, \quad y = 1, \quad z = 5$

$$\begin{bmatrix}3x+2y\\-z\\x+y+5z\end{bmatrix} = \boldsymbol{x}\begin{bmatrix}\mathbf{3}\\\mathbf{0}\\\mathbf{1}\end{bmatrix} + \boldsymbol{y}\begin{bmatrix}\mathbf{2}\\\mathbf{0}\\\mathbf{1}\end{bmatrix} + \boldsymbol{z}\begin{bmatrix}\mathbf{0}\\\mathbf{-1}\\\mathbf{5}\end{bmatrix}$$

Chapter three

Multivariable Normal Distribution

Multivariable Statistical analysis and proved (give)and example on it.?

Multivariable analysis consists of a collection of methods that can be used when several measurements are made on each individual or objects in one or more samples. we refer to the measurements as variables and to the individuals or objects as units (research units, sampling units, or experimental units) or observations. The below table contains some example of multivariable analysis.

1- **Univariate : when we have one variable in a function(p= 1)**

$P =$ number of variable, ژمارەی(variable)کانە

یانی کاتێک (p= 1)ئەوکاتە دەبێتە(univariate)

Then the p.d.f of X is $X \sim N(\mu, \sigma^2)$ Is given by: $f(x;\ \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\,\sigma^2}} \exp(-\frac{1}{2\sigma^2}(x-\mu)^2$

2- Bivariate: when we have two variables in a function(p= 2)

کاتێک دوو(variable)مان هەبێت.ئەگەر دوو (variable)مان هەبێت، دەبێت،دوو(mean)نیشمان هەبێت،وە دوو(μ)ماشمان هەبێت.وە ئەگەر دوو(variance)مان هەبێت،ئەوا(covariance)لە نێوانیاندا هەیە.نمونە،ئەوەی خەتی بەسەردا هاتووە،دەبنە (variance).

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix},\ \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$\sigma_{11}\ \sigma_{22} \rightarrow covariance$

Then the j.p.d.f of $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N\left\{\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}\right\}$ **is given by:**

$\frac{1}{\sqrt{2\pi\,\sigma^2}} exp(\frac{-1}{2\sigma^2}(x-\mu)^2$ دوای ئەوە دێین لە یاسا بەکاری دەهێنین

(σ^2)لەبەر ئەوەی ئێمە دوو(σ^2)مان هەیە یاسایەکە گۆڕانکاری بەسەردا دادێت،وە دواتر (σ^2)دەبەینە سەری کەرت. $\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}^{-1}$: وە دواتر $(x-\mu)^2$توان ناکرێت،دەبێت،بەم شێوەیە بینوسین،{ $x_1-\mu_1 \quad x_2-\mu_2$)،ئینجا دواتر $\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}^{-1}$: جارانی(vector)دەکەین،بەم شێوەی خوارەوەی لێدێت.

$$F(x_1,x_2,\mu_1;\mu_2,\sigma_{11},\sigma_{22}) = \frac{1}{2\pi|\Sigma|^{\frac{1}{2}}}\exp\left(-\frac{1}{2}(x_1-\mu_1 \quad x_2-\mu_2\}\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}^{-1}\begin{bmatrix}x_1-\mu_1\\ x_2-\mu_2\end{bmatrix}\right.$$

وە دەتوانین بەم شێوەیەش بنوسین.

$$F(x_1,x_2,\mu_1;\mu_2,\sigma_{11},\sigma_{22}) = \frac{1}{2\pi|\Sigma|^{\frac{1}{2}}}\exp\left(-\frac{1}{2}(x_1-\mu_1 \quad x_2-\mu_2\}\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}^{-1}\begin{bmatrix}x_1-\mu_1\\ x_2-\mu_2\end{bmatrix}\right.$$

$$F(\underline{x}:\underline{\mu},\underline{\Sigma}) = \frac{1}{2\pi|\Sigma|^{\frac{1}{2}}}\exp\left(-\frac{1}{2}(x-\mu)'\,\Sigma^{-1}\,(x-\mu)\right. \quad -\infty < x_1 < +\infty\,,\ -\infty < x_2 < +\infty$$

$$\Sigma^{-1} = \begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}^{-1}$$

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3) ***Multivariable***

Where we have P-variables in function (p = p)

یانی کاتێک زیاتر لە دوو(variable)مان هەبێت،وە دەبێت (mean)یش لە دوو زیاتر بێت،وە دەبێت(μ)یش لەدوو زیاتر بێت.

$$\boldsymbol{X} = \begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_p\end{bmatrix},\ \boldsymbol{\mu} = \begin{bmatrix}\mu_1\\ \mu_2\\ \vdots\\ \mu_p\end{bmatrix},\ \Sigma = \begin{bmatrix}\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p}\\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ \sigma_{31} & \sigma_{32} & \cdots & \sigma_{pp}\end{bmatrix}$$

Where Σ is square. Non-singular and symmetric matrix and (p*p) dimensional.

Then, the j.p.d.f of X ~N(μ,Σ) is given by:

$f(x_1, x_2, \dots, x_p;\ \mu_1, \mu_2, \dots, \mu_p; \sigma_{11}, \sigma_{22}, \dots, \sigma_{pp})$ =

$$\frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} exp\left(-\frac{1}{2}[x_1-\mu_1 \quad x_2-\mu_2 \quad \cdots \quad x_p-\mu_p]\begin{pmatrix} \boldsymbol{\sigma_{11}} & \boldsymbol{\sigma_{12}} & \cdots & \boldsymbol{\sigma_{1p}} \\ \boldsymbol{\sigma_{21}} & \boldsymbol{\sigma_{22}} & \cdots & \boldsymbol{\sigma_{2p}} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\sigma_{31}} & \boldsymbol{\sigma_{32}} & \cdots & \boldsymbol{\sigma_{pp}} \end{pmatrix}^{-1} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \\ \vdots \quad \vdots \\ x_p-\mu_p \end{bmatrix}\right)$$

$$f(\underline{X}: \underline{\mu}, \Sigma) = \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} exp\left(-\frac{1}{2}(\text{X} - \underline{\mu})' \Sigma^{-1} (\underline{X} - \underline{\mu})\right)$$

***Example**// **Find the j.p.d.f of Bivariate Normal dist** $(p = 2)$*

$\underline{\Sigma}$ لە شوێنی (σ) بەکاردەهێنین جونکە لێرەدا دوو(variable)مان هەیە، (p= 2)

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{bmatrix}, \qquad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu_1} \\ \boldsymbol{\mu_2} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \boldsymbol{\sigma_{11}} & \boldsymbol{\sigma_{12}} \\ \boldsymbol{\sigma_{21}} & \boldsymbol{\sigma_{22}} \end{bmatrix}$$

Solution//

$$f(\underline{x}: \underline{\mu}, \underline{\Sigma}) = \frac{1}{(2\pi)^{\frac{2}{2}}|\Sigma|^{\frac{1}{2}}} exp(-\frac{1}{2}(x - \underline{\mu})' \Sigma^{-1} (x - \underline{\mu})$$

یەکەم جار(inverse) دەدۆزینەوە.

$$\Sigma = \begin{bmatrix} \boldsymbol{\sigma_{11}} & \boldsymbol{\sigma_{12}} \\ \boldsymbol{\sigma_{21}} & \boldsymbol{\sigma_{22}} \end{bmatrix}, \qquad \rho = correlation \qquad \therefore \sigma_{12} = \rho_{12}, \sigma_{12} = covariance$$

$\sigma_1 = variance$

$$\therefore \boldsymbol{\rho_{12}} = \frac{\boldsymbol{\sigma_{12}}}{\boldsymbol{\sigma_1\sigma_2}} \Rightarrow \boldsymbol{\sigma_{12}} = \boldsymbol{\rho_{12}\sigma_1\sigma_2}$$

$$\Sigma = \begin{bmatrix} \boldsymbol{\sigma_{11}} & \boldsymbol{\rho_{12}\sigma_1\sigma_2} \\ \boldsymbol{\rho_{12}\sigma_1\sigma_2} & \boldsymbol{\sigma_{22}} \end{bmatrix}$$

دووەم جار(determent)دەدۆزینەوە

$$\boldsymbol{|\Sigma|} = \boldsymbol{\sigma_{11}} * \boldsymbol{\sigma_{22}} - \boldsymbol{\rho_{12}\sigma_1\sigma_2} * \boldsymbol{\rho_{12}\sigma_1\sigma_2\sigma_1\sigma_2} = \boldsymbol{\sigma_{11}\ \sigma_{22}} - \boldsymbol{\rho_{12}}^{2}\boldsymbol{\sigma_{11}\sigma_{22}}$$
$$= \boldsymbol{\sigma_{11}\sigma_{22}}(\boldsymbol{1} - \boldsymbol{\rho_{12}}^{2})$$

$$\Sigma^{-1} = \frac{adj\begin{bmatrix}\sigma_{11} & \rho_{12}\sigma_1\sigma_2\\ \rho_{12}\sigma_1\sigma_2 & \sigma_{22}\end{bmatrix}}{\begin{vmatrix}\sigma_{11} & \rho_{12}\sigma_1\sigma_2\\ \rho_{12}\sigma_1\sigma_2 & \sigma_{22}\end{vmatrix}}$$

$$adj(\Sigma) = \begin{bmatrix}\sigma_{22} & -\rho_{12}\sigma_1\sigma_2\\ -\rho_{12}\sigma_1\sigma_2 & \sigma_{11}\end{bmatrix}, \ |\Sigma| = \begin{vmatrix}\sigma_{11} & \rho_{12}\sigma_1\sigma_2\\ \rho_{12}\sigma_1\sigma_2 & \sigma_{22}\end{vmatrix} = \sigma_{11}\sigma_{22}(1-\rho_{12}{}^2)$$

$$, \Sigma^{-1} = \frac{1}{\sigma_{11}\sigma_{22}(1-\rho_{12}{}^2)}\begin{bmatrix}\sigma_{22} & -\rho_{12}\sigma_1\sigma_2\\ -\rho_{12}\sigma_1\sigma_2 & \sigma_{11}\end{bmatrix}$$

$$\Sigma^{-1}=\frac{1}{(1-\rho_{12}{}^2)}\begin{bmatrix}\frac{\sigma_{22}}{\sigma_{11}\sigma_{22}} & \frac{-\rho_{12}\sigma_1\sigma_2}{\sigma_{11}\sigma_{22}}\\ \frac{-\rho_{12}\sigma_1\sigma_2}{\sigma_{11}\sigma_{22}} & \frac{\sigma_{11}}{\sigma_{11}\sigma_{22}}\end{bmatrix}=\frac{1}{(1-\rho_{12}{}^2)}\begin{bmatrix}\sigma_{11}{}^{-1} & \frac{-\rho_{12}}{\sigma_1\sigma_2}\\ \frac{-\rho_{12}}{\sigma_1\sigma_2} & \sigma_{22}{}^{-1}\end{bmatrix}$$

Then the Quadratic from of X is:

$Q(X) = (x-\underline{\mu})'\,\Sigma^{-1}\,(x-\underline{\mu})$

$$Q(X)=[x_1-\mu_1 \quad x_2-\mu_2]\,\frac{1}{(1-\rho_{12}{}^2)}\begin{bmatrix}\sigma_{11}{}^{-1} & \frac{-\rho_{12}}{\sigma_1\sigma_2}\\ \frac{-\rho_{12}}{\sigma_1\sigma_2} & \sigma_{22}{}^{-1}\end{bmatrix}\begin{bmatrix}x_1 & - & \mu_1\\ x_2 & - & \mu_2\end{bmatrix}$$

$$Q(X)=\frac{1}{(1-\rho_{12}{}^2)}\ [x_1-\mu_1 \quad x_2-\mu_2]\begin{bmatrix}\sigma_{11}{}^{-1} & \frac{-\rho_{12}}{\sigma_1\sigma_2}\\ \frac{-\rho_{12}}{\sigma_1\sigma_2} & \sigma_{22}{}^{-1}\end{bmatrix}\begin{bmatrix}x_1 & - & \mu_1\\ x_2 & - & \mu_2\end{bmatrix}$$

ئینجا جارانی یەکتریان دەکەین.

$$\begin{bmatrix}\frac{x_1-\mu_1}{\sigma_{11}}-\frac{\rho_{12}(x_2-\mu_2)}{\sigma_1\sigma_2} & -\frac{\rho_{12}(x_1-\mu_1)}{\sigma_1\sigma_2}+\frac{x_1-\mu_1}{\sigma_{22}}\end{bmatrix}\begin{bmatrix}x_1 & - & \mu_1\\ x_2 & - & \mu_2\end{bmatrix}$$

$$Q(X)=\frac{1}{(1-\rho_{12}{}^2)}$$

$$=\left[(x_1-\mu_1)\left(\frac{x_1-\mu_1}{\sigma_{11}}-\frac{\rho_{12}(x_2-\mu_2)}{\sigma_1\sigma_2}\right)+(x_2-\mu_2)\left(-\frac{\rho_{12}(x_1-\mu_1)}{\sigma_1\sigma_2}+\frac{x_2-\mu_2}{\sigma_{22}}\right)\right]$$

$$Q(X)=\frac{1}{(1-\rho_{12}{}^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_{11}}-\frac{\rho_{12}(x_2-\mu_2)(x_1-\mu_1)}{\sigma_1\sigma_2}-\frac{\rho_{12}(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}+\frac{(x_2-\mu_2)^2}{\sigma_{22}}\right]$$

$$Q(X)=\frac{1}{(1-\rho_{12}{}^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_{11}}-2\frac{\rho_{12}(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}+\frac{(x_2-\mu_2)^2}{\sigma_{22}}\right]$$

$|\Sigma| = \sigma_{11}\sigma_{22}(1-\rho_{12}{}^2)$

$|\Sigma|^{\frac{1}{2}} \sqrt{\sigma_{11}\sigma_{22}(1-\rho_{12}{}^2)} = \sigma_1\sigma_2\sqrt{1-\rho_{12}{}^2}$

Then the j.b.d.f of X is:

$$f(\underline{X};\underline{\mu},\Sigma) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{12}{}^2}} \exp(\frac{1}{2}\cdot\frac{1}{1-\rho_{12}{}^2}\left[\frac{(x_1-\mu_1)^2}{\sigma_{11}} - \frac{\rho_{12}(x_2-\mu_2)(x_1-\mu_1)}{\sigma_1\sigma_2} - \frac{\rho_{12}(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_{22}}\right]$$

Or $f(\underline{X};\underline{\mu},\Sigma) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{12}{}^2}} \exp(\frac{1}{2} Q(\underline{X})$

Example// write te j.b.d.f of Bivariate Normal distribution when:

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \underline{\mu} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}, \Sigma = \begin{bmatrix} 9 & 16 \\ 16 & 64 \end{bmatrix}, \rho = 0.667$$

Solution// the j.p.d.f of **bivariate** normal distribution could be written a follows:

$$\sigma_2 = \sqrt{64} = 8 \quad \sigma_1 = \sqrt{9} = 3$$

$$\boldsymbol{f(\underline{X};\underline{\mu},\Sigma) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{12}{}^2}} \exp(\frac{1}{2}\cdot\frac{1}{1-\rho_{12}{}^2}\left[\frac{(x_1-\mu_1)^2}{\sigma_{11}} - \frac{\rho_{12}(x_2-\mu_2)(x_1-\mu_1)}{\sigma_1\sigma_2} - \frac{\rho_{12}(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_{22}}\right]}$$

$$f(\underline{X};\underline{\mu},\Sigma) = \frac{1}{2\pi(3)(8)\sqrt{1-(0.667)^2}} \exp\left(-\frac{1}{2}\frac{1}{1-(0.667)^2}\left[\frac{(x_1-5)^2}{9} - 2\frac{0.667(x_1-5)(x_2-10)}{(3)(8)} + \frac{(x_2-10)^2}{64}\right]\right)$$

$$f(\underline{X};\underline{\mu},\Sigma) = \frac{1}{48\pi\sqrt{1-(0.667)^2}} \exp(\left(-\frac{1}{2}\frac{1}{1-(0.667)^2}\left[\frac{(x_1-5)^2}{9} - 2\frac{0.667(x_1-5)(x_2-10)}{24} + \frac{(x_2-10)^2}{64}\right]\right)$$

$$f(\underline{X};\underline{\mu},\Sigma) = \frac{1}{13.23\pi} \exp\left(\frac{-1}{1.11}\left[\frac{(x_1-5)^2}{9} - 2\frac{0.667(x_1-5)(x_2-10)}{24} + \frac{(x_2-10)^2}{64}\right]\right)$$

$$f(\underline{X};\underline{\mu},\Sigma) = \frac{1}{13.23\pi} \exp\left(\frac{-1}{1.11}\left[\frac{(x_1^2-2x_1+25)}{9} - \frac{0.667\,(x_1x_2-10x_1-5x_2+50)}{12} + \frac{(x_2^2-2x_2+100)}{64}\right]\right)$$

Comparison between Univariate and Multivariate Normal distribution:

بەراود لە نێوان(Univariate)و(Multivariate)

1- the variable X has been transferred to vector of variables X: $X \rightarrow \underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_P \end{bmatrix}$

یەک(variable)هەیە لە(Univariate)بەڵام کە دەیکەینە(Multivariate)دەبێتە(vector)ێک لە(X)کان.

2- The mean of the variable X is μ has been transferred to the vector of means:

$$E(X) = \mu \rightarrow \underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_P \end{bmatrix}$$

یەک(μ)هەیە لە(Univariate).بەلام لە(Multivariate) جەندین(μ)لە خۆ دەگرێت.

3- the variance of the variable X is σ^2 has been transferred to the Var-Cov. Matrix its squared non-singular. And symmetric matrix of order (p * p):

V(x)= $\sigma^2 \rightarrow Var - Cov(x) = \sum, \sigma^2 \rightarrow \sum, \quad \sigma = |\sum|^{\frac{1}{2}}$

لە(Univariate)تەنها یەک(σ^2)مان هەیە،بەلام لە(Multivariate)دەبێتە(Multivariate)وە هەروەها(Multivariate)(normal distribution)ە،وە هەروەها(Multivariate)(non-singular)ە،واتە(محددە)ناکاتە سفر،وە هەروەها(Multivariate)(symmetric matrix)ە،یانی سێگۆشەی سەرەوەی یەکسانە بە سێگۆشەی مصفوفەی خوارەوە.

4) the square $(\frac{x-\mu}{\sigma})^2$ has been transferred to the quadratic from $(\underline{X} - \underline{\mu})' \sum^{-1} (\underline{X} - \underline{\mu})$

لە(Univariate)دەکاتە($(\frac{x-\mu}{\sigma})^2$)بەڵام (Multivariate)دەبێتە مصفوفە،یانی(Multivariate)هەم(Byveriate) وە(Univariate)یش،لەخۆ دەگرێت.

5) the p.d.f of X

$$f(x;\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma}}\exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

The j.b.d.f $\underline{X}$

$$f(\underline{X}:\underline{\mu},\Sigma)=\frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}}\exp\left(-\frac{1}{2}(\underline{X}-\underline{\mu})'\Sigma^{-1}(\underline{X}-\underline{\mu})\right)$$

---

4) ***Quadratic Form***

Def^n: if we have P variables $(X_1, X_2, \dots, X_P)$ OR $(\underline{X})$ and is (p* p) symmetric matrix where:

$$\underline{X}=\begin{bmatrix}X_1\\X_2\\\vdots\\X_P\end{bmatrix},\quad A=\begin{pmatrix}\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p}\\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ \sigma_{31} & \sigma_{32} & \cdots & \sigma_{pp}\end{pmatrix}$$

Then the Q($\underline{X}$) is called the quadratic from in $\underline{X}$ which is a function of the following from:

$$Q(\underline{X}) = \underline{X}'AX$$

$$Q(\underline{X})=[X_1\quad X_2\quad\cdots\quad X_P]\begin{pmatrix}\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p}\\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ \sigma_{31} & \sigma_{32} & \cdots & \sigma_{pp}\end{pmatrix}\begin{bmatrix}X_1\\X_2\\\vdots\\X_P\end{bmatrix}$$

$$Q(\underline{X})=\sum_{I=1}^{P}\sum_{j=1}^{p}Xi_{aij}\quad X_j$$

جیاوازی لە نێوان (Q($\underline{X}$))و(Normal Distribution)ئەوەیە،لە((Q($\underline{X}$))دەکاتە $\begin{bmatrix}X_1\\X_2\\\vdots\\X_P\end{bmatrix}$ بەڵام

لە(Normal Distribution)$(\underline{X}-\underline{\mu})$

Example// If A is a symmetric matrix by(2*2) dimension. Find Quadratic form.

Solution// $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, X' = [x_1 \quad x_2]$

$Q(\underline{X}) = \underline{X}'AX = [x_1 \quad x_2] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$=[a_{11}x_1 + a_{21}x_2 \quad a_{12}x_1 + a_{22}x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Q($\underline{X}$)=$[a_{11}{x_1}^2 + a_{21}x_1x_2 + \quad a_{12}x_1x_2 + a_{22}{x_2}^2]$ =$a_{11}{x_1}^2 + 2a_{21}x_1x_2 + a_{22}{x_2}^2$

Example// let $A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. find Quadratic form $Q(\underline{X})$:

مەرجە دەبێت،سێگۆشەکان(symmetric)بن.

Solution//

$Q(\underline{X}) = \underline{X}'AX = X =. [x_1 \quad x_2] \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [2x_1 + x_2 \quad x_1 + 4x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$=2{x_1}^2 + x_1x_2 + x_1x_2 + 4{x_2}^2 = 2{x_1}^2 + 4{x_2}^2 + 2x_1x_2$

---

example// If A is a symmetric matrix by (3× 3) dimension. Find Quadratic form.

Solution//

A=$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,$\underline{X}$=$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $[x_1 \quad x_2 \quad x_3]$

$Q(\underline{X}) = \underline{X}'AX = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Q($\underline{X}$)= $a_{11}{x_1}^2 + a_{22}{x_2}^2 + a_{33}{x_3}^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$

Example// find Quadratic form of matrix A= $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

Solution// A= $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ $, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $, [x_1 \quad x_2 \quad x_3]$

$$Q(\underline{X}) = \underline{X}'AX \;= 2x_1{}^2 - 6x_2{}^2 + 9x_3{}^2$$

مەرجە دەبێت (A)(symmetric)بێت.

Example// find Quadratic form of matrix A= $\begin{bmatrix} 5 & 2 & 3 \\ 2 & 7 & 1 \\ 3 & 1 & 8 \end{bmatrix}$

Solution//

$$Q(\underline{X}) = \underline{X}'AX = 5x_1{}^2 + 7x_2{}^2 + 8x_3{}^2 + 4x_1x_2 + 6x_1x_3 + 2x_2x_3$$

Example// if the quadratic form $\underline{X}'A\underline{X} = 2x_1{}^2 + 4x_3{}^2 + 2x_1x_2 + 6x_2x_3$

Solution// A= $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 3 & 4 \end{bmatrix}$

Q) 2022)// if the quadratic form $\underline{X}'A\underline{X}$**=** $\mathbf{5x_2{}^2 - 3x_1{}^2 + 2x_3{}^2 + 12x_1x_2 + 6x_2x_3 - 6x_1x_3}$

Solution// A= $\begin{bmatrix} 3 & 6 & -3 \\ 6 & 5 & 3 \\ -3 & 3 & 2 \end{bmatrix}$

Classification of Quadratic form.

1- *positive definite (p.d)*

The quadratic form of $Q(\underline{X})$is called p.d. if $\underline{X}'A\underline{X}$ >0 for all $\underline{X} \neq 0$

یانی کاتێک دەبێتە(positive definite)ئەگەر هاتوو $(\underline{X}'A\underline{X})$گەورەتر بوو لە(0)بە مەرجێک نابێت هەموو($\underline{X}$)یەکسان بن بەسفر.

Example// let $p = 2$ $then$ $\underline{X}$ =$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,$prove\ that\ the\ quadratic\ form\ \underline{X}$, $Q(\underline{X})$is p.d.

Solution// $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \underline{X}' = [x_1 \quad x_2]$

Q($\underline{X}$) = $\underline{X}'A\underline{X} = [x_1 \quad x_2].\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1{}^2 + x_2{}^2 > 0$

Not that: for any real vector $\underline{X} \neq 0$ that Q($\underline{X}$) will be positive, because the square of any number is positive, the coefficient of the squared terms are positive and the sum of positive numbers is always positive.

(positive definite)ە جونكە ($x_1{}^2$)توان دووە،وە هەروەها ناوەراستیان(+)یە.

وە هەر نیشانەیەک ج بە موجەب ج بەسالب لە شوێنی ($x_1{}^2 + x_2{}^2$) دابنێین ئەوا هەر موجەبە جونکە توان دووە.

Example// let A= $\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$ $prove\ the$ Q($\underline{X}$)is?

Solution// A= $\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$,X = $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,X$'$ = $[x_1 \quad x_2]$

Q($\underline{X}$) =$\underline{X}'A\underline{X} = [x_1 \quad x_2]\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$[x_1 \quad x_2]\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} = [2x_1 - x_2 \quad -x_1 + 4x_2].\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_1{}^2 - x_1x_2 - x_1x_2 + 4x_2{}^2$

=$2x_1{}^2 + 4x_2{}^2 - 2x_1x_2$

Q($\underline{X}$) =$\underline{X}'A\underline{X} = 2{x_1}^2 + 4{x_2}^2 - 2x_1x_2$

The first and second terms are clearly positive, but with $|\, x_1 \,| > |\, x_2 \,|$,$\left|\, 2{x_1}^2 \,\right| >$ $|\, 2x_1x_2 \,|$, so that first term is more positive than the third term ,and so the whole expression is positive. The same thing if $|\, x_1 \,| < |\, x_2 \,|$

∴ Q($\underline{X}$) > 0 ⇒ Q($\underline{X}$)is p.d

(positive definite)جونكه ($2{x_1}^2$)وه ناوەراستيان(+)وه($4{x_2}^2$)توان دووه.

2) ***positive semi –definite (p.s.d)***

The quadratic form is $\underline{X}'A\underline{X} \leq 0$ for all $\underline{X} \neq 0$

Example// $p = 2$ *then* $\underline{X}$ =$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$,*prove that the quadratic form*$\underline{X}$, Q($\underline{X}$)is p.d.

Solution// $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.$\mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,$\mathrm{X}' = [x_1 \quad x_2]$

$$[x_1 \quad x_2].\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = [x_1 - x_2 \quad -x_1 + x_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [{x_1}^2 - x_1x_2 - x_1x_2 + {x_2}^2]$$

Q($\underline{X}$) =$\underline{X}'A\underline{X} = {x_1}^2 + {x_2}^2 - 2x_1x_2 \geq 0$

Where : ${x_1}^2 + {x_2}^2 - 2x_1x_2 = (x_1 - x_2)^2 \geq 0$

Q($\underline{X}$)=is p.s.d

3) negative and negative semi-definite (n.d& n.s.d)

Negative definite and negative semi-definite quadratic forms are similarly defined meaning that: the quadratic form is n.d. if $\underline{X}'A\underline{X} < 0$ for all $\underline{X} \neq 0$

The quadratic form is n.d if $\underline{X}'A\underline{X} \leq 0$ for all $\underline{X} \neq 0$

Another Method: Eigenvalues to determine classification of the quadratic form

The basic equation is $A\underline{X} = \lambda\underline{X}$

We may find λ= 2 or $\frac{1}{2}$ or -1 or 1. Most 2 by 2 matrix have two eigenvector direction and two eigenvalues and eigenvectors.

Classification of the quadratic form

1. positive definite (p.d.)

The quadratic form of Q($\underline{X}$) is called p.d. if $\lambda_1, \lambda_2, \dots, \lambda_n > 0 \; for\ all\ \lambda \neq 0$

2) positive Semi-definite (p.s.d)

The quadratic form is p.s.d if $\lambda_1, \lambda_2, \dots, \lambda_n \geq 0 \; for\ all\ \lambda \neq 0$

Example// $\boldsymbol{p = 2\ then, A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, prove\ that\ the\ quadratic\ from\ \mathbf{Q}(\underline{X}) is\ p.s.d}$

Solution// $|[A - \lambda I]| = 0$

$$\Rightarrow \left| \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} 1-\lambda & 0-0 \\ 0-0 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 0 * 0 = (\lambda^2 - 5\lambda + 4) = 0$$

$$\Rightarrow (1-\lambda) = 1 , \; (4-\lambda) = 4$$

$$\therefore \lambda_1 = 1\ and\ \lambda_2 = 4 > 0 \qquad \therefore Q(\underline{X})\ is\ p.d$$

Example//

$p=3\ then, A=\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}, prove\ that\ the\ quadratic\ from\ \mathbf{Q}(\underline{X}) is\ p.s.d$

Solution// det $|[A-\lambda I]|=0$

(3×3) دەکاتە (A) جونکە $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ دەکاتە (I) ئێمە دەبێت بزانین

وە هەروەها دەبێت(A)سێگۆشەکانیان یەکسان بن.

یانی بەشێوەیەکی گشتی دەبێت($\mathrm{Q}(\underline{X})$)(square)بێت.

$$\left|\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}-\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right|=0$$ جارانی ناو(λ) دەکەین(I)

$$\Rightarrow\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}-\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}=0$$

$$\Rightarrow\begin{vmatrix}2-\lambda & -1 & -1\\ -1 & 2-\lambda & -1\\ -1 & -1 & 2-\lambda\end{vmatrix}=\begin{vmatrix}2-\lambda & -1 & -1\\ -1 & 2-\lambda & -1\\ -1 & -1 & 2-\lambda\end{vmatrix}\begin{matrix}2-\lambda & -1\\ -1 & 2-\lambda\\ -1 & -1\end{matrix}$$

$(2-\lambda).(2-\lambda).(2-\lambda)+(-1*-1*-1\ )+(-1*-1*-1)$

$-(-1*(2-\lambda)*-1)-(-1*-1*(2-\lambda)-(2-\lambda)*-1*-1$

$=[(2-\lambda)^3-1-1]-[(2-\lambda)+(2-\lambda)+(2-\lambda)]=0$

$=[(2-\lambda)^3-2-3(2-\lambda)]=0$

$(8-12\lambda+6\lambda^2-\lambda^3)-2-6+3\lambda$

$6\lambda^2-\lambda^3-9\lambda=0\quad\times-\ \Rightarrow\ -6\lambda^2+\lambda^3+9\lambda=0\Rightarrow\lambda^3-6\lambda^2+9\lambda$

$\lambda(\lambda^2-6\lambda+9)$

$\lambda_1=0,\qquad \lambda^2-6\lambda+9=0\Rightarrow(\lambda-3)(\lambda-3)\Rightarrow\lambda_2 and\lambda_3=3\ \geq 0$

$\mathrm{Q}(\underline{X})=P.s.d$

$\Rightarrow \lambda_1, \ \lambda_2 = \frac{-b\pm\sqrt{b^2-4ac}}{2a} \ = \frac{6\pm\sqrt{(-6)^2-4(1)(9)}}{2(1)}$=3 0r $(\lambda - 3)^2 = 0$

$\therefore \ \lambda_1$=0 and $\lambda_2 \ and \ \lambda_3 = 3 \geq 0$ $\quad \therefore \ Q(\underline{X}) \ is \ p.s.d$

3) Negative definite and Negative Semi-definite(n.s.d) and (n.d)

Negative definite and negative semi-definite quadratic forms are similarly defined ,meaning that:

The quadratic form is n.d if $\lambda_1 , \lambda_2, \dots, \lambda_n < 0 \ for \ all \ \lambda \neq 0$

The quadratic form is n.s.d if $\lambda_1 , \lambda_2, \dots, \lambda_n \leq 0 \ for \ all \ \lambda \neq 0$

Example// IF $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Solution// بەهەمان شێوە دەبێت(A)سێگۆشەکانیان(یەکسان)بێت.

یانی بەشێوەیەکی گشتی دەبێت($Q(\underline{X})$)یەکسان بێت،لەهەموو بابەتەکان

(A$-\lambda I$)=0

$$\begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 0 & 0 \\ 0 & -2-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$(-2-\lambda)^3 = 0$

The eigenvalues $\lambda = -2$

$\therefore \ Q(\underline{X})$ is n.d

ئەگەر هاتوو سفر یان بچوکتر لە سفر ($Q(\underline{X}) \geq 0$)دەرچوو ئەوا دەڵێین

Negative Semi-definite

بەڵام ئەگەر هاتوو سفر و گەورەتر لە سفر($Q(\underline{X}) \geq 0$) دەرچوو ئەوا دەڵێین

Positive semi-definite

Example// If $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ prove that $Q(\underline{X})$ is n.s.d.

Solution// $(A - \lambda I) = 0$

$$\left|\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right|$$

$$\left|\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right|$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = (-1-\lambda) * (-1-\lambda) - 1 * 1$$

$$\Rightarrow \lambda^2 + 2\lambda = 0$$

$$\Rightarrow \lambda(\lambda + 2) = 0$$

The eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 0$

$\therefore$ $Q(\underline{X})$ $is\ n.s.d$

H.W//

A) If $= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, show\ that\ Q(\underline{X}) is\ n.d$

B) If $= \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix},\ prove\ that\ Q(\underline{X}) is\ n.d$

C) If $= \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix},\ prove\ that\ Q(\underline{X}) is\ n.s.d$

A) If $= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, show\ that\ Q(\underline{X}) is\ n.d$

Solution//

$$\left|\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = \left|\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = \begin{bmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix}$$

$\Rightarrow (-1-\lambda)^2 = -1$,

The eigenvalues $\lambda = -1$

$\therefore$ Q$(\underline{X})$ is n.d

B) If $= \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, $prove\ that\ Q(\underline{X}) is\ n.d$

Solution//

(A$-\lambda I$)=0

$$\begin{vmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{vmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

$$= \begin{bmatrix} -2-\lambda & 1-0 & 0 \\ 1-0 & -2-\lambda & 0 \\ 0-0 & 0-0 & -2-\lambda \end{bmatrix} = \begin{bmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{bmatrix}$$

$(-2-\lambda)^3 - 1$

$= (-2-\lambda)(-2-\lambda)(-2-\lambda) - 1$

$(-2-\lambda)(-2-\lambda) = (-4 + 6\lambda\ + \lambda^2)(-2-\lambda) - 1$

$(8 - 12\lambda - 2\lambda + 4\lambda - 6\lambda^2\ -\ \lambda^3) - 1$

$(\ 7 - 10\lambda\ - 6\lambda^2\ -\ \lambda^3) \rightarrow \lambda(-3 - \lambda^2 - 6\lambda\ ) =\ \lambda = 0, \qquad \lambda = 3$

Positive semi-definite

C) If $= \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$, $prove\ that\ Q(\underline{X}) is\ n.s.d$

Solution// (A$-\lambda I$)=0

$$\left| \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}$$

$$\Rightarrow \lambda^2 + 4\lambda = 0$$

$$\Rightarrow \lambda(\lambda + 4) = 0$$

$$= -4$$

Negative Semi-definite

4) Determine the classification of quadratic form $Q(\underline{X})$

1) **positive definite.**

بەپێی ئەم یاسایە بڕیاری کۆتایی لەسەر دەدەین.

ئەگەر هاتوو مەصفوفەکان تاک بوون ئەوا دەبێت (determine<) بجوکتر بێت لە سفر،($1 \times 1,\ 3 \times 3,\ 5 \times 5$) وە ئەگەر مەصفوفەکان جوت بوون ئەوا دەبێت ($2 \times 2, 4 \times 4, 6 \times 6$),(determine> 0) بەم شێوەیە.

Example// $\begin{bmatrix} \mathbf{-2} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-2} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{-3} \end{bmatrix}$

$|-2| = -2 < 0$

$\begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = (-2 \times -2) - (0 \times 0) = 4 > 0$

$\begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & -1 \\ -1 & -1 & -3 \end{bmatrix}$

$\left[\begin{array}{ccc|cc} -2 & 0 & -1 & -2 & 0 \\ 0 & -2 & -1 & 0 & -2 \\ -1 & -1 & -3 & -1 & -1 \end{array}\right]$

$=(-2 \times -2 \times -3) + (0 \times -1 \times -1) + (-1 \times 0 \times -1) - (-1 \times -1 \times -2) - (-1 \times -1 \times -2) - (-3 \times 0 \times 0) = -8 < 0$ posative definite

کەواتە هەرسێ (determine)پێچەوانەی هیچ لەیاساکە نەبوو کەواتە دەبێتە.(definite).

ئەگەر هاتباو لە شوێنی (-8) موجەب هەشت با (8) ئەوکات دەمان وت(indefinite)جونکە ئێمە وتمان دەبێت،مەصفوفەی تاک بجوکتر بێت، لەسفر.

Example//

$$A = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$|-4| = -4 < 0$$

$$\begin{vmatrix} -4 & 0 \\ 0 & -1 \end{vmatrix} = (-4 \times -1) - (0 \times 0) = 4 > 0$$

$$\begin{vmatrix} -4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{vmatrix} = -4 \times -1 \times -3 = -12 < 0$$

Positive definite

کەواتە هەرسێ (determine)پێچەوانەی هیچ لەیاساکە نەبوو کەواتە دەبێتە.(definite).

ئەگەر هاتباو لە شوێنی (-4) موجەب جوار با (4) ئەوکات دەمان وت(indefinite)جونکە ئێمە وتمان دەبێت،مەصفوفەی تاک بجوکتر بێت، لەسفر.

Example//

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

$$|1| = 1 > 0$$

$$\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = (1 \times 1) - (-2 \times -2) = -3 < 0$$

Q)2022) **Determine** the classification of the following quadratic from $Q(\underline{X})$

A= $\begin{bmatrix} 4 & 2 \\ 2 & -1 \end{bmatrix}$

Solution:

Order $1 \rightarrow |4| = 4 > 0, oredr\ 2 \rightarrow \begin{vmatrix} 4 & 2 \\ 2 & -1 \end{vmatrix} = (4 \times -1) - (2 \times 2) = -8 < 0$

کەواتە لێرە دەبێتە(indefinite) جونکە مەصفوفەی تاک دەبێت بجوکتر بێت، لەسفر بە لێرەدا گەورەترە لە سفر $1 > 0$ بۆیە دەبێتە (indefinite)،وە دووەمیش بەهەمان شێوە پێجەوانەی یاسایە، وەئەگەر یەک دانەشیان پێجەوانەی یاسا بوو ئەوا هەر دەبێتە(indefinite)

ئەگەر هاتوو مەصفوفەکان تاک بوون ئەوا دەبێت (determine) بجوکتر بێت لە سفر،($1\times 1, 3 \times 3, 5 \times 5$) وە ئەگەر مەصفوفەکان جوت بوون ئەوا دەبێت (determine> 0),($2 \times 2, 4 \times 4, 6 \times 6$) بەم شێوەیە.

$$(1 \times 1), (3 \times 3), (5 \times 5) \dots . < 0$$

$$(2 \times 2), (4 \times 4), (6 \times 6) \dots . > 0$$

Example//

$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$|-1| = -1 < 0$

$\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = (-1 \times -1) - (0 \times 0) = 1 > 0$

Positive definite

Example//

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$| \ 1| = 1 > 0$

$$\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = (1 \times 1) - (-1 \times -1) = 0$$

indefinite

Example// Determine the classification of quadratic form .

$Q(\underline{X}) if \ A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & 0 \end{bmatrix}$ Using **eigenvalue** method.

Solution// Det$(A - \lambda I) = 0$

$$\begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & 0 \end{vmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & 0 \end{vmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

$$\begin{bmatrix} -2-\lambda & 0 & 0 \\ 0 & -2-\lambda & 2 \\ 0 & 2 & -\lambda \end{bmatrix} = 0$$

$(-2-\lambda)[(-2-\lambda)(-2-\lambda) - 4]$

$-2-\lambda = 0 \, , \qquad OR \ (-2-\lambda) - 4 = 0$

$-2-\lambda \ \to \lambda = -2$

$[(-2-\lambda) \ (-\lambda) - 4] = (\ 2\lambda + \lambda^2 - 4) = 0$

$$\lambda = -2 \quad OR \ \lambda_2, \lambda_3 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = -2 \quad OR \ \lambda_2, \lambda_3 = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \sqrt{5}$$

$Q(\underline{X}) = IS\ indefinite$

Example// $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution//

$|1| = 1 > 0$, $\quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = (1 \times 1) - (0 \times 0) = 1 > 0 \quad positive\ definite$

بەڵام ئەگەر هاتباو $1 > 0$ بەم شیوەیە دەرجوبایە یانی یەکسان بایە بە سفر، ئەوکاتە دەبووە
$positive\ semi\ definite$

Negative definite

Example:

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Solution:

$order\ 1 \rightarrow\ |-2| = -2 < 0$, $\quad order\ 2 \rightarrow \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = (-2 \times -2) - (1 \times 1) = 3 > 0$

$$order\ 3 \rightarrow \begin{vmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} \begin{matrix} -2 & 1 \\ 1 & -2 \\ 0 & 0 \end{matrix}$$

$$(-2 \times -2 \times -2) + (1 \times 0 \times 0) + (0 \times 1 \times 0) - (0 \times -2 \times 0) - (0 \times 0 \times -2) - (-2 \times 1 \times 1) = -6 < 0$$

Negative definite

ئەگەر $-6 < 0$ هاتباو یەکسان بوبایە بەسفر ئەوکاتە دەبووە **nagative semi difinte**

وە ئەگەر $-2 < 0$ بەم شێوەیە دەرجوبایە $2 > 0$ ئەوکاتە دەبووە posative semi difinte

Example:

$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Solution:

Order $1 \rightarrow |-1| = -1 < 0$

$oredr\ 2 \rightarrow \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = (-1 \times -1) - (0 \times 0) = 1 > 0$

Negative semi definite

بەم شیوازە دەتوانین دیاری بکەین کەوا ئەو (matrix)ە، (negative semi definite)ە.

ئەگەر هاتوو مەصفوفە تاکەکان یانی $(1 \times 1,\ 3 \times 3, 5 \times 5)$ بجوکتر و یەکسان بوون بەسفر ، وە مەصفوفە جوتەکان یانی$(2 \times 2\ ,4 \times 4\ ,6 \times 6)$گەورەتر و یەکسان بوون بە سفر، ئەوکاتە دەبێتە(negative semi definite).

مەصفوفە تاکەکان$\leq \mathbf{0}$

مەصفوفە جوتەکان≥ 0

Example// $A = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$

Solution// $|-2| = -2 < 0 \leftarrow order\ 1$

Order 2 $\rightarrow \begin{vmatrix} -2 & 2 \\ 2 & -2 \end{vmatrix} = (-2 \times -2) - (2 \times 2) = 0\ ,\ \ is\ negative\ semi\ definite$

لێرەدا هەم دالەی یەکەم یاساکەی گرتەوە، هەم دالەی دووەم یاساکەی گرتەوە، بۆیە

$is\ negative\ semi\ definite$دەبێتە

Example//

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution // order 1→ $|-1| = -1 < 0$

Order 2→ $\begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} = (-1 \times -2) - (0 \times 0) = 2 > 0$

Order 3→ $\begin{vmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{matrix} -1 & 0 \\ 0 & -2 \\ 0 & 0 \end{matrix}$

$$(-1 \times -2 \times 0) + (0 \times 0 \times 0) + (0 \times 0 \times 0) - (0 \times -2 \times 0) - (0 \times 0 \times -1) - (0 \times 0 \times 0) = 0$$

$$is\ negative\ semi\ definite$$

لێرەدا هەم دالەی یەکەم یاساکەی گرتەوە، هەم دالەی دووەم یاساکەی گرتەوە،هەم دالەی سێیەم یاساکەی گرتەوە، بۆیە دەبێتە $is\ negative\ semi\ definite$

ئەگەر $-1 < 0$ بەم شێوەیە دەرجوبایە $1 > 0$ ئەوکاتە دەبووە

$$is\ \ posative\ semi\ definite$$

Positive semi definite

Example//

$$A=\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}$$

Solution//

Order 1→ $|2| = 2 > 0$

Order 2→ $\begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix} = (2\times 2)-(-1\times -1) = 3 > 0$

Order 3→ $\begin{vmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{vmatrix}\begin{matrix}2 & -1\\ -1 & 2\\ -1 & -1\end{matrix}$

$$= (2\times 2\times 2)+(-1\times -1\times -1)+(-1\times -1-1)-(-1\times 2\times -1)-(-1\times -1\times 2)-(2\times -1\times -1)=0$$

$$\textit{is posative semi definite}$$

ئەگەر $2 > 0$ بەم شیوەیە دەرجوبایە، $-2 > 0$ ئەوکاتە دەبووە *is negative semi definite*

Example//

$$A=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Solution//

Order 1→ $|\ 1| = 1 > 0$

Order 2→ $\begin{vmatrix}1 & -1\\ -1 & 1\end{vmatrix} = (\ 1\times 1)-(-1\times -1) =\ 0$

$$\textit{is posative semi definite}$$

ئەگەر $0 < 1$ بەم شیوەیە دەرجوبایە $0 < 1-$ ئەوکاتە دەبووە $is\ negative\ semi\ definite$

وە ئەگەر (2 Order) گەورەتر دەرجوبایە لە سفر ئەوکاتە دەبووە positive definite

Chapter four

Partition of vectors and matrix

Dif[th]: let $\underline{X}$ be p-random vector with $\underline{X} \sim \left(\underline{\mu}, \Sigma\right)$ then $\underline{X}$ has been partitioned into sum- vectors, as follows.

واته(vector)ی(x) دابەش کراوەتە سەر(vector)ی بچووک تر .

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ \cdots \\ x_r+1 \\ x_r+2 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{bmatrix},\ \underline{X}^{(1)} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix}, \underline{X}^{(2)} = \begin{bmatrix} x_1+1 \\ x_1+2 \\ \vdots \\ x_p \end{bmatrix}, \underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_r \\ \cdots \\ \mu_r+1 \\ \mu_r+2 \\ \vdots \\ \mu_p \end{bmatrix} = \begin{bmatrix} \underline{\mu}^{(1)} \\ \underline{\mu}^{(2)} \end{bmatrix}, \underline{\mu}^{(1)} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_r \end{bmatrix}, \underline{\mu}^{(2)} = \begin{bmatrix} \mu_1+1 \\ \mu_1+2 \\ \vdots \\ \mu_p \end{bmatrix}$$

یانی هەر یەکێک لە($\underline{X}^{(1)}$) و ($\underline{X}^{(2)}$)،(μ)ی،تایبەت بە خۆیەوە هەیە.

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1\,r} & \vdots & \sigma_{1\,r+1} & \sigma_{1\,r+2} & \cdots & \sigma_{1\,p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2\,r} & \vdots & \sigma_{2\,r+1} & \sigma_{2\,r+2} & \cdots & \sigma_{2\,p} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{r\,1} & \sigma_{r\,2} & \cdots & \sigma_{r\,r} & \vdots & \sigma_{r\,r+1} & \sigma_{r\,r+2} & \cdots & \sigma_{r\,p} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{r+11} & \sigma_{r+12} & \cdots & \sigma_{r+1\,r} & \vdots & \sigma_{r+1r+1} & \sigma_{r+1r+2} & \cdots & \sigma_{r+1\,p} \\ \sigma_{r+21} & \sigma_{r+22} & \cdots & \sigma_{r+2\,r} & \vdots & \sigma_{r+2r+1} & \sigma_{r+2r+2} & \cdots & \sigma_{r+2p} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{p\,r} & \vdots & \sigma_{p\,r+1} & \sigma_{p\,r+2} & \cdots & \sigma_{pp} \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

$$\Sigma_{11} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1\,r} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2\,r} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{r\,1} & \sigma_{r\,2} & \cdots & \sigma_{r\,r} \end{bmatrix} = var\left(\underline{X}^{(1)}\right) \quad ,\Sigma_{21} = \begin{bmatrix} \sigma_{r+11} & \sigma_{1\,r+2} & \cdots & \sigma_{r+1\,r} \\ \sigma_{r+21} & \sigma_{2\,r+2} & \cdots & \sigma_{r+2\,r} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{P\,1} & \sigma_{P\,2} & \cdots & \sigma_{P\,r} \end{bmatrix} = cov(\underline{X}^{(1)}, \underline{X}^{(2)})$$

$$\Sigma_{12} = \begin{bmatrix} \sigma_{1\,r+1} & \sigma_{1\,r+2} & \cdots & \sigma_{1\,p} \\ \sigma_{2\,r+1} & \sigma_{2\,r+2} & \cdots & \sigma_{2\,p} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{rr+1} & \sigma_{rr+2} & \cdots & \sigma_{r\,p} \end{bmatrix} = cov(\underline{X}^{(1)}, \underline{X}^{(2)}),$$

$$\Sigma_{22} = \begin{bmatrix} \sigma_{r+1r+1} & \sigma_{r+1r+2} & \cdots & \sigma_{r+1\,p} \\ \sigma_{r+2r+1} & \sigma_{r+2\,r+1} & \cdots & \sigma_{r+2\,p} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{pr+1} & \sigma_{pr+2} & \cdots & \sigma_{p\,p} \end{bmatrix} = cov(\underline{X}^{(2)})$$

$$\Sigma = \begin{bmatrix} var(\underline{X}^{(1)}) & cov(\underline{X}^{(1)}, \underline{X}^{(2)}) \\ cov(\underline{X}^{(1)}, \underline{X}^{(2)}) & cov(\underline{X}^{(2)}) \end{bmatrix}$$

Var - cov($\underline{X}$)=Σ

$\Sigma = E(\underline{X} - \underline{\mu})(\underline{X} - \underline{\mu})'$

$$\Sigma_{11} = var - cov(\underline{X}^{(1)}) = E(\underline{X}^{(1)} - \underline{\mu}^{(1)})(\underline{X}^{(1)} - \underline{\mu}^{(1)})'$$

$$\Sigma_{22} = var - cov(\underline{X}^{(2)}) = E(\underline{X}^{(2)} - \underline{\mu}^{(2)})(\underline{X}^{(2)} - \underline{\mu}^{(2)})'$$

$$\Sigma_{12} = cov(\underline{X}^{(1)}, \underline{X}^{(2)}) = E(\underline{X}^{(1)} - \underline{\mu}^{(1)})(\underline{X}^{(2)} - \underline{\mu}^{(2)})'$$

$\underline{X} \sim N(\underline{\mu}, \Sigma)$

$\underline{X}^{(1)} \sim N(\underline{\mu}^{(1)}, \Sigma_{11})$

$\underline{X}^{(2)} \sim N(\underline{\mu}^{(2)}, \Sigma_{22})$

Example// $(x_1 x_2 | x_3)$ =($\underline{x}^{(1)} | \underline{x}^{(2)}$)

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_3 \end{bmatrix} \sim \left(\begin{bmatrix} \mu_1 \\ \mu_2 \\ \cdots \\ \mu_3 \end{bmatrix}, \begin{bmatrix} \sigma_{11} & \sigma_{12} & \vdots & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \vdots & \sigma_{23} \\ \cdots & \cdots & \vdots & \cdots \\ \sigma_{31} & \sigma_{32} & \cdots & \sigma_{33} \end{bmatrix} \right), \underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_3 \end{bmatrix} = \begin{bmatrix} \underline{x}^{(1)} \\ \underline{x}^{(2)} \end{bmatrix}, \underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \cdots \\ \mu_3 \end{bmatrix} = \begin{bmatrix} \underline{\mu}^{(1)} \\ \underline{\mu}^{(2)} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \vdots & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \vdots & \sigma_{23} \\ \cdots & \cdots & \vdots & \cdots \\ \sigma_{31} & \sigma_{32} & \cdots & \sigma_{33} \end{bmatrix}$$

$\Sigma_{11} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = var\left(\underline{X}^{(1)}\right)$, $\Sigma_{12} = \begin{bmatrix} \sigma_{13} \\ \sigma_{23} \end{bmatrix} = cov(\underline{X}^{(1)}, \underline{X}^{(2)})$, , $\Sigma_{21} = [\sigma_{31} \quad \sigma_{32}] =$ $cov(\underline{X}^{(1)}, \underline{X}^{(2)})$ =, $\Sigma_{22} = [\sigma_{33}] = cov(\underline{X}^{(2)})$

$$\Sigma = \begin{bmatrix} var\left(\underline{X}^{(1)}\right) & cov(\underline{X}^{(1)}, \underline{X}^{(2)}) \\ cov(\underline{X}^{(1)}, \underline{X}^{(2)}) & cov(\underline{X}^{(2)}) \end{bmatrix}$$

Var-cov($\underline{X}$)=Σ

$\Sigma = E(\underline{X} - \underline{\mu})(\underline{X} - \underline{\mu})'$

$\Sigma_{11} = var - cov\left(\underline{X}^{(1)}\right) = E(\underline{X}^{(1)} - \underline{\mu}^{(1)})(\underline{X}^{(1)} - \underline{\mu}^{(1)})'$

$\Sigma_{22} = var - cov\left(\underline{X}^{(2)}\right) = E(\underline{X}^{(2)} - \underline{\mu}^{(2)})(\underline{X}^{(2)} - \underline{\mu}^{(2)})'$

$\Sigma_{12} = cov(\underline{X}^{(1)}, \underline{X}^{(2)}) = E(\underline{X}^{(1)} - \underline{\mu}^{(1)})(\underline{X}^{(2)} - \underline{\mu}^{(2)})'$

$\underline{X} \sim N(\underline{\mu}, \Sigma)$

$\underline{X}^{(1)} \sim N(\underline{\mu}^{(1)}, \Sigma_{11})$

$\underline{X}^{(2)} \sim N(\underline{\mu}^{(2)}, \Sigma_{22})$

Functions of Multivariate Normal Distribution

I- Marginal Distribution function

Let the j.p.d.f of two r.v.s X and Y is given by $f(x, y)$then the m.p.d.f of X

$$f(x) = \int_{Ry} f(x, y)dy$$

And the marginal p.d.f of Y is denoted by $g(y)$and is define as

$$g(y) = \int_{RX} f(x, y)dx$$

Is X and Y are independent, then $f(x,y) = f(x).f(y)$

In general case of$(x_1, x_2, \dots, x_p)$with j.p.d.f is $f(x_1, x_2, \dots, x_p) or f(\underline{X})$

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ \cdots \\ x_r + 1 \\ x_r + 2 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{bmatrix}, \qquad \underline{X}^{(1)} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix}, \underline{X}^{(2)} = \begin{bmatrix} x_1 + 1 \\ x_1 + 2 \\ \vdots \\ x_p \end{bmatrix}$$

Then the j.p.d.f of $\underline{X}^{(1)} is$

$$\int_1 \left(\underline{X}^{(1)}\right) = \int_{R\underline{X}^{(1)}} f\left(\underline{X}^{(1)}, \underline{X}^{(2)}\right) d\underline{X}^{(2)}$$

$$\int_1 (x_1, x_2, \dots, x_r) = \int_{RXr+1} \cdots \int_{R\,XP} f(x_1, x_2, \dots, x_p) dXp \dots dXr + 1$$

And the j.p.d.f of $\underline{X}^{(2)}$

$$\int_2 \left(\underline{X}^{(2)}\right) = \int_{R\underline{X}^{(1)}} f\left(\underline{X}^{(1)}, \underline{X}^{(2)}\right) d\underline{X}^{(1)}$$

$$\int_2 (x_{r+1}, x_{r+2}, \dots, x_p) = \int_{RX_R} \cdots \int_{R\,X1} f(x_1, x_2, \dots, x_p) dX_1 \dots dXr$$

If $\underline{X}^{(1)} and\ \underline{X}^{(2)}\ are\ independet\ f(\underline{X}^{(1)}, \underline{X}^{(2)}) = f(\underline{X}^{(1)}).f(\underline{X}^{(2)})$

II. The moment of Multivariable

A. The Expected value of random vector $\underline{X}\ is\ the\ vector\ of\ expectation\ of\ its\ elements$

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} \Rightarrow E(\underline{X}) = \begin{bmatrix} E(x_1) \\ E(x_2) \\ \vdots \\ E(x_p) \end{bmatrix} = \begin{bmatrix} \int x_1 f(x_1) dx_1 \\ \int x_2 f(x_2) dx_2 \\ \vdots \\ \int x_p f(x_p) dx_p \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix}$$

B. the Expected value of random vector (z) *is the* vector of expectation of in elements.

$$z = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{13} \\ x_{21} & x_{22} & \cdots & x_{23} \\ \cdots & \cdots & \ddots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{n3} \end{bmatrix} \Rightarrow E(Z) = \begin{bmatrix} E(x_{11}) & E(x_{12}) & \cdots & E(x_{13}) \\ E(x_{21}) & E(x_{22}) & \cdots & E(x_{23}) \\ \cdots & \cdots & \ddots & \cdots \\ E(x_{n1}) & E(x_{n2}) & \cdots & E(x_{n3}) \end{bmatrix}$$

C. $E\left(x_1^r x_2^r \cdots x_p^r\right) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^r x_2^r \cdots f\left(x_1, x_2, \dots, x_p\right) dx_p \cdots dx_1$

D. if $A\&B$ are two constant matrix, then $E(AZB) = AE(Z)B$

E. if $\underline{X} = T\underline{Y}$ where $\underline{X}$ *and* $\underline{Y}$ are two random vectors, and T is constant, then.

$$E(\underline{X}) = TE(\underline{Y})$$

F- the correlation coefficient between X_i *and* X_j

$$Y_{ij} = \frac{cov(X_i, X_j)}{\sqrt{f(X_i)}\sqrt{f(X_j)}} = \frac{E(X_i - E(X_i)(X_j - E(X_j)}{\sqrt{E(X_i - E(X_i)^2}\sqrt{(X_j - E(X_j)^2}} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{jj}}} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$$

And the matrix of population correlation.

$$\rho = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1\,p} \\ \rho_{21} & 1 & \rho_{13} & \cdots & \rho_{2\,p} \\ \rho_{31} & \rho_{32} & 1 & \cdots & \rho_{2\,p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{p\,1} & \rho_{p\,2} & \rho_{p\,3} & \cdots & 1 \end{bmatrix}_{p*p}$$

هەرەدەم(correlation matrix)سێگۆشەکانیان یەکسانن،واتە(symmetric matrix)

III. The Statistical Independence

Let X & Y are two r.v.s with j.p.d.f $f(x,y)$ and they are said to be independent if:

$$f(x,y) = f(x) * f(y)$$

Where $f(x)$&$f(y)$ are the marginal distribution function of (x,y) respectively,

Ten the j.p.d.f of $(x_1, x_2, \dots, x_p) is f(x_1, x_2, \dots, x_p)$

Then the set of r.v'.s are said to be independent if :

$$f(x_1, x_2, \dots, x_p) = f_1(x_1) * f_2(x_2) * \dots * f_p(x_p) = \prod_{i=1}^{p} f_i(x_i)$$

Where $f_i(x_i)$ is the m.p.d.f of x_i $where\ i = 1,2,\dots,p$

Theorem : let $\underline{X} \sim N(\underline{\mu}, \Sigma)$where $\underline{X} = \begin{bmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{bmatrix}, \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$

1- show that $\underline{X}^{(1)}$ and $\underline{X}^{(2)}$ are independent if $\Sigma_{12} = 0$

2- if $\underline{X}^{(1)}$ and $\underline{X}^{(2)}$ are independent if $\Sigma_{12} = 0$

Proof:

1- the joint p.d.f. of $\underline{X} \sim N(\underline{\mu}, \Sigma)$

$$f\left(\underline{X}; \underline{\mu}, \Sigma\right) = \frac{1}{(2\pi)^{\frac{p}{2}} \, |\Sigma|^{\frac{1}{2}}} \, exp\left(-\frac{1}{2}(\underline{X} - \underline{\mu})' \, \Sigma^{-1}(\underline{X} - \underline{\mu})\right)$$

$$Q(\underline{X}) = (\underline{X} - \underline{\mu})'\Sigma^{-1}(\underline{X} - \underline{\mu})$$

$$Q(\underline{X}) = (\underline{X}^{(1)} - \underline{\mu}^{(1)} \quad \underline{X}^{(2)} - \underline{\mu}^{(2)})' \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}^{-1} \begin{pmatrix} \underline{X}^{(1)} - \underline{\mu}^{(1)} \\ \underline{X}^{(2)} - \underline{\mu}^{(2)} \end{pmatrix}$$

$$\Sigma^{-1} = \begin{bmatrix} \boldsymbol{\Sigma_{11}^{-1}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma_{12}^{-1}} \end{bmatrix}$$

$$Q(\underline{X}) = (\underline{X} - \underline{\mu})'\Sigma^{-1}(\underline{X} - \underline{\mu})$$

$$Q(\underline{X}) = \left[(\underline{X}^{(1)} - \underline{\mu}^{(1)})' \quad (\underline{X}^{(2)} - \underline{\mu}^{(2)})'\right] \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}^{-1} \begin{bmatrix} \underline{X}^{(1)} - \underline{\mu}^{(1)} \\ \underline{X}^{(2)} - \underline{\mu}^{(2)} \end{bmatrix}$$

$$Q(\underline{X}) = (\underline{X}^{(1)} - \underline{\mu}^{(1)})'\Sigma_{11}^{-1} \; (\underline{X}^{(2)} - \underline{\mu}^{(2)})'\Sigma_{22}^{-1} \begin{bmatrix} \underline{X}^{(1)} - \underline{\mu}^{(1)} \\ \underline{X}^{(2)} - \underline{\mu}^{(2)} \end{bmatrix}$$

$$Q(\underline{X}) = (\underline{X}^{(1)} - \underline{\mu}^{(1)})'\Sigma_{11}^{-1}\underline{X}^{(1)} - \underline{\mu}^{(1)} + (\underline{X}^{(2)} - \underline{\mu}^{(2)})'\Sigma_{22}^{-1}\underline{X}^{(2)} - \underline{\mu}^{(2)}$$

$$Q(\underline{X}) = Q(\underline{X}^{(1)}) + Q(\underline{X}^{(2)})$$

The j.p.d.f of $\underline{X}^{(1)}$ and $\underline{X}^{(2)} is\ a\ follows$

$$f\left(\underline{X}^{(1)}, \underline{X}^{(2)}; \underline{\mu}^{(1)}, \underline{\mu}^{(2)}, \Sigma_{11}, \Sigma_{22}\right) = \frac{1}{(2\pi)^{\frac{r}{2}}(2\pi)^{\frac{p-r}{2}}|\Sigma_{11}|^{\frac{1}{2}}|\Sigma_{22}|^{\frac{1}{2}}} exp\left(-\frac{1}{2}(Q(\underline{X}^{(1)}) + Q(\underline{X}^{(2)})\right)$$

$$f\left(\underline{X}^{(1)},\underline{X}^{(2)};\underline{\mu}^{(1)},\underline{\mu}^{(2)},\Sigma_{11},\Sigma_{22}\right)=\frac{1}{(2\pi)^{\frac{r}{2}}|\Sigma_{11}|^{\frac{1}{2}}}Exp\left(-\frac{1}{2}(Q(\underline{X}^{(1)})\right)\frac{1}{(2\pi)^{\frac{p-r}{2}}|\Sigma_{22}|^{\frac{1}{2}}}exp\left(\frac{-1}{2}Q(\underline{X}^{(2)})\right)$$

$$f\left(\underline{X}^{(1)},\underline{X}^{(2)};\underline{\mu}^{(1)},\underline{\mu}^{(2)},\Sigma_{11},\Sigma_{22}\right)=f\left(\underline{X}^{(1)}\right)*f(\underline{X}^{(2)})$$

proved

$$\Sigma^{-1}=\begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{bmatrix}^{-1}=\begin{bmatrix}\boldsymbol{\Sigma_{11}^{-1}} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Sigma_{12}^{-1}}\end{bmatrix}$$

$$let\ B=\begin{bmatrix}B_{11} & 0\\ 0 & B_{22}\end{bmatrix}\ It\ the\ inverse\ of\ the\ \Sigma\ matrix\ then\ \begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{bmatrix}.\begin{bmatrix}B_{11} & 0\\ 0 & B_{22}\end{bmatrix}$$

$$\begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{bmatrix}.\begin{bmatrix}B_{11} & 0\\ 0 & B_{22}\end{bmatrix}=\begin{bmatrix}\Sigma_{11}*B_{11}+\Sigma_{12}*0 & \Sigma_{11}*0+\Sigma_{12}*B_{22}\\ \Sigma_{21}*B_{11}+B_{22}*0 & \Sigma_{21}*0+\Sigma_{22}*B_{22}\end{bmatrix}=\begin{bmatrix}\Sigma_{11}B_{11} & 0\\ 0 & \Sigma_{22}B_{22}\end{bmatrix}$$
$$=\begin{bmatrix}I_{11} & 0\\ 0 & I_{22}\end{bmatrix}$$

$$B_{11}=\boldsymbol{\Sigma_{11}^{-1}}I_{11}=B_{11}=\boldsymbol{\Sigma_{11}^{-1}},\quad B_{22}=\boldsymbol{\Sigma_{22}^{-1}}I_{22},B_{22}=\boldsymbol{\Sigma_{22}^{-1}}$$

2- if $\underline{X}^{(1)}$ *and* $\underline{X}^{(2)}$ are independent if $\Sigma_{12}=0$

$$\Sigma_{11}=E(\underline{X}^{(1)}-\underline{\mu}^{(1)})\ (\underline{X}^{(2)}-\underline{\mu}^{(2)})'$$

Let X_i be a variable from the first subset $\underline{X}^{(1)}$ where $i=1,2,\dots,r$

And X_j be a variable from the secont subset $\underline{X}^{(2)}$ where $j=r+1,r+2,\dots,p$

Now , we wont to prove that $cov(X_i,X_j)=0$

$$\sigma_{ij}=cov(X_i,X_j)=E(X_i-\mu_i)(X_j-\mu_j)$$

$$E(x)=\int xf(x)dx$$

$$E(X_i-\mu_i)(X_j-\mu_j)=\int_{-\infty}^{\infty}\ddot{p}\int_{-\infty}^{\infty}(X_i-\mu_i)(X_j-\mu_j)f(x_1,x_2,\dots,x_p)dx_p\dots dx_1$$

$\because$ $\underline{X}^{(1)}$ *and* $\underline{X}^{(2)}$are independent by assumption

$$f\left(\underline{X}^{(1)}, \underline{X}^{(2)}\right) = f\left(\underline{X}^{(1)}\right) * f\left(\underline{X}^{(2)}\right)$$

$$f\left(x_1, x_2, \dots, x_p\right) = f(x_1, x_2, \dots, x_r) * f(x_{r+1}, x_{r+2}, \dots, x_p)$$

$$\sigma_{ij} = \int_{-\infty}^{\infty} \overset{p}{\cdots} \int_{-\infty}^{\infty} (X_i - \mu_i)\left(X_j - \mu_j\right) f(x_1, x_2, \dots, x_r) * f(x_{r+1}, x_{r+2}, \dots, x_p)$$

$$\sigma_{ij} = \int_{-\infty}^{\infty} \overset{r}{\cdots} \int_{-\infty}^{\infty} (X_i - \mu_i) f(x_1, x_2, \dots, x_r) dx_r \dots dx_1$$
$$* \int_{-\infty}^{\infty} \overset{p-r}{\cdots} \int_{-\infty}^{\infty} \left(X_j - \mu_j\right) f\left(x_{r+1}, x_{r+2}, \dots, x_p\right) dx_p \dots dx_{r+1}$$

$$\sigma_{ij} = E(X_i - \mu_i) E\left(X_j - \mu_j\right)$$

$$\sigma_{ij} = (E(X_i) - \mu_i)\left(E(X_j) - \mu_j\right)$$

$$\sigma_{ij} = (\mu_i - \mu_i)\left(\mu_j - \mu_j\right) = 0 * 0 = 0$$

$$\therefore = 0 \Rightarrow \sum_{21} = 0 \quad PROVED$$

H.W// for the following data that are normally distributed $\underline{X} \sim N(\underline{\mu}, \Sigma)$ find mean vector , var-covar matrix , correlation matrix , and prove or disprove that

$\underline{X}^{(1)}$ and $\underline{X}^{(2)}$ are independent ,if:

$\underline{X}^{(1)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\underline{X}^{(2)} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$

x_1	35	35	40	10	6	20	35	35	35	30
x_2	3.5	4.5	30	2.8	2.7	2.8	4.6	10.9	8	1.6
x_3	2.8	2.7	4.38	3.21	2.73	2.81	2.88	2.9	3.28	3.2
x_4	1	2	2	4	3	2	1	0	0	2

Solution//

$$X = \begin{bmatrix} 35 & 3.5 & 2.8 & 1 \\ 35 & 4.5 & 2.7 & 2 \\ 40 & 30 & 4.38 & 2 \\ 10 & 02.8 & 3.21 & 4 \\ 6 & 2.7 & 2.73 & 3 \\ 20 & 2.8 & 2.81 & 2 \\ 35 & 4.6 & 2.88 & 1 \\ 35 & 10.9 & 2.9 & 0 \\ 35 & 8 & 3.28 & 0 \\ 30 & 1.6 & 3.2 & 2 \end{bmatrix}$$

$$\underline{\overline{X}} = \begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \\ \overline{X}_3 \\ \overline{X}_4 \end{bmatrix} = \begin{bmatrix} 28.1 \\ 7.18 \\ 3.08 \\ 1.7 \end{bmatrix}$$

$$\overline{X}_1 = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{(35 + 35 + 40 + 10 + 6 + 20 + 35 + 35 + 35 + 30)}{10} = \frac{281}{10} = 28.1$$

$$\overline{X}_2 = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{(3.5 + 4.5 + 30 + 2.8 + 2.7 + 2.8 + 4.6 + 10.9 + 8 + 1.6)}{10} = \frac{71.8}{10} = 7.18$$

$$\overline{X}_3 = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{(2.8 + 2.7 + 4.38 + 3.21 + 2.73 + 2.81 + 2.88 + 2.9 + 3.28 + 3.2)}{10} = \frac{30.8}{10} = 3.08$$

$$\overline{X}_4 = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{(1+2+2+4+3+2+1+0+0+2)}{10} = \frac{17}{10} = 1.7$$

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{bmatrix} = \begin{bmatrix} 140.54 & 49.7 & 1.94 & -11.08 \\ 49.7 & 75.25 & 3.68 & -1.78 \\ 1.94 & 3.68 & 0.25 & 0.04 \\ -11.08 & -1.78 & 0.04 & 1.57 \end{bmatrix}$$

$$s_{11} = \frac{\sum_{i=1}^{n} (x_{i1} - \overline{X}_1)^2}{n-1}$$

$$s_{11} = \sum_{i=1}^{10} \frac{(35-28.1)^2 + (35-28.1)^2 + (40-28.1)^2 + (10-28.1)^2 + \cdots + (30-28.1)^2}{9}$$

$$\frac{1264.86}{9} = 140.54$$

$$s_{22} = \frac{\sum_{i=1}^{n} (x_{i2} - \overline{X}_2)^2}{n-1}$$

$$= \sum_{i=1}^{10} \frac{(3.5-7.18)^2 + (4.5-7.18)^2 + (30-7.18)^2 + (2.8-7.18)^2 + \cdots + (1.6-7.18)^2}{10-1} = \frac{677.25}{9}$$

$$= 75.25$$

$$s_{33} = \frac{\sum_{i=1}^{n} (x_{i3} - \overline{X}_3)^2}{n-1}$$

$$= \sum_{i=1}^{10} \frac{(2.8-3.08)^2 + (2.7-3.08)^2 + (4.38-3.08)^2 + (3.21-3.08)^2 + \cdots + (3.2-3.08)^2}{10-1}$$

$$= \frac{2.25}{9} = 0.25$$

$$s_{44} = \frac{\sum_{i=1}^{n} (x_{i4} - \overline{X}_4)^2}{n-1}$$

$$= \frac{(1-1.7)^2 + (2-1.7)^2 + (2-1.7)^2 + (4-1.7)^2 + (3-1.7)^2 + (2-1.7)^2 + \cdots + (2-1.7)^2}{10-1}$$

$$= \frac{14.13}{9} = 1.57$$

$$s_{12} = \frac{\sum_{i=1}^{n} (x_{i1} - \overline{X}_1)(x_{j2} - \overline{X}_2)}{n-1}$$

$$= \frac{(35-28.1)(3.5-7.18)+(35-28.1)(4.5-7.18)+(40-28.1)(30-7.18)+(10-28.1)}{10-1}$$

$$\frac{(2.8-7.18)+(6-28.1)(2.7-7.18)+(20-28.1)(2.8-7.18)+(35-28.1)(4.6-7.18)+(35-28.1)}{9}$$

$$\frac{(10-7.18)+(35-28.1)(8.-7.18)+(30-28.1)(1.6-7.18)}{9} = \frac{444.274}{9} = 49.7$$

$$s_{1\,3} = \frac{\sum_{i=1}^{n}(x_{i1}-\bar{X}_1)(x_{j\,3}-\bar{X}_3)}{n-1}$$

$$= \frac{(35-28.1)(2.8-3.08)+(35-28.1)(2.7-3.08)+(40-28.1)(4.38-3.08)+(10-28.1)(3.21-3.08)}{10-1}$$

$$\frac{+(6.28-28.1)(2.73-3.08)+(20-28.1)(2.81-3.08)+(35-28.1)(2.88-3.08)+(35-28.1)(2.9-3.08)}{9}$$

$$+\frac{35-28.1)(3.28-3.03)+(30-28.1)(3.2-2-3.08)}{9} = \frac{17.471}{9} = 1.94$$

$$s_{1\,4} = \frac{\sum_{i=1}^{n}(x_{i1}-\bar{X}_1)(x_{j\,4}-\bar{X}_4)}{n-1}$$

$$= \frac{(35-28.1)(1-1.7)+(35-28.1)(2-1.7)+(40-28.1)(2-1.7)+\cdots+(30-28.1)(2-1.7)}{10-1}$$

$$= \frac{-106.2}{9} = -11.8$$

$$s_{21} = \frac{\sum_{i=1}^{n}(x_{i1}-\bar{X}_1)(x_{j2}-\bar{X}_2)}{n-1}$$

$$= \frac{(3.5-7.18)(35-28.1)+(4.5-7.18)(35-7.18)+(30-7.18)(40-28.1)+\cdots+(1.6-7.18)(30-28.1)}{10-1}$$

$$= \frac{447.3}{9} = 49.7$$

$$s_{23} = \frac{\sum_{i=1}^{n}(x_{i\,2}-\bar{X}_2)(x_{j\,3}-\bar{X}_3)}{n-1}$$

$$= \frac{(3.5-7.18)(2.8-3.08)+(4.5-7.18)(2.7-3.08)+(30-7.18)(4.38-3.08)+\cdots+(1.6-7.18)(3.2-3.08)}{10-1}$$

$$= \frac{33.12}{9} = 3.68$$

$$s_{2\,4} = \frac{\sum_{i=1}^{n}(x_{i\,2}-\bar{X}_2)(x_{j\,4}-\bar{X}_4)}{n-1}$$

$$= \frac{(3.5-7.18)(1-1.7)+(4.5-7.18)(2-1.7)+(30-7.18)(4.38-1.7)+\cdots+(1.6-7.18)(2-1.7)}{10-1}$$

$$= \frac{-16.02}{9} = -1.78$$

$$s_{34} = \frac{\sum_{i=1}^{n}(x_{i3} - \overline{X}_2)(x_{j4} - \overline{X}_4)}{n-1}$$

$$= \frac{(2.8 - 3.08)(\ 1 - 1.7) + (\ 2.7 - 3.08)(\ 2 - 1.7) + (\ 4.38 - 3.08)(\ 1 - 1.7) + \cdots + (\ 3.2 - 3.08)(\ 2 - 1.7)}{10 - 1}$$

$$= \frac{0.36}{9} = 0.04$$

Correlation

وە نابێت ئەنجامەکان لە یەک گەورەتر بن وە ناشبێت لە لە سالب یەک بجوکتر بن جونکە (correlation) لە نێوان سالب یەک و یەکدایە.

$$s_{11} = v(x_1)\ , \qquad s_{22} = v(x_2), \quad s_{33} = v(x_3), \quad s_{44} = v(x_4)$$

$$R = \begin{bmatrix} r_{12} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{bmatrix} = \begin{bmatrix} 1 & r_{12} & r_{13} & r_{14} \\ r_{21} & 1 & r_{23} & r_{24} \\ r_{31} & r_{32} & 1 & r_{34} \\ r_{41} & r_{42} & r_{43} & 1 \end{bmatrix}$$

$$r_{12} = \frac{cov(x_1, x_2)}{\sqrt{v(x_1)v(x_2)}.} = \frac{cov(x_1, x_2)}{\sqrt{s_{11} * s_{22}}} = \frac{49.7}{\sqrt{140.54 * 75.25}} = \frac{49.7}{100.75} = 0.49$$

$$r_{13} = \frac{cov(x_1, x_3)}{\sqrt{v(x_1)v(x_3)}.} = \frac{cov(x_1, x_3)}{\sqrt{s_{11} * s_{33}}} = \frac{s_{13}}{\sqrt{s_{11} * s_{33}}} = \frac{1.94}{\sqrt{140.54 * 0.04}} = \frac{1.94}{2.371} = 0.82$$

$$r_{14} = \frac{cov(x_1, x_4)}{\sqrt{v(x_1)v(x_4)}.} = \frac{cov(x_1, x_4)}{\sqrt{s_{11} * s_{44}}} = \frac{s_{13}}{\sqrt{s_{11} * s_{44}}} = \frac{-11.8}{\sqrt{140.5 * 1.57}} = \frac{-11.08}{14.85} = -0.75$$

$$r_{24} = \frac{cov(x_2, x_4)}{\sqrt{v(x_2)v(x_4)}.} = \frac{cov(x_2, x_4)}{\sqrt{s_{22} * s_{44}}} = \frac{s_{24}}{\sqrt{s_{22} * s_{44}}} = \frac{-1.78}{\sqrt{75.25 * 1.57}} = \frac{-1.78}{10.87} = -0.017$$

$$r_{34} = \frac{cov(x_3, x_4)}{\sqrt{v(x_3)v(x_4)}.} = \frac{cov(x_3, x_4)}{\sqrt{s_{33} * s_{44}}} = \frac{s_{34}}{\sqrt{s_{33} * s_{44}}} = \frac{0.04}{\sqrt{0.25 * 1.57}} = \frac{0.04}{0.627} = 0.07$$

$$r_{23} = \frac{cov(x_2, x_3)}{\sqrt{v(x_2)v(x_3)}.} = \frac{cov(x_2, x_3)}{\sqrt{s_{22} * s_{33}}} = \frac{s_{23}}{\sqrt{s_{22} * s_{33}}} = \frac{3.68}{\sqrt{75.25 * 0.25}} = \frac{3.68}{4.34} = 0.85$$

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0.49 & 0.81 & -0.75 \\ 0.49 & 1 & 0.85 & -0.017 \\ 0.81 & 0.85 & 1 & 0.07 \\ -0.75 & -0.017 & 0.07 & 1 \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \\ \underline{X}_3 \\ \underline{X}_4 \end{bmatrix}, \overline{\underline{X}}^{(1)} = \begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \end{bmatrix}, \overline{\underline{X}}^{(2)} = \begin{bmatrix} \underline{X}_3 \\ \underline{X}_4 \end{bmatrix}$$

$$S = \left[\begin{array}{cc|cc} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ \hline s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0.49 & 0.81 & -0.75 \\ 0.49 & 1 & 0.85 & -0.017 \\ \hline 0.81 & 0.85 & 1 & 0.07 \\ -0.75 & -0.017 & 0.07 & 1 \end{array}\right]$$

Q//for the following data that are normally distributed $\underline{X} \sim N(\underline{\mu}, \Sigma)$ find mean vector , var-covar matrix , correlation matrix , and prove or disprove that $\underline{X}^{(1)}$ *and* $\underline{X}^{(2)}$ are independent ,if:

x_1	1	2	3	4	5
x_2	1	4	0	1	10
x_3	2	3	0	2	-2

Solution

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, it\ is\ possible\ to\ write\ it\ as\ a\ matix: X = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 3 \\ 3 & 0 & 0 \\ 4 & 1 & 2 \\ 5 & 10 & -2 \end{bmatrix}$$

$$\overline{X}_1 = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{(1+2+3+4+5)}{5} = \frac{15}{5} = 3$$

$$\overline{X}_2 = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{(1+4+0+1+10)}{5} = \frac{16}{5} = 3.2$$

$$\overline{X}_3 = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{(2+3+0+2+(-2))}{5} = \frac{5}{5} = 1$$

$$\underline{\overline{X}} = \begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \\ \overline{X}_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3.2 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} 2.5 & 3.75 & -2.25 \\ 3.75 & 16.7 & -5 \\ -2.25 & -5 & 4 \end{bmatrix}$$

$$s_{11} = \frac{\sum_{i=1}^{n}(x_{i1} - \bar{X}_1)^2}{n-1}$$

$$s_{11} = \sum_{i=1}^{5} \frac{(1-3)^2 + (2-3)^2 + (3-3)^2 + (4-3)^2 + (5-3)^2}{5-1} = \frac{10}{4} = 2.5$$

$$s_{22} = \frac{\sum_{i=1}^{n}(x_{i2} - \bar{X}_2)^2}{n-1}$$

$$= \sum_{i=1}^{5} \frac{(1-3.2)^2 + (4-3.2)^2 + (0-3.2)^2 + (1-3.2)^2 + (10-3.2)^2}{5-1} = \frac{66.8}{4} = 16.7$$

$$s_{33} = \frac{\sum_{i=1}^{n}(x_{i3} - \bar{X}_3)^2}{n-1}$$

$$= \sum_{i=1}^{5} \frac{(2-1)^2 + (3-1)^2 + (0-1)^2 + (2-1)^2 + (-2-1)^2}{5-1} = \frac{16}{4} = 4$$

$$s_{12} = \frac{\sum_{i=1}^{n}(x_{i1} - \bar{X}_1)(x_{j2} - \bar{X}_2)}{n-1}$$

$$= \frac{(1-3)(1-3.2) + (2-3)(4-3.2) + (3-3)(0-3.2) + (4-3)(1-3.2) + (5-3)(10-3.2)}{5-1}$$

$$= \frac{15}{4} = 3.75$$

$$s_{13} = \frac{\sum_{i=1}^{n}(x_{i1} - \bar{X}_1)(x_{j3} - \bar{X}_3)}{n-1}$$

$$= \frac{(1-3)(2-1) + (2-3)(3-1) + (3-3)(0-1) + (4-3)(2-1) + (5-3)(-2-1)}{5-1} = \frac{-9}{4}$$

$$= -2.25$$

$$s_{23} = \frac{\sum_{i=1}^{n}(x_{i2} - \bar{X}_2)(x_{j3} - \bar{X}_3)}{n-1}$$

$$= \frac{(1-3.2)(2-1) + (4-3.2)(3-1) + (0-3.2)(0-1) + (1-3.2)(2-1) + (10-3.2)(-2-1)}{5-1}$$

$$= \frac{-20}{4} = -5$$

$$S = \left[\begin{array}{c|cc} 2.5 & 3.75 & -2.25 \\ \hline 3.75 & 16.7 & -5 \\ -2.25 & -5 & 4 \end{array}\right]$$

$$cov\left(\underline{X}^{(1)}, \underline{X}^{(2)}\right) = S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} 16.7 & -5 & \vdots & 3.75 \\ -5 & 4 & \vdots & -2.25 \\ \cdots & \cdots & \vdots & \cdots \\ 3.75 & -2.25 & \vdots & 2.5 \end{bmatrix}$$

$$Since\ s_{11} \neq 0, then \underline{X}^{(1)} and\ \underline{X}^{(2)} are\ not\ independent$$

Correlation

وە نابێت ئەنجامەکان لە یەک گەورەتر بن وە ناشبێت لە لە سالب یەک بجوکتر بن جونکە (correlation) لە نێوان سالب یەک و یەکدایە.

$$s_{11} = v(x_1)\,, \qquad s_{22} = v(x_2), \quad s_{33} = v(x_3),$$

$$R = \begin{bmatrix} r_{12} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & 1 \end{bmatrix}$$

$$r_{12} = \frac{cov(x_1, x_2)}{\sqrt{v(x_1)v(x_2)}.} = \frac{cov(x_1, x_2)}{\sqrt{s_{11} * s_{22}}} = \frac{3.75}{\sqrt{2.5 * 16.75}} = \frac{3.75}{6.47} = 0.580$$

$$r_{13} = \frac{cov(x_1, x_3)}{\sqrt{v(x_1)v(x_3)}.} = \frac{cov(x_1, x_3)}{\sqrt{s_{11} * s_{33}}} = \frac{s_{13}}{\sqrt{s_{11} * s_{33}}} = \frac{-2.25}{\sqrt{2.5 * 4}} = \frac{-2.25}{3.163} = 0.712$$

$$r_{23} = \frac{cov(x_2, x_3)}{\sqrt{v(x_2)v(x_3)}.} = \frac{cov(x_2, x_3)}{\sqrt{s_{22} * s_{33}}} = \frac{s_{23}}{\sqrt{s_{22} * s_{33}}} = \frac{-5}{\sqrt{16.7 * 4}} = \frac{-5}{8.173} = -0.618$$

$$R = \begin{bmatrix} r_{12} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0.580 & -0.712 \\ 0.580 & 1 & -0.618 \\ -0.712 & -0.618 & 1 \end{bmatrix}$$

Q//for the following data that are normally distributed $\underline{X} \sim N(\underline{\mu}, \Sigma)$ find mean vector , var-covar matrix , correlation matrix , and prove or disprove that

$\underline{X}^{(1)} = [x_2]$ *and* $\underline{X}^{(2)} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$ are independent ,if:

x_1	-1	-2	0	-2
x_2	3	0	0	1
x_3	1	2	2	4

Solution

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \textit{it is possible to write it as a matix}: X = \begin{bmatrix} -1 & 3 & 1 \\ -2 & 0 & 2 \\ 0 & 0 & 2 \\ -2 & 1 & 4 \end{bmatrix}$$

$$\overline{X}_1 = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{(-1 + -2 + 0 + \ -2\ )}{4} = \frac{-5}{4} = -1.25$$

$$\overline{X}_2 = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{(3 + 0 + 0 + 1\ )}{4} = \frac{4}{4} = 1$$

$$\overline{X}_3 = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{(1 + 2 + 2 + 4\ )}{4} = \frac{9}{4} = 2.25$$

$$\underline{\overline{X}} = \begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \\ \overline{X}_3 \end{bmatrix} = \begin{bmatrix} -1.25 \\ 1 \\ 2.25 \end{bmatrix}$$

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} 0.917 & 0 & -0.583 \\ 0 & 2 & -0.667 \\ -0.583 & -0.667 & 1.583 \end{bmatrix}$$

$$s_{11} = \frac{\sum_{i=1}^{n} (x_{i1} - \overline{X}_1)^2}{n-1}$$

$$s_{11} = \sum_{i=1}^{4} \frac{(-1 - (-1.25)^2 + (-2 - (-1.25)^2 + (0 - (-1.25)^2 + (-2 - (-1.25)^2}{4-1} = \frac{2.7501}{3} \; 0.917$$

$$s_{22} = \frac{\sum_{i=1}^{n} (x_{i2} - \overline{X}_2)^2}{n-1}$$

$$= \sum_{i=1}^{4} \frac{(3 - 1)^2 + (0 - 1)^2 + (0 - 1)^2 + (1 - 1)^2 +}{4-1} = \frac{6}{3} = 2$$

$$s_{33} = \frac{\sum_{i=1}^{n} (x_{i3} - \overline{X}_3)^2}{n-1}$$

$$= \sum_{i=1}^{4} \frac{(1 - 2.25)^2 + (2 - 2.25)^2 + (2 - 2.25)^2 + (4 - 2.25)^2}{4-1} = \frac{4.75}{3} = 1.583$$

$$s_{12} = \frac{\sum_{i=1}^{n} (x_{i1} - \overline{X}_1)(x_{j2} - \overline{X}_2)}{n-1}$$

$$= \frac{(-1 - (-1.25)(3 - 1) + (-2 - (-1.25)(0 - 1) + (0 - (-1.25)(0 - 1) + (-2 - (-1.25)(1 - 1)}{4-1} = 0$$

$$s_{13} = \frac{\sum_{i=1}^{n} (x_{i1} - \overline{X}_1)(x_{j3} - \overline{X}_3)}{n-1}$$

$$= \frac{(-1 - (-1.25)(1 - 2.25) + (-2 - (-1.25)(2 - 2.25) + (0 - (-1.25)(2 - 2.25) + (-2 - (-1.25)(4 - 2.25)}{4-1}$$

$$= -0.583$$

$$s_{23} = \frac{\sum_{i=1}^{n} (x_{i2} - \overline{X}_2)(x_{j3} - \overline{X}_3)}{n-1}$$

$$= \frac{(3 - 1)(1 - 2.25) + (0 - 1)(2 - 2.25) + (0 - 1)(2 - 2.25) + (1 - 1)(4 - 2.25)}{4-1} = \frac{-2}{3}$$

$$= -0.667$$

$$S = \begin{matrix} & X_3 & X_1 & X_2 \\ X_3 & 0.917 & 0 & -0.583 \\ X_1 & 0 & 2 & -0.667 \\ X_2 & -0.583 & -0.667 & 1.583 \end{matrix} = \left[\begin{array}{c|cc} 2 & -0.667 & 0 \\ \hline -0.667 & 1.583 & -0.583 \\ 0 & -0.583 & 0.917 \end{array}\right]$$

$$cov(\underline{X}^{(1)}, \underline{X}^{(2)}) = S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \left[\begin{array}{c|cc} 2 & -0.667 & 0 \\ \hline -0.667 & 1.583 & -0.583 \\ 0 & -0.583 & 0.917 \end{array}\right]$$

$$Since\ s_{11} \neq 0, then \underline{X}^{(1)} and\ \underline{X}^{(2)} are\ not\ independent$$

Correlation

وه نابێت ئەنجامەكان لە یەک گەورەتر بن وە ناشبێت لە لە سالب یەک بجوكتر بن جونكە (correlation) لە نێوان سالب یەک و یەكدایە.

$$s_{11} = v(x_1), \qquad s_{22} = v(x_2), \quad s_{33} = v(x_3),$$

$$R = \begin{bmatrix} r_{12} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & 1 \end{bmatrix}$$

$$r_{12} = \frac{cov(x_1, x_2)}{\sqrt{v(x_1)v(x_2).}} = \frac{cov(x_1, x_2)}{\sqrt{s_{11} * s_{22}}} = \frac{0}{\sqrt{0.917 * 2}} = 0$$

$$r_{13} = \frac{cov(x_1, x_3)}{\sqrt{v(x_1)v(x_3).}} = \frac{cov(x_1, x_3)}{\sqrt{s_{11} * s_{33}}} = \frac{s_{13}}{\sqrt{s_{11} * s_{33}}} = \frac{-0.583}{\sqrt{0.917 * 1.583}} = -0.484$$

$$r_{23} = \frac{cov(x_2, x_3)}{\sqrt{v(x_2)v(x_3).}} = \frac{cov(x_2, x_3)}{\sqrt{s_{22} * s_{33}}} = \frac{s_{23}}{\sqrt{s_{22} * s_{33}}} = \frac{-0.667}{\sqrt{2 * 1.583}} = -0.375$$

$$R = \begin{bmatrix} r_{12} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -0.484 \\ 0 & 1 & -0.375 \\ -0.484 & -0.375 & 1 \end{bmatrix}$$

Chapter five

Transformations

Transformation of variables.

Let the j.p.d.f of $x_1, x_2, \dots, x_p$ be $f(x_1, x_2, \dots, x_p)$, consider the real- valued function:

$$Y_i = y_i(y_1, y_2, \dots, y_p), \qquad i = 1, 2, \dots, p$$

Then, the transformation from the X −space to the Y −(one to one) is the inverse transformation, which it $X_i = X_i(y_1, y_2, \dots, y_p)$

Let the random variables, $y_1, y_2, \dots, y_p$ are defined an:
$Y_i = y_i(y_1, y_2, \dots, y_p)$ then the j.p.d.f of $y_1, y_2, \dots, y_p$

$$g(\underline{y}) = f(w(\underline{y}))\,|J|$$

بەکاری دەهێنین ئەگەر هاتوو(P.d.f)مان هەبووجۆن بیگۆڕین بۆ(p.d.f)ی دیکە ،ئەگەر هاتوو احتیمالمان کردە سەر هاوکێشەیەک وەکو (Y_i).

Transformation of Quadratic form:

Theorem: let p-dimensional random vector $\underline{X} \sim N(\underline{\mu}, \Sigma)$ *and let* $\underline{y} = c\underline{x}$

Where c is non- singular matrix ($|c| \neq 0$), find the distribution function of $\underline{y}$

Proof: the j.p.d.f of $\underline{y}$

$$g\left(\underline{y}\right) = f\left(w(\underline{y})\right) \mid J \mid$$

$\underline{y} = c\underline{x}$ به زیادکردنی (c^{-1}) بۆ هەردوو لا

$$c^{-1}\underline{y} = c^{-1}c\underline{x}$$

$$c^{-1}\underline{y} = \underline{x} \rightarrow \underline{x} = c^{-1}\underline{y}$$

$\mid J \mid = \left|\frac{dx}{dy}\right| =, i, e.$ **We took absolute of the Jacobian.**

داتاشراوی $(\underline{x} = c^{-1}\underline{y})$ وەردەگرین.

$$\underline{x} = c^{-1}$$

$$\mid J \mid = \mid c^{-1} \mid = \frac{1}{|c\,|} = \frac{1}{\sqrt{c^2}} = \frac{1}{\sqrt{c\,c'}}$$

$\frac{|\Sigma|^{\frac{1}{2}}}{|\Sigma|^{\frac{1}{2}}}$ جارانی دەکەین.

$$\frac{1}{\sqrt{c\,c'}} * \frac{|\Sigma|^{\frac{1}{2}}}{|\Sigma|^{\frac{1}{2}}}$$

$$\mid J \mid = \frac{|\Sigma|^{\frac{1}{2}}}{\sqrt{c\sum c'}}$$

Since the j.p.d.f of $\underline{X} \sim N(\underline{\mu}, \Sigma)$

$$f(\underline{X}; \underline{\mu}, \Sigma) = \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} exp\left(-\frac{1}{2}\left(\underline{X} - \underline{\mu}\right)' \Sigma^{-1} \left(\underline{X} - \underline{\mu}\right)\right)$$

$$Q(\underline{X}) = (\underline{X} - \underline{\mu})'\Sigma^{-1}\left(\underline{X} - \underline{\mu}\right)$$

$$\underline{X} = \underline{x} = c^{-1}\underline{y}$$

$$Q(\underline{X}) = (c^{-1}\underline{y} - \underline{\mu})'\Sigma^{-1}\left(c^{-1}\underline{y} - \underline{\mu}\right)$$

به زیادکردنی(cc^{-1})بۆ هەردوو لا بۆ ئەوەی بتوانین (c^{-1}) هاوبەش دەربهێنین.

$$Q(\underline{X}) = (c^{-1}\underline{y} - cc^{-1}\underline{\mu})'\Sigma^{-1}\left(c^{-1}\underline{y} - cc^{-1}\underline{\mu}\right)$$

به هاوبەش دەریدەهێنینc^{-1}

$$Q(\underline{X}) = \left(c^{-1}(\underline{y} - c\,\underline{\mu})\right)'\Sigma^{-1}\left(c^{-1}(\underline{y} - c\,\underline{\mu}\right)$$

ئینجا(transpose) دەکەینەوە.

$$Q(\underline{X}) = \left(\underline{y} - c\,\underline{\mu}\right)'(c^{-1})'\Sigma^{-1}c^{-1}\left(\underline{y} - c\,\underline{\mu}\right)$$

$$\therefore\ (c^{-1})'\Sigma^{-1}c^{-1} = (c\Sigma c')^{-1}$$

$$\Rightarrow Q(\underline{X}) = \left(\underline{y} - c\,\underline{\mu}\right)'(c\Sigma c')^{-1}\left(\underline{y} - c\,\underline{\mu}\right)$$

$$\therefore\ Q(\underline{X}) = Q(\underline{Y}) = \left(\underline{y} - c\,\underline{\mu}\right)'(c\Sigma c')^{-1}\left(\underline{y} - c\,\underline{\mu}\right)$$

کەواتە (X) لە نەخشەکەدا نەما .

$$g\left(\underline{y}\right) = f\left(w(\underline{y})\right)|J|$$

$$w\left(\underline{y}\right) = \left(\underline{y} - c\,\underline{\mu}\right)'(c\Sigma c')^{-1}\left(\underline{y} - c\,\underline{\mu}\right), \qquad |J| = \frac{|\Sigma|^{\frac{1}{2}}}{\sqrt{c\Sigma c'}}$$

$$g\left(\underline{y}\right) = \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}}exp\left(-\frac{1}{2}\left(\underline{y} - c\,\underline{\mu}\right)'(c\Sigma c')^{-1}\left(\underline{y} - c\,\underline{\mu}\right)\right)\cdot\frac{|\Sigma|^{\frac{1}{2}}}{\sqrt{c\Sigma c'}}$$

$$g\left(\underline{y}\right) = \frac{1}{(2\pi)^{\frac{p}{2}}|c\Sigma c'|^{\frac{1}{2}}}exp\left(-\frac{1}{2}\left(\underline{y} - c\,\underline{\mu}\right)'(c\Sigma c')^{-1}\left(\underline{y} - c\,\underline{\mu}\right)\right)$$

$$\therefore\ \underline{Y} \sim N(c\,\underline{\mu}, c\Sigma c')$$

Example: let $\underline{X} \sim N\begin{bmatrix}0\\0\end{bmatrix}.\begin{bmatrix}6 & -3\\-3 & 4\end{bmatrix}$ where $\underline{y} = c\underline{x}$ find the distribution

$y_1 = x_1 - x_2$

Solution:

$\underline{y} = c\underline{x}$, $y_1 = x_1 - x_2$, let $y_2 = x_2$

$x_1 - x_2 \rightarrow x_1 = 1, x_2 = -1$ لە هاوکێشەی یەکەم $y_2 = x_2 \rightarrow x_1 = 0$ ، $x_2 = 1$ لە هاوکێشەی دووەم

$$\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$c = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix},\ c' = \begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix}$$

$$\underline{Y} \sim N(c\,\underline{\mu}, c\textstyle\sum c')$$

$$E(\underline{y}) = c\,\underline{\mu} = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = \underline{0}$$

$$var(\underline{y}) = c\textstyle\sum c' = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}.\begin{bmatrix}6 & -3\\-3 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}.\begin{bmatrix}6 & -3\\-3 & 4\end{bmatrix} = \begin{bmatrix}1\times 6 + (-1\times -3) & (1\times -3 + (-1\times 4))\\ 0\times 6 + 1\times -3 & 0\times -3 + 1\times 4\end{bmatrix} = \begin{bmatrix}9 & -7\\-3 & 4\end{bmatrix}$$

$$\begin{bmatrix}9 & -7\\-3 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix} = \begin{bmatrix}9\times 1 + -7\times -1 & 9\times 0 + (-7\times 1)\\ -3\times 1 + 4\times -1 & -3\times 0 + 4\times 1\end{bmatrix}$$

$$= \begin{bmatrix}16 & -7\\-7 & 4\end{bmatrix}\ \begin{bmatrix}v(y_1) & cov(y_1, y_2)\\ cov(y_1, y_2) & v(y_2)\end{bmatrix}$$

$y_1 \sim N(0, 16)$, $y_2 \sim N(0, 4)$

$$\therefore f(\underline{X}, \underline{\mu}, \textstyle\sum) = \frac{1}{(2\pi)^{\frac{p}{2}}|c\sum c'|^{\frac{1}{2}}} exp\left(-\frac{1}{2}\left(\underline{y} - \underline{\mu}\right)'(c\textstyle\sum c')^{-1}\left(\underline{y} - \underline{\mu}\right)\right)$$

$$(16)^{-1} = \frac{1}{16}$$

$$g(y_1) = \frac{1}{(2\pi)^{\frac{p}{2}}|c\sum c'|^{\frac{1}{2}}} exp\left(-\frac{1}{2}\left(\underline{y} - 0\right)'(16)^{-1}\left(\underline{y} - 0\right)\right)$$

$$g(y_1) = \frac{1}{(2\pi)^{\frac{p}{2}}|c\sum c'|^{\frac{1}{2}}} exp\left(-\frac{{y_1}^2}{32}\right)$$

$$g(y_1) = \begin{cases} \dfrac{1}{|\ 32\pi|^{\frac{1}{2}}} exp\left(-\dfrac{{y_1}^2}{32}\right) & -\infty < y_1 < \infty \\ 0 & 0.w \end{cases}$$

Example:

Let $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \sim N(\underline{\mu}, \Sigma)$ *where* $\mu' = [4 \quad 3 \quad -1], \Sigma = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 5 & 2 \\ -1 & 2 & 1 \end{bmatrix}$

Find:1. The p.d.f of $y_2 = 2x_1 - 4x_2 + 5x_3$

2. the j p.d.f of $\underline{y} = 5x_1 - 4x_2 + x_3$

Solution:1

دەبێت، بە پێی ئەمە داواکاری بدۆزینەوە.

$\underline{Y} \sim N(c\,\underline{\mu}, c\Sigma\, c')$

پێویستە نرخی(c) بدۆزینەوە، بەم شێوەیە.

$y_2 = 2x_1 - 4x_2 + 5x_3 \quad , \quad y_1 = x_1, \quad y_3 = x_3$

لەبەر ئەوەی هاوکێشەی (y_1،y_3) لە پرسیاردا نەدراوە ئەوە بە خۆمان گریمانەی بۆ دەکەین.

دۆزینەوەی نرخەکانی(c)ی، بەم شێوەیە دەبێت.

ئینجا نرخەکان لە ستونی یەکەم دادەنێین. $y_1 = x_1 \Rightarrow x_1 = 1, x_2 = 0, x_3 = 0$.

ئینجا نرخەکان لە ستونی دووەم دادەنێین. $y_2 = 2x_1 - 4x_2 + 5x_3 \Rightarrow x_1 = 2$، $x_2 = -4$، $x_3 = 5$

ئینجا نرخەکان لە ستونی سێیەم دادەنێین. $y_3 = x_3 \Rightarrow x_1 = 0$، $x_2 = 0$، $x_3 = 1$

یانی بەم شێوەیەی لێ دێت.

$c = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -4 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

ئینجا بەکارهێنانی یاسا

ئامانجی سەریەکیمان ئەوەیە نابێت(x) لە هاوکێشەکەدا هەبێت.یانی لە شوێنی (x) دەبێت،(y)ی،دەربهێنین.

$\underline{y} = c\underline{x}$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -4 & 5 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$c' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -4 & 5 \\ 0 & 0 & 1 \end{bmatrix}' = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

$$\boldsymbol{E}\left(\underline{\boldsymbol{y}}\right) = \boldsymbol{c}\,\underline{\boldsymbol{\mu}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -4 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \times 4 + 0 \times 3 + 0 \times -1 \\ 2 \times 4 + (-4 \times 3) + 5 \times 3 \\ 0 \times 4 + 0 \times 3 + 1 \times -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -9 \\ -1 \end{bmatrix}$$

$$var - cov\left(\underline{y}\right) = \boldsymbol{c}\sum c' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -4 & 5 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 3 & 0 & -1 \\ 0 & 5 & 2 \\ -1 & 2 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -4 & 5 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 3 & 0 & -1 \\ 0 & 5 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 3 + 0 \times 0 + 0 \times -1 & 1 \times 0 + 0 \times 5 + 0 \times 2 & 1 \times -1 + 0 \times 2 + 0 \times 1 \\ 2 \times 3 + -4 \times 0 + 5 \times -1 & 2 \times 0 + (-4 \times 5 + 5 \times 2) & 2 \times -1 + (-4 \times 2 + 5 \times 1) \\ 0 \times 3 + 0 \times 0 + 1 \times -1 & 0 \times 0 + 0 \times 5 + 1 \times 2 & 0 \times -1 + 0 \times 2 + 1 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & -1 \\ 1 & -10 & -5 \\ -1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & -1 \\ 1 & -10 & -5 \\ -1 & 2 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 0 \\ 0 & 5 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 3 \times 1 + 0 \times 0 + (-1 \times 0) & 3 \times 2 + 0 \times -4 + -1 \times 5 & 3 \times 0 + 0 \times 0 + (-1 \times 1) \\ 1 \times 1 + (-10 \times 0 + (-5 \times 0) & 1 \times 2 + (-10 \times -4 + (-5 \times 5) & 1 \times 0 + (-10 \times 0 + (-5 \times 1) \\ -1 \times 1 + 2 \times 0 + 1 \times 0 & -1 \times 2 + 2 \times -4 + 1 \times 5 & -1 \times 0 + 2 \times 0 + 1 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 17 & -5 \\ -1 & -5 & 1 \end{bmatrix}$$

$\boldsymbol{y_1 \sim N(4, 3)}$

$\boldsymbol{y_2 \sim N(-9, 17)}$

$\boldsymbol{y_3 \sim N(-1, 1)}$

$\boldsymbol{\therefore j.p.d.f \ of \ \underline{y} \ is}$

$$\therefore f(\underline{X}, \underline{\mu}, \Sigma) = \frac{1}{(2\pi)^{\frac{p}{2}}|c\Sigma c'|^{\frac{1}{2}}} exp\left(-\frac{1}{2}\left(\underline{y} - \underline{\mu}\right)'(c\Sigma c')^{-1}\left(\underline{y} - \underline{\mu}\right)\right)$$

$$g(y_2) = \frac{1}{(2\pi)^{\frac{p}{2}}|c\Sigma c'|^{\frac{1}{2}}} exp\left(-\frac{1}{2}\left(\underline{y} + 9\right)'(\ 17)^{-1}\left(\underline{y} + 9\right)\right)$$

$$g(y_2) = \frac{1}{(2\pi)^{\frac{p}{2}}|\ 1\ 7|^{\frac{1}{2}}} exp\left(-\frac{1}{2}\ (y_2 + 9)^2\ \right)$$

$$g(y_2) = \begin{cases} \frac{1}{|\ 34\pi|^{\frac{1}{2}}} exp\left(-\frac{1}{2}\ (y_2 + 9)^2\ \right) & -\infty < y_2 < \infty \\ 0 & 0.w \end{cases}$$

داواکاری دووەم

2. the j p.d.f of $\underline{y} = 5\,x_1 - 4\,x_2 + \ x_3$

لێرەدا پێویست ناکات(y_3)و(y_1) بدۆزینەوە جونکە لێرەدا(y)ی(vector)ە.

$\underline{y} = 5\,x_1 - 4\,x_2 + \ x_3$

$\underline{y} = c = 5\ , \quad -4, \quad 1$

$\underline{y} = c\underline{x}$

$$\Rightarrow \underline{y} = c\underline{x} = [5 \quad -4 \quad 1].\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\therefore E\left(\underline{y}\right) = c\,\underline{\mu} = [5 \quad -4 \quad 1].\begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} = 5 \times 4 - (-4 \times 3) - 1 \times -1) = 7$$

$$\text{And } Var - cov\left(\underline{y}\right) = c\sum.c' = [5 \quad -4 \quad 1].\begin{bmatrix} 3 & 0 & -1 \\ 0 & 5 & 2 \\ -1 & 2 & 1 \end{bmatrix}.\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} = 130$$

$$[5 \quad -4 \quad 1].\begin{bmatrix} 3 & 0 & -1 \\ 0 & 5 & 2 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 5(3) = \ 1\,5 & -4 * 0 = 0 & 1 * -5 = -5 \\ -4 * 0 = 0 & -4 * (5) = -20 & -4 * 2 = -8 \\ 1 * -1 = -1 & 1 * 2 = 2 & 1 * (1) = 1 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 0 & -5 \\ 0 & -20 & -8 \\ -1 & 2 & 1 \end{bmatrix}.\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 0 & -5 \\ 0 & -20 & -8 \\ -1 & 2 & 1 \end{bmatrix} . \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

$$15 * 5 + (0 * (-4) + (-5 * 1) + (0 * 5 + (-20 * -4 + (-8 * 1) + (-1 * 5 + 2 * -4 + 1 * 1) = 130$$

$$\underline{y} \sim N(7, 130)$$

$$\therefore f(\underline{X}, \underline{\mu}, \Sigma) = \frac{1}{(2\pi)^{\frac{p}{2}} |c \Sigma c'|^{\frac{1}{2}}} exp\left(-\frac{1}{2}\left(\underline{y} - \underline{\mu}\right)' (c\Sigma c')^{-1} \left(\underline{y} - \underline{\mu}\right)\right)$$

$$g\left(\underline{y}\right) = \frac{1}{(2\pi)^{\frac{p}{2}} |c \Sigma c'|^{\frac{1}{2}}} exp\left(-\frac{1}{2}\left(\underline{y} + 7\right)' (130)^{-1} \left(\underline{y} + 7\right)\right)$$

$$g\left(\underline{y}\right) = \frac{1}{(2\pi)^{\frac{p}{2}} |130|^{\frac{1}{2}}} exp\left(-\frac{1}{2}\left(\underline{y} + 7\right)^2\right)$$

Example:

Let $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \sim N\left(\underline{\mu}, \Sigma.\right), where\ \underline{\mu}' = [3 \quad 2 \quad 1], \underline{\mu} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$Q(\underline{X}) = \underline{X}' A \underline{X} = 2x_1^2 + 3x_3^2 + 4x_2^2 - 4x_1x_2 + 6x_2x_3$

Find the p.d.f of $y_1 = x_1 - 2x_2 + x_3$

Solution:

$2x_1^2 + 3x_3^2 + 4x_2^2 - 4x_1x_2 + 6x_2x_3$.دەبێت (∑) لەم هاوکێشەیە بدۆزینەوە

$2x_1^2 = a_{11} = 2$, $\quad -4x_1x_2 = a_{12}, a_{21} = -2,$ $\quad 6x_2x_3 = a_{23}, a_{32} = 3$

$3x_3^2 = a_{33} = 3 \quad 4x_2^2 = a_{22} = 4$

a_{13}, a_{31} یەکسانە بە سفر جونکە لە هاوکێشەدا نیتمان

$$\Sigma = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & 3 \\ 0 & 3 & 3 \end{bmatrix}$$

Since $y_1 = x_1 - 2x_2 + x_3$

دەبێتە گریمانە دابنێین بۆ (y_2, y_3) جونکە هاوکێشەمان نییە بۆ(y_2, y_3)

$y_2 = x_2, \qquad y_3 = x_3$

ئینجا دەبێت نرخی(c)لەم سێ هاوکێشەیە دەربهێنین.

. $\boldsymbol{y_1 = x_1 - 2x_2 + x_3}$, $\boldsymbol{y_2 = x_2,} \qquad \boldsymbol{y_3 = x_3}$

دۆزینەوەی نرخەکانی(c)ی، بەم شێوەیە دەبێت.

. $\boldsymbol{y_1 = x_1 \Rightarrow x_1 = 1 , x_2 = -2, \quad x_3 = 1}$. ئینجا نرخەکان لە ستونی یەکەم دادەنێین.

$\boldsymbol{y_2 = x_2 \Rightarrow x_1 = 0 , x_2 = 1, \quad x_3 = 0}$ ئینجا نرخەکان لە ستونی دووەم دادەنێین.

$\boldsymbol{y_3 = x_3 \Rightarrow x_1 = 0، x_2 = 0، \quad x_3 = 1}$ ئینجا نرخەکان لە ستونی سێیەم دادەنێین

$$c = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = c' = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$E\left(\underline{y}\right) = c\,\underline{\mu} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \times 3 + (-2 \times 2) + 1 \times 1 \\ 0 \times 2 + 1 \times 2 + 0 \times 1 \\ 0 \times 3 + 0 \times 2 + 1 \times 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$Var\left(\underline{y}\right) = \boldsymbol{c}\sum c' = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & 3 \\ 0 & 3 & 3 \end{bmatrix} . \left[\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\right]$$

$$= \left[\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right] . \begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & 3 \\ 0 & 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} \times \mathbf{2} + (\mathbf{-2} \times \mathbf{-2}) + \mathbf{1} \times \mathbf{0} & \mathbf{1} \times \mathbf{-2} + (\mathbf{-2} \times \mathbf{4}) + \mathbf{1} \times \mathbf{3}\ ) & \mathbf{1} \times \mathbf{0} + (\mathbf{-2} \times \mathbf{3}) + \mathbf{1} \times \mathbf{3}) \\ \mathbf{0} \times \mathbf{2} + \mathbf{1} \times \mathbf{-2} + \mathbf{0} \times \mathbf{0} & \mathbf{0} \times \mathbf{-2} + \mathbf{1} \times \mathbf{4} + \mathbf{0} \times \mathbf{3} & \mathbf{0} \times \mathbf{0} + \mathbf{1} \times \mathbf{3} + \mathbf{0} \times \mathbf{3} \\ \mathbf{0} \times \mathbf{2} + \mathbf{0} \times \mathbf{-2} + \mathbf{1} \times \mathbf{0} & \mathbf{0} \times \mathbf{-2} + \mathbf{0} \times \mathbf{4} + \mathbf{1} \times \mathbf{3} & \mathbf{0} \times \mathbf{0} + \mathbf{0} \times \mathbf{3} + \mathbf{1} \times \mathbf{3} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{6} & \mathbf{-7} & \mathbf{-3} \\ \mathbf{-2} & \mathbf{4} & \mathbf{3} \\ \mathbf{0} & \mathbf{3} & \mathbf{3} \end{bmatrix}$$

$$\begin{bmatrix} 6 & -7 & -3 \\ -2 & 4 & 3 \\ 0 & 3 & 3 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6\times 1 + (-7\times -2) + (-3\times 1) & 6\times 0 + (-7\times 1 + (-3\times 0) & 6\times 0 + (-7\times 0) + -3\times 1) \\ -2\times 1 + 4\times -2 + 3\times 1 & -2\times 0 + 1\times 4 + 3\times 0 & -2\times 0 + 4\times 0 + 3\times 1 \\ 0\times 1 + 3\times -2 + 3\times 1 & 0\times 0 + 3\times 1 + 3\times 0 & 0\times 0 + 3\times 0 + 3\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & -7 & -3 \\ -7 & 4 & 3 \\ -3 & 3 & 3 \end{bmatrix}$$

$$\therefore f(\underline{X}, \underline{\mu}, \Sigma) = \frac{1}{(2\pi)^{\frac{p}{2}} |c\Sigma c'|^{\frac{1}{2}}} exp\left(-\frac{1}{2}\left(\underline{y} - \underline{\mu}\right)' (c\Sigma c')^{-1} \left(\underline{y} - \underline{\mu}\right)\right)$$

$$g(y_1) = \frac{1}{(2\pi)^{\frac{p}{2}} |c\Sigma c'|^{\frac{1}{2}}} exp\left(-\frac{1}{2}\left(\underline{y} + 0\right)' (17)^{-1} \left(\underline{y} + 0\right)\right)$$

$$g(y_1) = \frac{1}{(2\pi)^{\frac{p}{2}} \sqrt{17}} exp\left(-\frac{y_1^{\ 2}}{34}\right)$$

$$g(y_1) = \begin{cases} \frac{1}{(2\pi)^{\frac{p}{2}} \sqrt{17}} exp\left(-\frac{y_1^{\ 2}}{34}\right) & -\infty < y_1 < \infty \\ 0 & 0.w \end{cases}$$

Theorem: **if $\underline{X} \sim N\left(\underline{\mu}, \Sigma,\right)$ $and\ let\ \underline{X}$ in partitioned $\underline{X}^{(1)}$ $and\ \underline{X}^{(1)}$ which they are independent, and let**

$$\left.\begin{matrix} \underline{Y}^{(1)} = \underline{X}^{(1)} + B\underline{X}^{(2)} \\ \underline{Y}^{(2)} = \underline{X}^{(2)} \end{matrix}\right\} \Rightarrow \underline{Y} = C\,\underline{X}$$

Where B is a matrix that most statisfy the eqution.

Find:

1. $B\ when\ \underline{Y}^{(1)}\ and\ \underline{Y}^{(2)}$ are uncorrelated (independent)(i.e.$cov\left(\underline{Y}^{(1)}, \underline{Y}^{(2)}\right) = 0$ and what is the value of C in this transformation?

2. $prove\ that\ cov\left(\underline{Y}^{(1)}, \underline{Y}^{(2)}\right) = 0$

Solution:

1. $cov(\underline{Y}^{(1)}, \underline{Y}^{(2)})$

به وەرگرتنی (E)

$$E\left(\underline{Y}^{(1)} - E(\underline{Y}^{(1)})\right)\left(\underline{Y}^{(2)} - E(\underline{Y}^{(2)})\right)'$$

$\underline{Y}^{(1)} = \underline{X}^{(1)} + B\underline{X}^{(2)}$, $\underline{Y}^{(2)} = \underline{X}^{(2)}$

$$\Sigma_{12} = cov(\underline{X}^{(1)}, \underline{X}^{(2)}) = E(\underline{X}^{(1)} - \underline{\mu}^{(1)})(\underline{X}^{(2)} - \underline{\mu}^{(2)})'$$

لە بەشی چوارەم دا باسکراوە

ئینجا تەوعیضیان دەکەینەوە.

$$\Rightarrow E\left(\underline{X}^{(1)} + B\underline{X}^{(2)} - E(\underline{X}^{(1)} + B\underline{X}^{(2)})\right)\left(\underline{X}^{(2)} - E(\underline{X}^{(2)})\right)' = 0$$

ئینجا (E) داخیلی ناوکەوانە دەکەین.

$$\Rightarrow E\left(\underline{X}^{(1)} + B\underline{X}^{(2)} - E(\underline{X}^{(1)} + B\underline{X}^{(2)})\right)\left(\underline{X}^{(2)} - E(\underline{X}^{(2)})\right)' = 0$$

$$\Rightarrow E(\underline{X}^{(1)} - E(\underline{X}^{(1)} + B\underline{X}^{(2)} - BE(\underline{X}^{(2)})\left(\underline{X}^{(2)} - E(\underline{X}^{(2)})\right)' = 0$$

بە هاوبەش وەردەگرین

$$\Rightarrow E(\underline{X}^{(1)} - E(\underline{X}^{(1)} + B(\underline{X}^{(2)} - E(\underline{X}^{(2)})\left(\underline{X}^{(2)} - E(\underline{X}^{(2)})\right)' = 0$$

$E(\underline{X}^{(1)}) = \underline{\mu}^{(1)}$, $E(\underline{X}^{(2)}) = \underline{\mu}^{(2)}$

$$\Rightarrow E(\underline{X}^{(1)} - \underline{\mu}^{(1)} + B(\underline{X}^{(2)} - \underline{\mu}^{(2)})\left(\underline{X}^{(2)} - \underline{\mu}^{(2)}\right)' = 0$$

ئینجا) $\left(\underline{X}^{(2)} - \underline{\mu}^{(2)}\right)'$ جارانی ناوکەوانە دەکەین.

$$\Rightarrow E(\underline{X}^{(1)} - \underline{\mu}^{(1)})\left(\underline{X}^{(2)} - \underline{\mu}^{(2)}\right)' + B(\underline{X}^{(2)} - \underline{\mu}^{(2)})\left(\underline{X}^{(2)} - \underline{\mu}^{(2)}\right)' = 0$$

$$\Rightarrow E(\underline{X}^{(1)} - \underline{\mu}^{(1)})\left(\underline{X}^{(2)} - \underline{\mu}^{(2)}\right)' + BE(\underline{X}^{(2)} - \underline{\mu}^{(2)})\left(\underline{X}^{(2)} - \underline{\mu}^{(2)}\right)' = 0$$

$$\Rightarrow Cov(\underline{X}^{(1)}, \underline{X}^{(2)}) + B\,Var(\underline{X}^{(2)})$$

$$\Rightarrow \Sigma_{12} + B\Sigma_{22} = 0 \Rightarrow B = -\Sigma_{12}\Sigma_{22}^{-1}$$

$$\underline{Y}^{(1)} = \underline{X}^{(1)} + B\underline{X}^{(2)}$$

$$\underline{Y}^{(1)} = \underline{X}^{(1)} - \Sigma_{12}\Sigma_{22}{}^{-1}\underline{X}^{(2)}$$

And $\underline{Y}^{(2)} = \underline{X}^{(2)}$

$$\therefore\ \underline{Y} = C\,\underline{X} \Rightarrow C = \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}{}^{-1} \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} \underline{Y}^{(1)} \\ \underline{Y}^{(2)} \end{bmatrix} = \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}{}^{-1} \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{bmatrix}$$

$$\underline{Y}^{(1)} = \underline{X}^{(1)} - \Sigma_{12}\Sigma_{22}{}^{-1}\underline{X}^{(2)}$$

داتاشراو وەردەگرین

$$\underline{X}^{(1)} = I\ , -\Sigma_{12}\Sigma_{22}{}^{-1}\underline{X}^{(2)} = -\Sigma_{12}\Sigma_{22}{}^{-1}$$

داتاشراو وەردەگرین

$$\underline{Y}^{(2)} = \underline{X}^{(2)}$$

$$\underline{Y}^{(2)} = 0, \qquad \underline{X}^{(2)} = I$$

2. *prove that* $cov(\underline{Y}^{(1)}, \underline{Y}^{(2)}) = 0$

since $Cov(\underline{Y}^{(1)}, \underline{Y}^{(2)}) = E\left(\underline{Y}^{(1)} - E(\underline{Y}^{(1)})\right)\left(\underline{Y}^{(2)} - E(\underline{Y}^{(2)})\right)'$

$\underline{Y}^{(1)} = \underline{X}^{(1)} + B\underline{X}^{(2)}, \qquad \underline{Y}^{(2)} = \underline{X}^{(2)}$

$$\Rightarrow E\left(\underline{X}^{(1)} + B\underline{X}^{(2)} - E(\underline{X}^{(1)} + B\underline{X}^{(2)})\right)\left(\underline{X}^{(2)} - E(\underline{X}^{(2)})\right)' = 0$$

$$\Rightarrow Cov(\underline{Y}^{(1)}, \underline{Y}^{(2)}) = cov(\underline{X}^{(1)}, \underline{X}^{(2)}) + B\,Var(\underline{X}^{(2)})$$

$$\Rightarrow Cov(\underline{Y}^{(1)}, \underline{Y}^{(2)}) = \Sigma_{12} + B\Sigma_{22} \Rightarrow B = -\Sigma_{12}\Sigma_{22}{}^{-1}$$

$$\therefore Cov(\underline{Y}^{(1)}, \underline{Y}^{(2)}) = \Sigma_{12} - \Sigma_{12}\Sigma_{22}{}^{-1}\Sigma_{22} = 0$$

Example: if $\underline{X} = \begin{bmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{bmatrix} \sim N\left(\begin{bmatrix} \underline{\mu}^{(1)} \\ \underline{\mu}^{(2)} \end{bmatrix} . \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right)$

Let $\underline{Y}^{(1)} = \underline{X}^{(1)} - \Sigma_{12}\Sigma_{22}{}^{-1}$

$\underline{Y}^{(2)} = \underline{X}^{(2)}$

Find j.p.d.f of $\underline{Y}^{(1)}$ *and* $\underline{Y}^{(2)}$

Solution: *since* $\underline{Y} = C\underline{X} \sim N(C\underline{\mu}, C\Sigma c')$

$$\begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}{}^{-1} \\ 0 & I \end{bmatrix} . \begin{bmatrix} \underline{\mu}^{(1)} \\ \underline{\mu}^{(2)} \end{bmatrix}$$

$$\begin{bmatrix} I * \underline{\mu}^{(1)} + (-\Sigma_{12}\Sigma_{22}{}^{-1} * \underline{\mu}^{(2)} \\ 0 * \underline{\mu}^{(1)} + I\ \underline{\mu}^{(2)} \end{bmatrix}.$$

$$\begin{bmatrix} \underline{\mu}^{(1)} - \underline{\mu}^{(2)}\Sigma_{12}\Sigma_{22}{}^{-1} \\ \underline{\mu}^{(2)} \end{bmatrix}$$

$C\underline{\mu} = ?$

$$\Rightarrow E\left(\underline{y}\right) = c\underline{\mu} = \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}{}^{-1} \\ 0 & I \end{bmatrix} . \begin{bmatrix} \underline{\mu}^{(1)} \\ \underline{\mu}^{(2)} \end{bmatrix} = \begin{bmatrix} \underline{\mu}^{(1)} - \underline{\mu}^{(2)}\Sigma_{12}\Sigma_{22}{}^{-1} \\ \underline{\mu}^{(2)} \end{bmatrix}$$

$Var - cov\left(\underline{y}\right) = C\Sigma c'?$

$$c' = \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}{}^{-1} \\ 0 & I \end{bmatrix}' = \begin{bmatrix} I & 0 \\ -\Sigma_{12}\Sigma_{22}{}^{-1} & I \end{bmatrix}$$

$$Var - cov\left(\underline{y}\right) = C\Sigma c' = \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}{}^{-1} \\ 0 & I \end{bmatrix} . \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} . \begin{bmatrix} I & 0 \\ -\Sigma_{11}\Sigma_{22}{}^{-1} & I \end{bmatrix}$$

$$\begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}{}^{-1} \\ 0 & I \end{bmatrix} . \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} I\Sigma_{11} + (-\Sigma_{12}\Sigma_{22}{}^{-1}\Sigma_{21} & I\Sigma_{12} + (-\Sigma_{12}\Sigma_{22}{}^{-1}\Sigma_{22}) \\ 0 * \Sigma_{11} + I\Sigma_{21} & 0 * \Sigma_{12} + I\Sigma_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}{}^{-1}\Sigma_{21} & \Sigma_{12} - \Sigma_{12}\Sigma_{22}{}^{-1}\Sigma_{22} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}{}^{-1}\Sigma_{21} & \mathbf{0} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}{}^{-1}\Sigma_{21} & 0 \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} . \begin{bmatrix} I & 0 \\ -\Sigma_{12}\Sigma_{22}{}^{-1} & I \end{bmatrix} =$$

$$\begin{bmatrix} I\Sigma_{11} - \Sigma_{12}\Sigma_{22}{}^{-1}\Sigma_{21} + 0 & \Sigma_{11} - \Sigma_{12}\Sigma_{22}{}^{-1}\Sigma_{21} * 0 + 0 * I \\ \Sigma_{21}I + -\Sigma_{12}\Sigma_{22}{}^{-1}\Sigma_{22} & \Sigma_{21} * 0 + \Sigma_{22}I \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}{}^{-1}\Sigma_{21} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}$$

$\underline{Y} = C\underline{X} \sim N(C\underline{\mu}, C\sum c')$

$\underline{y}^{(1)} \sim N(\underline{\mu}^{(1)} - \underline{\mu}^{(2)} \sum_{11} \sum_{22}^{-1}, \sum_{11} - \sum_{12} \sum_{22}^{-1} \sum_{21})$

$\underline{y}^{(1)} \sim N(\underline{\mu}^{(2)}, \sum_{22})$

Example: let $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \sim N(\mu, \sum) where\ \underline{\mu}' = [4 \quad 3 \quad -1], \sum = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 5 & 2 \\ -1 & 2 & 1 \end{bmatrix}$

Let $\underline{X}^{(1)} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}, \underline{X}^{(2)} = [x_2],\ \underline{Y}^{(1)} = \underline{X}^{(1)} - \sum_{12} \sum_{22}^{-1} \underline{X}^{(2)}$

$\underline{Y}^{(2)} = \underline{X}^{(2)}$

Solution:

since $\underline{X} = \begin{bmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \\ [x_2] \end{bmatrix} \Rightarrow \underline{\mu} = \begin{bmatrix} \underline{\mu}^{(1)} \\ \underline{\mu}^{(2)} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} \\ [3] \end{bmatrix}$

ئینجا دێین(سەفی یەکەم بە(x_1)دادەنێین،بەهەمان شێوەش عامودی یەکەم بە (x_1)دادەنێین، وە سەفی دووەم بە(x_3)دادەنێین،بەهەمان شێوەش عامودی دووەم بە(x_3)دادەنێین،جونکە ($\underline{X}^{(1)}$)بریتییە لە (x_1وx_3)وە سەفی سێ یەم بە((x_2))دادەنێین، بەهەمان شێوەش عامودی سێ یەم (x_2)،جونکە ($\underline{X}^{(2)}$)بریتیە لە (x_2)

$$\text{and} \sum = \begin{matrix} & x_1 & x_3 & x_2 \\ x_1 & 3 & 0 & -1 \\ x_3 & 0 & 5 & 2 \\ x_2 & -1 & 2 & 1 \end{matrix} = \begin{bmatrix} 3 & -1 & \vdots & 0 \\ -1 & 1 & \vdots & 2 \\ \cdots & \cdots & \vdots & \cdots \\ 0 & 2 & \vdots & 5 \end{bmatrix} = \begin{bmatrix} \sum_{11} & \sum_{12} \\ \sum_{21} & \sum_{22} \end{bmatrix}$$

x_1 لەگەڵ x_1 دەکاتە (3)،

ئینجا دێینە سەر x_1 لەگەڵx_2دەکاتە (-1)ئینجا لە عامودی دووەم دادەنێین.

ئینجا دێینە سەر x_1 لەگەڵ x_3دەکاتە (0)لە عاموودی سێ یەم دادەنێین.

دەبێت بە تەسەسول بێت نابێت بڵێین،لە عاموودی دووەم (2) دادەنێین،جونکە دەبێت یەکەم جار x_1 لەگەڵ x_2
وەربگرین لە عاموودی دووەم دادەنێین، ئینجا دێینە سەر x_1 لەگەڵ x_3 کە دەکاتە(0)لە عاموودی سێ یەم دادەنێین.ئەمە سەفی یەکەم تەواو بوو،ئینجا دێینە سەر سەفی دووەم.

یانی (a_{11}وa_{12} وa_{13})مان دۆزیەوە،وە(a_{21})دەکاتە(-1)جونکە(x_2 لەگەڵ x_1)دەکاتە(-1)

وە هەروەها(a_{22})دەکاتە (1)جونکە(x_2 لەگەڵ x_2)دەکاتە(1).

ئینجا (a_{23})دەکاتە(2) جونکە(x_3 لەگەڵ x_2)دەکاتە(2).

ئینجا دێینە سەر سەفی سێ یەم، (a_{31})دەکاتە (0) جونکە(x_3 لەگەڵ x_1)دەکاتە(0).

وە(a_{32})دەکاتە(2) جونکە(x_3 لەگەڵ x_2)دەکاتە(2).

وە (a_{33})دەکاتە(5) جونکە(x_3 لەگەڵ x_3)دەکاتە(5).

ئینجا دێین لەم یاسایە بەکاری دەهێنین.

$$\boldsymbol{since\ \underline{Y} = c\underline{X} \sim N\left(c\underline{\mu}, c\textstyle\sum c'\right)}$$

$$\Rightarrow E\left(\underline{y}\right) = c\underline{\mu} = \begin{bmatrix} \underline{\mu}^{(1)} - \underline{\mu}^{(2)} \sum_{,12} \sum_{,22}^{-1} \\ \underline{\mu}^{(2)} \end{bmatrix}$$

$$\underline{\mu}^{(1)} = \begin{bmatrix} 4 \\ -1 \end{bmatrix},\ \underline{\mu}^{(2)} = [3],\ \textstyle\sum_{12} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},\ \sum_{,22}^{-1} = 5 = \frac{1}{5}$$

ئینجا دێین لە شوێنیان تەوعیض دەکەینەوە.

$$\Rightarrow E\left(\underline{y}\right) = c\underline{\mu} = \begin{bmatrix} \underline{\mu}^{(1)} - \underline{\mu}^{(2)} \sum_{,12} \sum_{,22}^{-1} \\ \underline{\mu}^{(2)} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} - 3.\begin{bmatrix} 0 \\ 2 \end{bmatrix}.\frac{1}{5} \\ [3] \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix} - 3.\begin{bmatrix} 0 \\ 2 \end{bmatrix}.\frac{1}{5} \Rightarrow \begin{bmatrix} 4 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 6 \end{bmatrix}.\frac{1}{5} \Rightarrow \begin{bmatrix} 4 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{6}{5} \end{bmatrix} \Rightarrow \begin{bmatrix} 4-0 \\ -1-\frac{6}{5} \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{-11}{5} \end{bmatrix}$$

$$E\left(\underline{y}\right) = c\underline{\mu} = \begin{bmatrix}\begin{bmatrix}4\\-11\\\hline 5\end{bmatrix}\\ [3]\end{bmatrix} = \begin{bmatrix}4\\-11\\\hline 5\\ \cdots\\ 3\end{bmatrix} = \begin{bmatrix}\underline{\mu}_y^{(1)}\\ \underline{\mu}_y^{(2)}\end{bmatrix}$$

$$\boldsymbol{var-cov}\left(\underline{\boldsymbol{y}}\right) = \boldsymbol{c}\sum\boldsymbol{c}' = \begin{bmatrix}\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & \mathbf{0}\\ \mathbf{0} & \Sigma_{22}\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{3} & \mathbf{-1} & \vdots & \mathbf{0}\\ \mathbf{-1} & \mathbf{1} & \vdots & \mathbf{2}\\ \cdots & \cdots & \vdots & \cdots\\ \mathbf{0} & \mathbf{2} & \vdots & \mathbf{5}\end{bmatrix}$$

$$\Sigma_{11} = \begin{bmatrix}3 & -1\\-1 & 1\end{bmatrix}, \Sigma_{12} = \begin{bmatrix}0\\2\end{bmatrix}, \Sigma_{22}^{-1} = 5 = \frac{1}{5}, \Sigma_{21} = [0 \quad 2], \Sigma_{22} = 5$$

$$= \begin{bmatrix}\begin{bmatrix}\mathbf{3} & \mathbf{-1}\\ \mathbf{-1} & \mathbf{1}\end{bmatrix} - \begin{bmatrix}\mathbf{0}\\ \mathbf{2}\end{bmatrix}.\frac{\mathbf{1}}{\mathbf{5}}.[\mathbf{0} \quad \mathbf{2}] & \mathbf{0}\\ \mathbf{0} & \mathbf{5}\end{bmatrix}$$

$$= \begin{bmatrix}\mathbf{0}\\ \mathbf{2}\end{bmatrix}.[\mathbf{0} \quad \mathbf{2}].\frac{\mathbf{1}}{\mathbf{5}} = \begin{bmatrix}\mathbf{0*0} & \mathbf{0*2}\\ \mathbf{2*0} & \mathbf{2*2}\end{bmatrix}.\frac{\mathbf{1}}{\mathbf{5}} = \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{4}\end{bmatrix}.\frac{\mathbf{1}}{\mathbf{5}} = \begin{bmatrix}\mathbf{0}*\frac{\mathbf{1}}{\mathbf{5}} & \mathbf{0}*\frac{\mathbf{1}}{\mathbf{5}}\\ \mathbf{0}*\frac{\mathbf{1}}{\mathbf{5}} & \mathbf{4}*\frac{\mathbf{1}}{\mathbf{5}}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \frac{\mathbf{4}}{\mathbf{5}}\end{bmatrix}$$

$$\boldsymbol{var-cov}\left(\underline{\boldsymbol{y}}\right) = \boldsymbol{c}\sum\boldsymbol{c}' = \begin{bmatrix}\begin{bmatrix}\mathbf{3} & \mathbf{-1}\\ \mathbf{-1} & \mathbf{1}\end{bmatrix} - \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \frac{\mathbf{4}}{\mathbf{5}}\end{bmatrix} & \mathbf{0}\\ \mathbf{0} & \mathbf{5}\end{bmatrix}$$

$$= \begin{bmatrix}\mathbf{3} & \mathbf{-1}\\ \mathbf{-1} & \mathbf{1}\end{bmatrix} - \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \frac{\mathbf{4}}{\mathbf{5}}\end{bmatrix} = \begin{bmatrix}\mathbf{3-0} & \mathbf{-1-0}\\ \mathbf{-1-0} & \mathbf{1}-\frac{\mathbf{4}}{\mathbf{5}}\end{bmatrix} = \begin{bmatrix}\mathbf{3} & \mathbf{-1}\\ \mathbf{-1} & \frac{\mathbf{1}}{\mathbf{5}}\end{bmatrix}$$

$$\boldsymbol{var-cov}\left(\underline{\boldsymbol{y}}\right) = \boldsymbol{c}\sum\boldsymbol{c}' = \begin{bmatrix}\begin{bmatrix}\mathbf{3} & \mathbf{-1}\\ \mathbf{-1} & \frac{\mathbf{1}}{\mathbf{5}}\end{bmatrix} & \mathbf{0}\\ \mathbf{0} & \mathbf{5}\end{bmatrix}$$

$$\therefore \underline{\boldsymbol{Y}} = \begin{bmatrix}\underline{\boldsymbol{Y}}^{(1)}\\ \underline{\boldsymbol{Y}}^{(2)}\end{bmatrix} \sim \boldsymbol{N}\left(\begin{bmatrix}\mathbf{4}\\ \mathbf{-11}\\\hline \mathbf{5}\\ \cdots\\ \mathbf{3}\end{bmatrix}, \begin{bmatrix}\begin{bmatrix}\mathbf{3} & \mathbf{-1}\\ \mathbf{-1} & \frac{\mathbf{1}}{\mathbf{5}}\end{bmatrix} & \mathbf{0}\\ \mathbf{0} & \mathbf{5}\end{bmatrix}\right)$$

Q) 2022// $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \sim N(\mu, \Sigma) where\ \underline{\mu}' = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}, \Sigma = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 5 & 2 \\ -1 & 2 & 1 \end{bmatrix}$

Let $\underline{X}^{(1)} = \begin{bmatrix} x_3 \\ x_2 \end{bmatrix}, \underline{X}^{(2)} = [x_1],\ \underline{Y}^{(1)} = \underline{X}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{X}^{(2)}$

$$\underline{Y}^{(2)} = \underline{X}^{(2)}$$

Find the j.p.d.f of $\underline{Y}^{(1)} and \underline{Y}^{(2)}$

Solution:

since$\underline{X} = \begin{bmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} x_3 \\ x_2 \end{bmatrix} \\ [x_1] \end{bmatrix} \Rightarrow \underline{\mu} = \begin{bmatrix} \underline{\mu}^{(1)} \\ \underline{\mu}^{(2)} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\ [-1] \end{bmatrix} = \begin{bmatrix} \underline{\mu}^{(1)} \\ \cdots \\ \underline{\mu}^{(2)} \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ \cdots \\ 4 \end{bmatrix}$

ئینجا دێین(سەفی یەکەم بە(x_3)دادەنێین،بەهەمان شێوەش عامودی یەکەم بە (x_3)دادەنێین، وە سەفی دووەم بە(x_2)دادەنێین،بەهەمان شێوەش عامودی دووەم بە(x_2)دادەنێین،جونکە ($\underline{X}^{(1)}$)بریتیە لە (x_3وەx_2)وە سەفی سێ یەم بە(x_1))دادەنێین، بەهەمان شێوەش عامودی سێ یەم (x_1)،جونکە ($\underline{X}^{(2)}$)بریتیە لە (x_1)

$$And \Sigma = \begin{matrix} & x_3 & x_2 & x_1 \\ x_3 & 3 & 0 & -1 \\ x_2 & 0 & 5 & 2 \\ x_1 & -1 & 2 & 1 \end{matrix} = \begin{bmatrix} 1 & 2 & \vdots & -1 \\ 2 & 5 & \vdots & 0 \\ \cdots & \cdots & \vdots & \cdots \\ -1 & 0 & \vdots & 3 \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

$\Sigma_{11} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \Sigma_{12} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \Sigma_{21} = [-1 \quad 0], \Sigma_{22} = [3]$

x_1 لەگەل x_1 دەکاتە(1) واتە a_{11}

x_1 لەگەل x_2 دەکاتە(2) واتە a_{12}

x_1 لەگەل x_3 دەکاتە(-1) واتە a_{13}

x_2 لەگەل x_1 دەکاتە(2) واتە a_{21}

x_2 لەگەل x_2 دەکاتە(5) واتە a_{22}

x_2 لەگەل x_3 دەکاتە(0) واتە a_{23}

x_3 لەگەل x_1 دەکاتە(-1) واتە a_{31}

x_3 لەگەل x_2 دەکاتە(0) واتە a_{32}

x_3 لەگەل x_3 دەکاتە(3) واتە a_{33}

ئینجا دێین لەم یاسایە بەکاری دەهێنین.

$$\textbf{since } \underline{Y} = c\underline{X} \sim N\left(c\underline{\mu}, c\Sigma c'\right)$$

$$\Rightarrow E\left(\underline{y}\right) = c\underline{\mu} = \begin{bmatrix} \underline{\mu}^{(1)} - \underline{\mu}^{(2)} \Sigma_{,12} \Sigma_{,22}^{-1} \\ \underline{\mu}^{(2)} \end{bmatrix}$$

$$\underline{\mu}^{(1)} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \underline{\mu}^{(2)} = [\,4], \ \Sigma_{12} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \Sigma_{,22}^{-1} = 3 = \frac{1}{3}$$

ئینجا دێین لە شوێنیان تەوضیع دەکەینەوە.

$$\Rightarrow E\left(\underline{y}\right) = c\underline{\mu} = \begin{bmatrix} \underline{\mu}^{(1)} - \underline{\mu}^{(2)} \Sigma_{,12} \Sigma_{,22}^{-1} \\ \underline{\mu}^{(2)} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} - 4.\begin{bmatrix} -1 \\ 0 \end{bmatrix}.\frac{1}{3} \\ [4] \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 3 \end{bmatrix} - 4.\begin{bmatrix} -1 \\ 0 \end{bmatrix}.\frac{1}{3} \Rightarrow \begin{bmatrix} -1 \\ 3 \end{bmatrix} - \begin{bmatrix} -4 \\ 0 \end{bmatrix}.\frac{1}{3} \Rightarrow \begin{bmatrix} \mathbf{-1} \\ \mathbf{3} \end{bmatrix} - \begin{bmatrix} \frac{-4}{3} \\ \mathbf{0} \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{-1} - \frac{-4}{3} \\ \mathbf{3} - \mathbf{0} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \mathbf{3} \end{bmatrix}$$

$$E\left(\underline{y}\right) = c\underline{\mu} = \begin{bmatrix} \begin{bmatrix} \frac{1}{3} \\ 3 \end{bmatrix} \\ [\,4] \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 3 \\ \cdots \\ 4 \end{bmatrix} = \begin{bmatrix} \underline{\mu}_y^{(1)} \\ \underline{\mu}_y^{(2)} \end{bmatrix}$$

$$\boldsymbol{var - cov}\left(\underline{y}\right) = c\Sigma c' = \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}{}^{-1}\Sigma_{21} & \mathbf{0} \\ \mathbf{0} & \Sigma_{22} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & \vdots & -1 \\ 2 & 5 & \vdots & 0 \\ \cdots & \cdots & \vdots & \cdots \\ -1 & 0 & \vdots & 3 \end{bmatrix}$$

$$\Sigma_{11} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \Sigma_{12} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \Sigma_{22}{}^{-1} = 3 = \frac{1}{3}, \Sigma_{21} = [-1 \quad 0], \Sigma_{22} = 3$$

$$= \begin{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{5} \end{bmatrix} - \begin{bmatrix} -\mathbf{1} \\ \mathbf{0} \end{bmatrix} \cdot \frac{\mathbf{1}}{\mathbf{3}} \cdot [-\mathbf{1} \quad \mathbf{0}] & \mathbf{0} \\ \mathbf{0} & \mathbf{3} \end{bmatrix}$$

$$= \begin{bmatrix} -\mathbf{1} \\ \mathbf{0} \end{bmatrix} . [-\mathbf{1} \quad \mathbf{0}] . \frac{\mathbf{1}}{\mathbf{3}} = \begin{bmatrix} -\mathbf{1} * -\mathbf{1} & -\mathbf{1} * \mathbf{0} \\ \mathbf{0} * -\mathbf{1} & \mathbf{0} * \mathbf{0} \end{bmatrix} . \frac{\mathbf{1}}{\mathbf{5}} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} . \frac{\mathbf{1}}{\mathbf{3}} = \begin{bmatrix} \mathbf{1} * \frac{\mathbf{1}}{\mathbf{3}} & \mathbf{0} * \frac{\mathbf{1}}{\mathbf{3}} \\ \mathbf{0} * \frac{\mathbf{1}}{\mathbf{3}} & \mathbf{0} * \frac{\mathbf{1}}{\mathbf{3}} \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{1}}{\mathbf{3}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{var - cov}\left(\underline{\boldsymbol{y}}\right) = \boldsymbol{c}\Sigma\boldsymbol{c}' = \begin{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{5} \end{bmatrix} - \begin{bmatrix} \frac{\mathbf{1}}{\mathbf{3}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{5} \end{bmatrix} - \begin{bmatrix} \frac{\mathbf{1}}{\mathbf{3}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{1} - \frac{\mathbf{1}}{\mathbf{3}} & \mathbf{2} - \mathbf{0} \\ \mathbf{2} - \mathbf{0} & \mathbf{5} - \mathbf{0} \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{2}}{\mathbf{3}} & \mathbf{2} \\ \mathbf{2} & \mathbf{5} \end{bmatrix}$$

$$\boldsymbol{var - cov}\left(\underline{\boldsymbol{y}}\right) = \boldsymbol{c}\Sigma\boldsymbol{c}' = \begin{bmatrix} \begin{bmatrix} \frac{\mathbf{2}}{\mathbf{3}} & \mathbf{2} \\ \mathbf{2} & \mathbf{5} \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} \end{bmatrix}$$

$$\therefore \underline{\boldsymbol{Y}} = \begin{bmatrix} \underline{\boldsymbol{Y}}^{(1)} \\ \underline{\boldsymbol{Y}}^{(2)} \end{bmatrix} \sim \boldsymbol{N}\left(\begin{bmatrix} \frac{\mathbf{1}}{\mathbf{3}} \\ \mathbf{3} \\ \cdots \\ \mathbf{4} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} \frac{\mathbf{2}}{\mathbf{3}} & \mathbf{2} \\ \mathbf{2} & \mathbf{5} \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} \end{bmatrix} \right)$$