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Class : Third Stage

Time Series

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Time series

Time series is a set of observations for a particular phenomenon and for a period of time and usually the unit of time equal and it symbolizes the series by (Z_t).

A time series is a collection of observations of well-defined data items obtained through repeated measurements over time. For example, measuring the value of retail sales each month of the year would comprise a time series.

Time series are classified to:

Continuous time series : the series that are continuous recording her observations of the time and could be seen the value of the series in every moment of time (such as temperature , price … etc)

Discrete time series : are those time series that take the observations of the phenomenon at the points of previous time of equal periods and may be an hour or a moment or year … etc (such as rain , when is measured daily)

Time series consist of four components

- Trend component
- Seasonal component
- Cyclical component
- Irregular component

Time series Analysis requires the factors affecting it has been found that the time series are affected by the following factors

The general trend

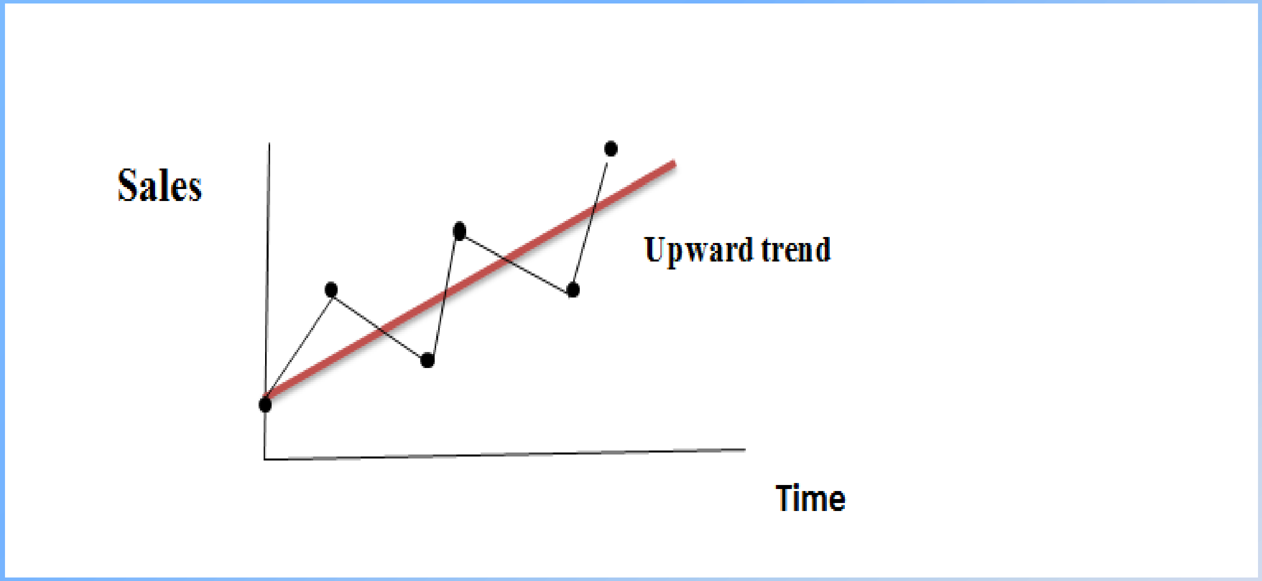
Is the tendency of values of the phenomenon to increase or decrease during the period of the time series and the overall trend is positive when increasing the value of the phenomenon over time for example , population growth is a positive trend time series .

The general trend is negative when decreasing the value of the phenomenon over time, such as the time series of mortality from the disease of small pox in Iraq.

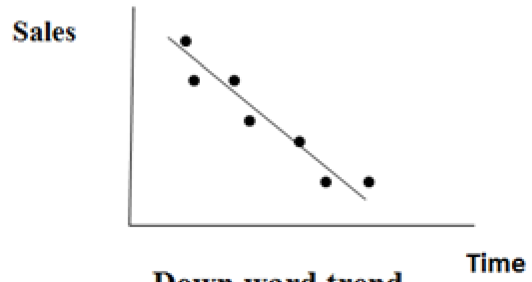
- **Trend component:**

1- Long-run increase or decrease over time (overall upward or downward)

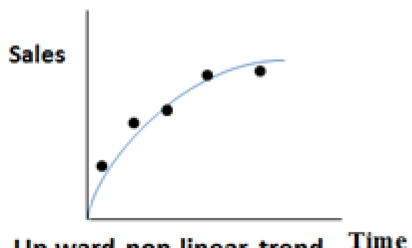
2- Data taken over a long period of time.



- Trend can be up ward or down ward.
- Trend can be linear or non-linear



Down ward trend



Up ward non-linear trend

Seasonal change:

Can be define as seasonal as a pattern repeats it self during certain time periods for example , sales of oil and gas are high in winter and low in summer and that the change experienced by time series be in the time periods fixed such as hours and days and months and seasons knows change seasonality. For example rain fall in winter in a given year and not falling in the winter in other year.

Cyclical Change :

A change that happens in great cycles and may increase the range , each for ten years as the phenomenon back to her first after this period such as a solar eclipse.

Irregular Change :

The changes that occur due to reason for an unexpected emergency random, such as earthquakes , wars and revolutions as the cause of such changes turbulent movement and irregular in their direction disappear after a short period of time and return the series to the movement of regular shall and take as a new direction.

Types of model in time series :

Additive model : in the additive model the observed time series (Y_t) is considered to be sum of independent component ; the seasonal St , the trend Tt , the cyclical Ct and the Irregular It.

Observed series = Trend + Seasonal + Cyclical + Irregular

$$Y_t = T_t + S_t + C_t + I_t$$

Multiplicative model

In the multiplicative model the original time series in expressed as the product of trend , seasonal , cyclical and irregular components.

Observed time series = Trend *Seasonal * Cyclical *Irregular

$$Y_t = T_t * S_t * C_t * I_t$$

Chapter Two
Measurement of Trend

The increase or decrease in the movements of a time series is called trend. A time series data may show upward trend or downward trend for a period of years and this may be due to factors like :

- Increase in population .
- Change in technological progress .
- Large scale shift in consumers demands .

Trend is measured by the following mathematical methods.

1- Graphical method.

2- Method of semi averages.

3- Method of moving averages.

4- Methods of minimum least squares.

Graphical method:

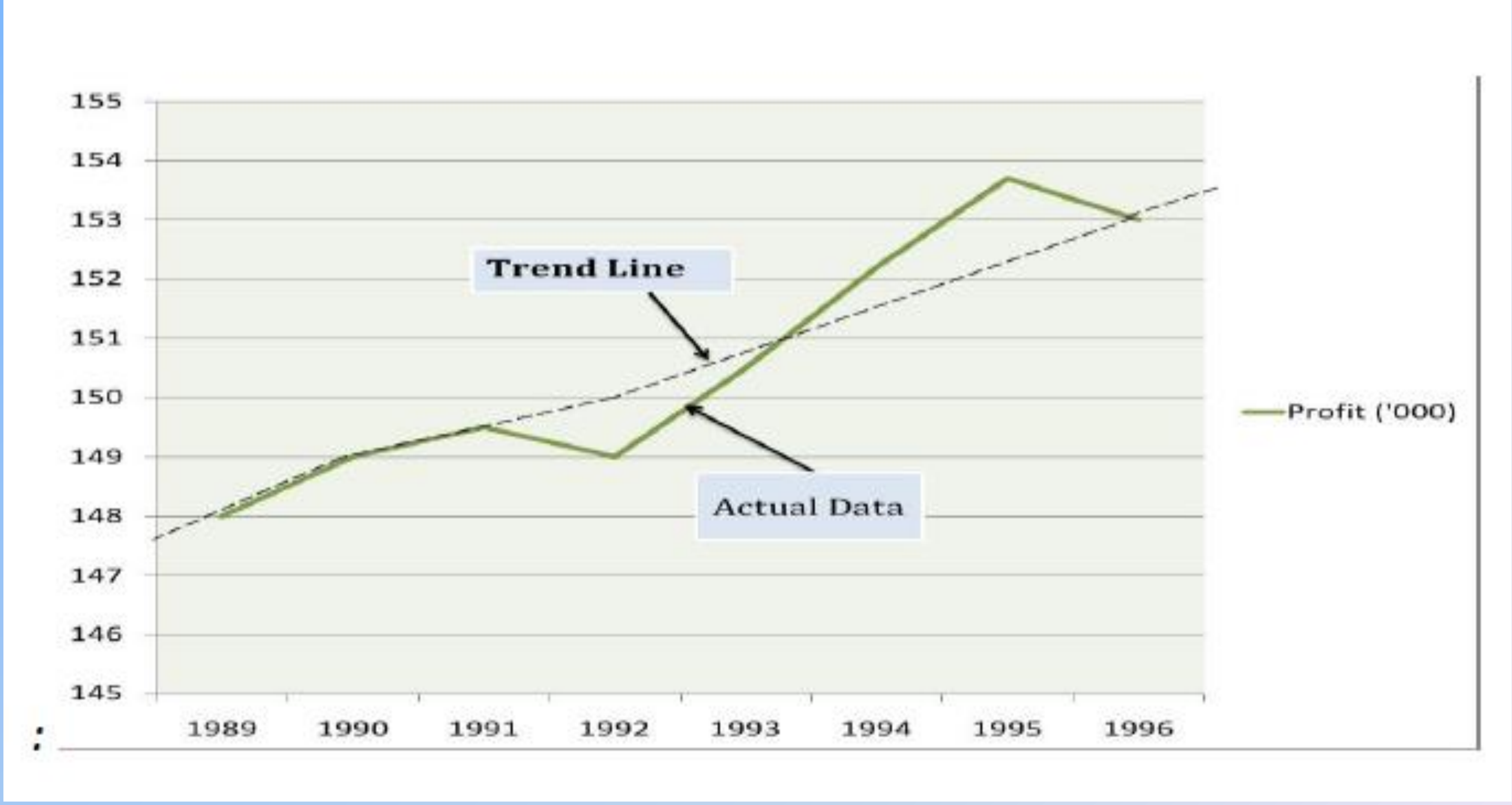
This is the easiest and simplest method of measuring trend. In this method , given data must be plotted on the graph taken time on the horizontal axis and values on the vertical axis. Draw a smooth curve which will show the direction of the trend while fitting a trend line the following important points should be noted to get a perfect trend line.

In this method the data is denoted on graph paper . We take "Time " on X-axis and "data" on Y-axis . On graph there will be a point for every point of time . We make a smooth hand curve with the help of this plotted points .

Ex2-1// Fit a trend line to the following data by using graphical method.

No.	Year	Profit
1	1989	148
2	1990	149
3	1991	149.5
4	1992	149
5	1993	150.5
6	1994	152.2
7	1995	153.7
8	1996	153

Solution



155
154
153
152
151
150
149
148
147
146
145
Trend Line
Actual Data
1989
1990
1991
1992
1993
1994
1995
1996
Profit ('000)

Method of semi average

In this method the given data in divided into **two** parts in the sequence data we have even in this case the in divided into two parts, but if the data sequence 3,5,7,9, … , etc (it means that odd) in this case the data to divided in two equal parts with neglect the number in the middle.

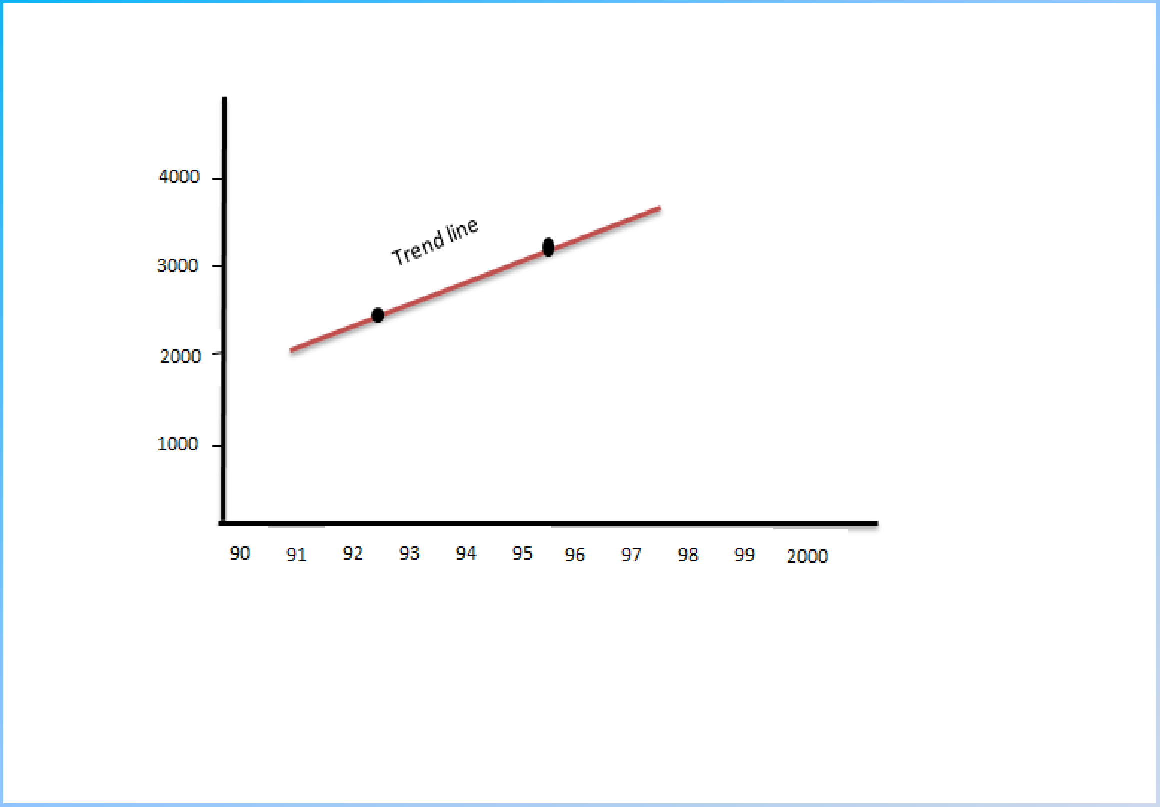
Ex2-2// fit a trend line to the following data by using semi averages.

No.	Year	Y	
1	1991	1812	$\overline{y_1} = \sum \frac{y_i}{n} 2388$
2	1992	2721	
3	1993	3271	
4	1994	1944	
5	1995	2193	

Ex2-2// fit a trend line to the following data by using semi averages.

6	1996	2478	
7	1997	3139	
8	1998	3617	$\overline{y_2} = 3391$
9	1999	3613	
10	2000	4110	

Ex2-2// fit a trend line to the following data by using semi averages.

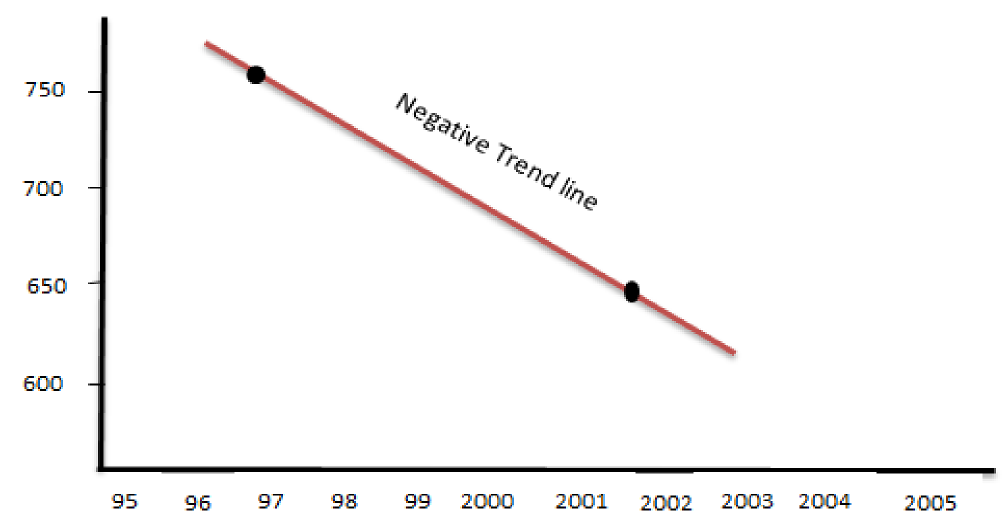


Ex2-3// fit a trend line to the following data by using semi averages.

No.	Year	Y	
1	1995	778	
2	1996	747	
3	1997	795	$\overline{y_1} = 753$
4	1998	735	
5	1999	708	
6	2000	720	اهمال دةكةتين
7	2001	603	
8	2002	650	
9	2003	621	$\overline{y_2} = 645$
10	2004	693	
11	2005	660	

$$\overline{y_1} = \frac{778 + 747 + 795 + 735 + 708}{5} = 753$$

$$\overline{y_1} = \frac{603 + 650 + 621 + 693 + 660}{5} = 645$$


750
700
650
600
Negative Trend line
95
96
97
98
99
2000
2001
2002
2003
2004
2005

Advantages and Disadvantages of Semi Average method:

Advantages:

1- This method is simple to understand as compare to other methods for measuring the secular trend.

2- Every one to apply this method will get the same result.

Disadvantages:

1- The method assumes a straight line relationship between the plotted point without considering the fact whether that relationship exists or not.

2- If we have more data to the original data then we have to do the complete process again for the new data to get the trend value and the trend line also changes.

Method of least squares (LSD)

This is one of the most important methods of fitting a mathematical trend the fitted trend is termed as the best in the sense that the sum of squares of deviations of observations from it is minimized.

This method is most widely in practice .When this method is applied, a trend line is fitted to data in such a manner that the following two conditions are satisfied :

The sum of deviations of the actual values of Y and computed values of Y (Yc) is zero:

- Fitting of linear trend

Given the data (y_t,t) from n periods where t denotes time period such as year , month , day, …, etc we have the values of two constants, "a" , and "b" of the linear equation.

$$y = y_t$$

$$y = a + b_t$$

$\hat{y} = \hat{\alpha} + \hat{\beta}_{ti}$ (Estimate model) (Trend equation)

$$\hat{\beta} = \frac{\sum t_i y_i - n\overline{t\bar{y}}}{\sum t_i{}^2 - n\bar{t}^2}$$

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{t}$$

Ex2-4// If we have the following data production.

No.	Year	yi
1	1998	10
2	1999	8
3	2000	12
4	2001	11
5	2002	4
6	2003	8
7	2004	3
$\sum yi$		56

Required:

- Find trend equation (estimation model) by use least square method (LSD).
- Draw trend line by use estimate model.
- Find Mean Square Error (MSE).

t_i	$t_i y_i$	ti^2
1	10	1
2	16	4
3	36	9
4	44	16
5	20	25
6	48	36
7	21	49
$\sum$ 28	195	140

- Solution:

$$\hat{y} = \hat{\alpha} + \hat{\beta}_{ti}$$

$$\hat{\beta} = \frac{\sum t_i y_i - n\overline{ty}}{\sum t_i^{\,2} - n\bar{t}^2}$$

$$\bar{t} = \frac{\sum t_i}{n} = \frac{28}{7} = 4$$

$$\bar{y} = \frac{\sum y_i}{n} = \frac{56}{7} = 8$$

$$\hat{\beta} = \frac{195 - (7)(4)(8)}{140 - (7)(4^2)} = -1.036$$

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{t}$$

$$= 8 - (-1.036)(4) = 12.144$$

$$\hat{y} = 12.144 - 1.036\, t_i$$

2 - $\hat{y}_1 = 12.144 - 1.036\,(1) = 11.108$

$\hat{y}_2 = 12.144 - 1.036\,(2)$ =10.72

$\hat{y}_3 = 12.144 - 1.036\,(3)$=9.036

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$\hat{y}_7 = 12.144 - 1.036\,(7)$=4.892

3- MSE $= \frac{\sum {e_i}^2}{n-1}$

$= \frac{43.046}{7-1} =$ **7.17**

$e_i = y_i - \hat{y}_i$	e_i^2
-1.108	1.228
-2.72	7.398
2.044	8.769
3	9
-2.944	8.785
2.072	4.293
-1.892	3.858
$\sum e_i^2$	43.046

H.W // if we have the following data production.

No.	Year	yi
1	1991	4
2	1992	2
3	1993	3
4	1994	4
5	1995	5
6	1996	7
7	1997	10
8	1998	8
9	1999	12
10	2000	11

- **Required:**

1- Find trend equation (estimation model) by use least square method (LSD).

2- Draw trend line by use estimate model.

3- Find Mean Square Error (MSE).

Moving average method (MA)

Ex2-6// The following data represent the number of people visiting the Rizgari hospital during the years (1998 – 2005) who safer from tooth decay.

Required:

Estimate number of patients visiting during (2006) by moving average method on the duration of two years.

- Solution:

No.	Year	No. of people (MA)	
1	1998	180	-
2	1999	500	-
3	2000	420	340
4	2001	320	460
5	2002	191	370
6	2003	601	255.5
7	2004	860	396
8	2005	750	730
9	2006	$\frac{860+750}{2} = 805$	805

H.W//

If you have the following data.

Month	yi	MA 3	MA
1	200		
2	135		
3	195		
4	197		
5	310		
6	175		
7	155		
8	130		
9	220		
10	277		
11	235		
12	?		

Required: : Estimate the number of sales during the 12 by using moving average method on the duration (3) months and (5) months.

Trend equation by use semi average method:

In this way turns (t_i) in a form the general trend $(\hat{y} = \hat{\alpha} + \hat{\beta}_{ti})$ to n_i a regression equation becomes as follows:

$$\hat{y} = \hat{a} + \hat{b}n_i$$

$$\hat{b} = \frac{\bar{y}_2 - \bar{y}_1}{\bar{n}_2 - \bar{n}_1}$$

$$\hat{a} = \bar{y}_1 - \hat{b}\bar{n}_1$$

Or $\hat{a} = \bar{y}_2 - \hat{b}\bar{n}_2$

Ex2-5// the following data represent the time series of the average production in a given planet.

No.	Year	yi
1	1955	4
2	1956	8
3	1957	9
4	1958	10
5	1959	12
6	1960	12
7	1961	13
8	1962	15
9	1963	12
10	1964	13

- **Requirement:**

1- Find trend equation (estimation model) by use semi average method (LSD).

$$\hat{y} = \hat{a} + \hat{b}n_i$$

$$\hat{b} = \frac{\bar{y}_2 - \bar{y}_1}{\bar{n}_2 - \bar{n}_1}$$

$$\bar{y}_1 = \frac{4 + 8 + 9 + 10 + 12}{5} = \mathbf{8.6}$$

$$\bar{y}_2 = \frac{12 + 13 + 15 + 12 + 13}{5} = \mathbf{13}$$

$$\bar{n}_1 = \frac{1955 + 1956 + 1957 + 1958 + 1959}{5} = \mathbf{1957}$$

$$\bar{n}_1 = \frac{1960 + 1961 + 1962 + 1963 + 1964}{5} = \mathbf{1962}$$

$$\hat{\boldsymbol{b}} = \frac{13 - 8.6}{1962 - 1957} = \frac{4.4}{5} = \mathbf{0.88}$$

$$\hat{a} = \bar{y}_1 - \hat{b}\bar{n}_1$$

$$= 8.6 - (0.88)(1957)$$

= -1713.5

$\hat{y} = -1713.5 + 0.88n_i$ the trend equation

- **The analysis of time series is based on the assumption that successive values in the data file represent consecutive measurements taken at equally spaced time intervals.**

There are two main goals of time series analysis:

- (a) identifying the nature of the phenomenon represented by the sequence of observations.
- (b) forecasting (predicting future values of the time series variable).

In time series analysis it is assumed that the data consist of a systematic pattern (usually a set of identifiable components) and random noise (error) which usually makes the pattern difficult to identify.

- Most time series patterns can be described in terms of two basic classes of components: trend and seasonality.

- **The Simple Moving Average (SMA): The smoothed statistic S_t is just the mean of the k observations:**

- $S_t = \frac{1}{k}\sum_{n=0}^{k-1} X_{t+\frac{k-1}{2}-n}$ $\quad$ *if k is odd*
- $S_{t+\frac{1}{2}} = \frac{1}{k}\sum_{n=0}^{k-1} X_{t+\frac{k}{2}-n}$ $\quad$ *if k is even*

- Where the choice of an integer *k > 1* is arbitrary, and we start with $t = \frac{k+1}{2}$ if *k* is odd (or *t=k/2* if *k* is even)
- adding one to *t* for each step when we find the values of $\{S_t\}$.

Ex-2-6: Consider the following data of the monthly actual demand on a certain commodity to find the three and four month moving average.

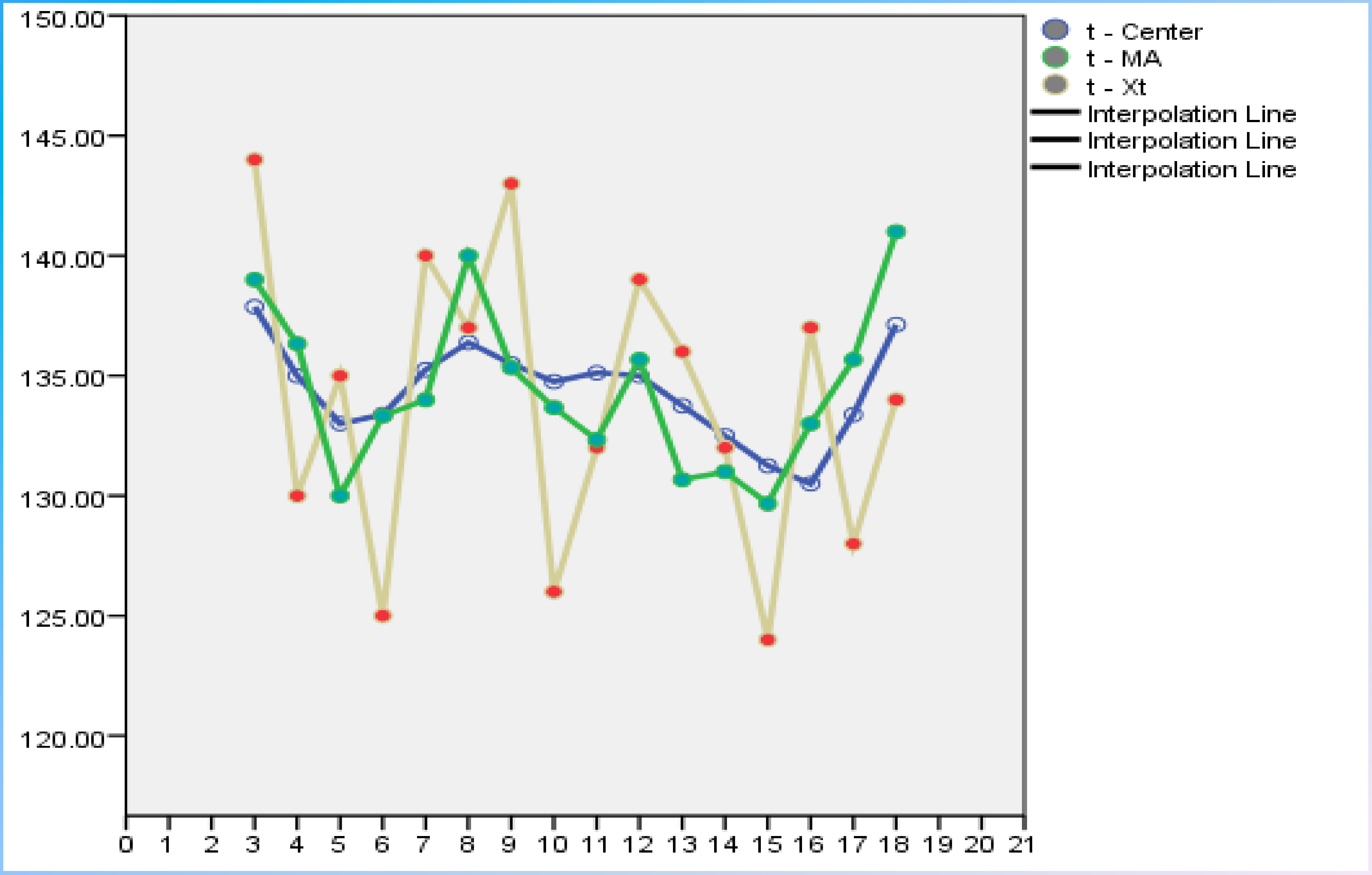
Month	1	2	3	4	5	6	7	8	9	10
Demand	134	143	144	130	135	125	140	137	143	126
Month	11	12	13	14	15	16	17	18	19	20
Demand	132	139	136	132	124	137	128	134	145	146

Sol.: $k = 3 \Rightarrow for\ example\ to\ find\ S_2$

$$S_2 = \frac{1}{3}\sum_{n=0}^{2} X_{3-n} = \frac{1}{3}[X_3 + X_2 + X_1] = \frac{1}{3}[134 + 143 + 144] = 140.33$$

Using the same procedure to find the rest values of the three and four (centred) months moving average.

t	X_t	3month M.A.	4month M.A.	centred
1	134			
2	143	140.33		
			137.75	
3	144	139		137.875
			138	
4	130	136.33		135.75
			133.5	
5	135	130		133
			132.5	
6	125	133.33		133.375
			134.25	
7	140	134		135.25
			136.25	
8	137	140		136.375
			136.5	
9	143	135.33		135.5
			134.5	
10	126	133.67		134.75
			135	
11	132	132.33		135.125
			133.25	
12	139	135.67		134
			134.75	
13	136	135.67		133.75
			132.75	
14	132	130.67		132.5
			132.25	
15	124	131		131.25
			130.25	
16	137	129.67		130.5
			130.75	
17	128	133		133.375
			136	
18	134	135.67		137.125
			138.25	
19	145	141.67		
20	146			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t - Center
t - MA
t - Xt
Interpolation Line
Interpolation Line
Interpolation Line

➢ *The Weighted Moving Average (WMA)*:

- A slightly more intricate method for smoothing a raw time series $\{X_t\}$ is to calculate a weighted moving average by first choosing a set of weighting factors provide that the sum of weights equal one and then using these weights to calculate the smoothed statistics $\{S_t\}$ as described in *SMA* above after product each value $\{ X_t \}$ by its weight . In practice the weighting factors are often chosen to give more weight to the most recent terms in the time series and less weight to older data.

- Ex-2-7: Using data of Ex-2-6 to find the weighted three month moving average with weights 0.2, 0.3 and 0.5.

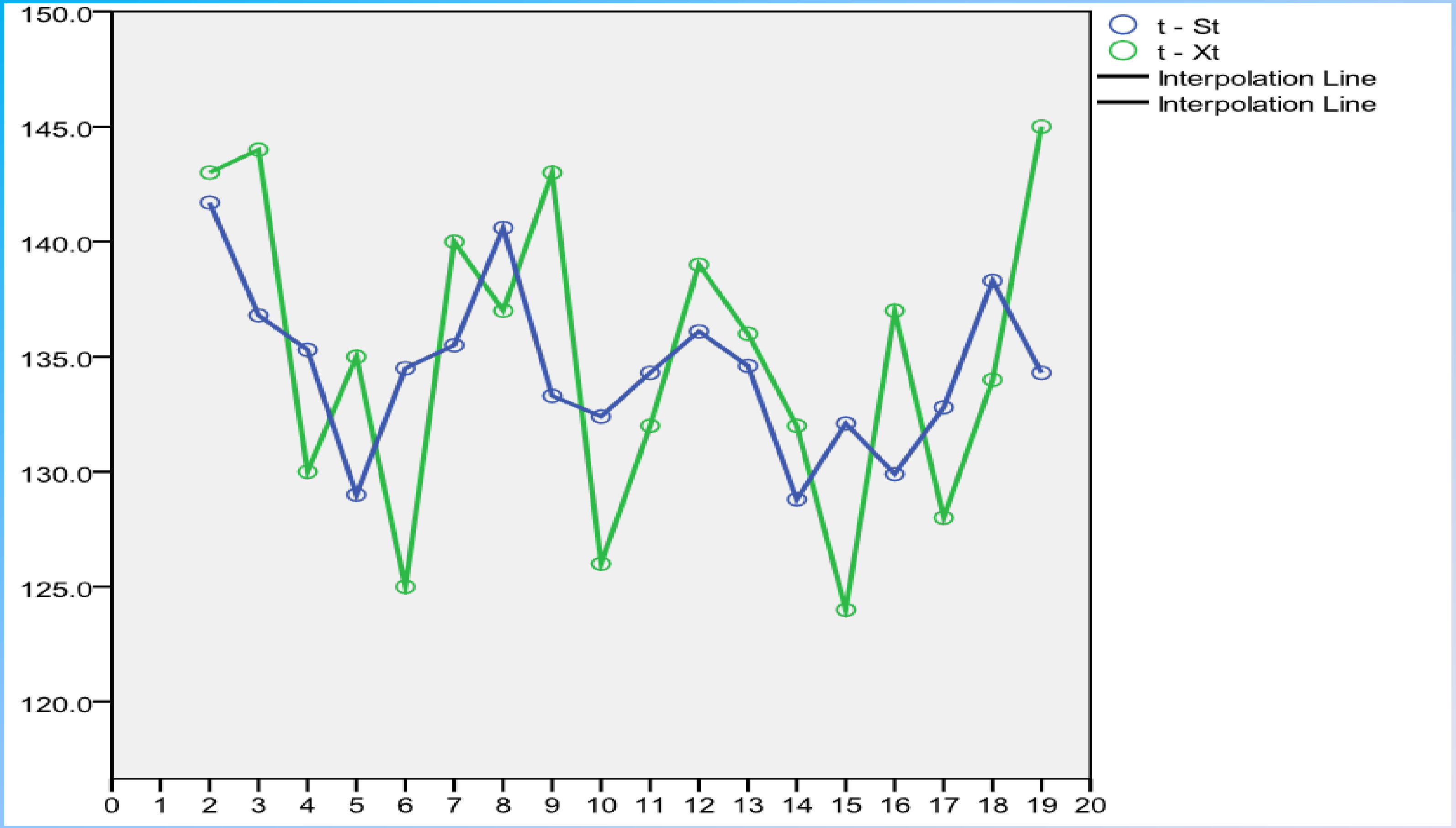
- Sol.: K=3 ⇒ for example to find S_2 and S_3:

$$S_2 = 0.2 * 134 + 0.3 * 143 + 0.5 * 144 = 141.7$$

$$S_3 = 0.2 * 143 + 0.3 * 144 + 0.5 * 130 = 136.8$$

And so on, hence the table below shows all values of the weighted three month moving average:

t	X_t	S_t		t	X_t	S_t
1	134			11	132	134.3
2	143	141.7		12	139	136.1
3	144	136.8		13	136	134.6
4	130	135.3		14	132	128.8
5	135	129		15	124	132.1
6	125	134.5		16	137	129.9
7	140	135.5		17	128	132.8
8	137	140.6		18	134	138.3
9	143	133.3		19	145	134.3
10	126	132.4		20	146	


150.0
145.0
140.0
135.0
130.0
125.0
120.0
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
t - St
t - Xt
Interpolation Line
Interpolation Line

The Exponential Moving Average (EMA):

Exponential smoothing is a rule of thumb technique for smoothing time series data, particularly for recursively applying as many as 3 Low-pass filters with exponential window functions.

The raw data sequence is often represented by {Xt} beginning at time t = 0, and the output of the exponential smoothing algorithm is commonly written as {St}, which may be regarded as a best estimate of what the next value of X will be. When the sequence of observations begins at time t = 0, the simplest form of exponential smoothing is given by the following:

$$S_1 = X_1 \quad \textit{or the average of } X_t$$

$$S_t = \alpha X_{t-1} + (1-\alpha) S_{t-1} \quad ; \quad t > 0$$

Where α is the smoothing factor, and $0 < \alpha < 1$.

Ex-2-8: Find the values of the time series of data given in Ex-2-6, by using exponential smoothing MA when $\alpha = 0.3$ and $\alpha = 0.8$.

Sol.: $S_1 = \frac{134+143+\cdots+146}{20} = 135.5$

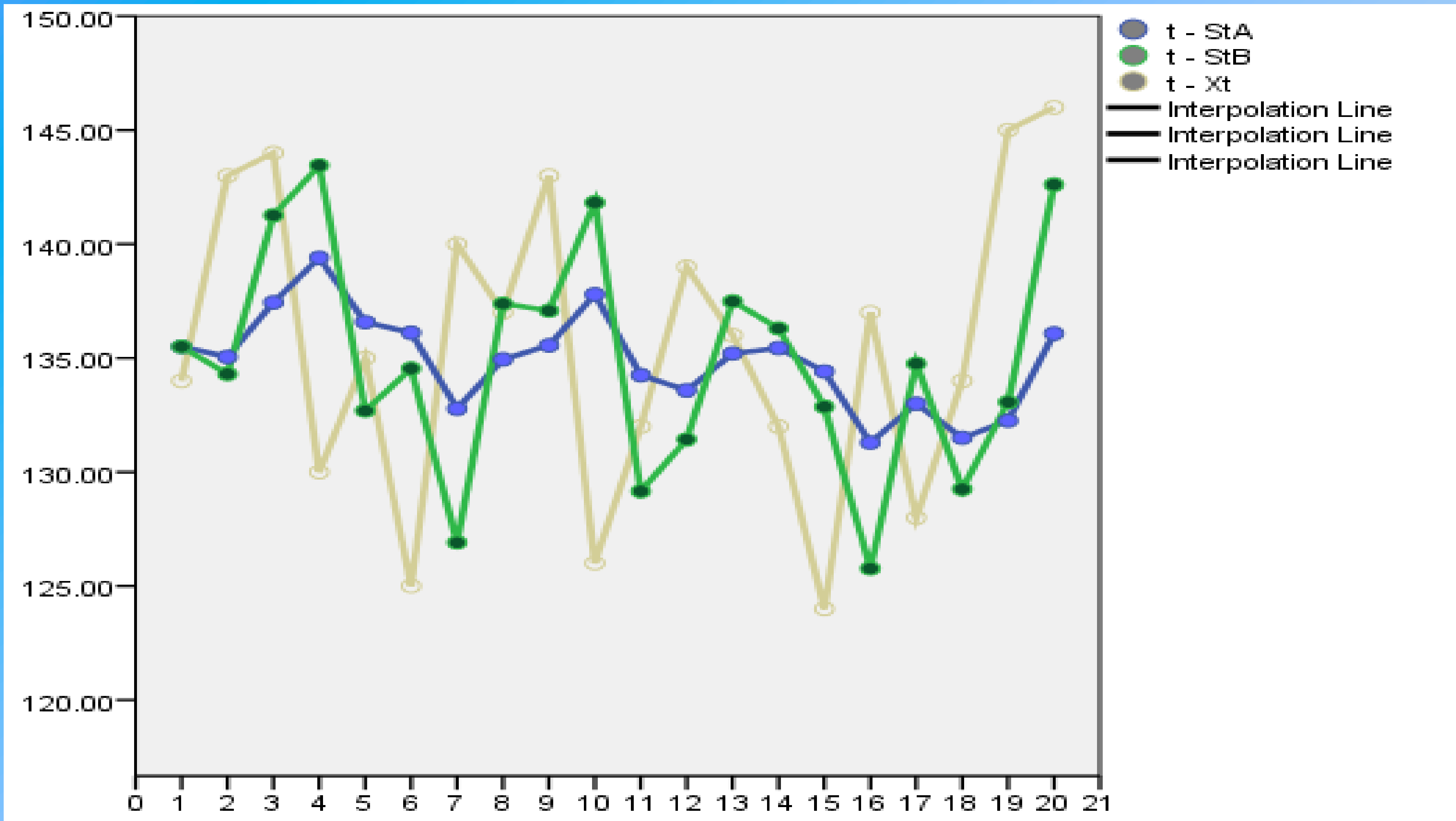
For example, we find S_2 at $\alpha = 0.3$ and $\alpha = 0.8$ as follow:

$S_2 = 0.3 * X_1 + (1-0.3) * S_1 = 0.3 * 134 + 0.7 * 135.5 = 135.05$

$S_2 = 0.8 * X_1 + (1-0.8) * S_1 = 0.8 * 134 + 0.2 * 135.5 = 134.30$

And so on, we can find all rest values as shown in table below :

t	X_t	S_t			t	X_t	S_t	
		$\alpha = 0.3$	$\alpha = 0.8$				$\alpha = 0.3$	$\alpha = 0.8$
1	134	135.5	135.5		11	132	134.25	129.16
2	143	135.05	134.3		12	139	133.58	131.43
3	144	137.44	141.26		13	136	135.2	137.49
4	130	139.4	143.45		14	132	135.44	136.3
5	135	136.58	132.69		15	124	134.41	132.86
6	125	136.11	134.54		16	137	131.29	125.77
7	140	132.78	126.91		17	128	133	134.75
8	137	134.94	137.38		18	134	131.5	129.35
9	143	135.56	137.08		19	145	132.25	133.07
10	126	137.79	141.82		20	146	136.08	142.61



150.00
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t - StB
t - Xt
Interpolation Line
Interpolation Line
Interpolation Line

Chapter Three
Seasonal Variation

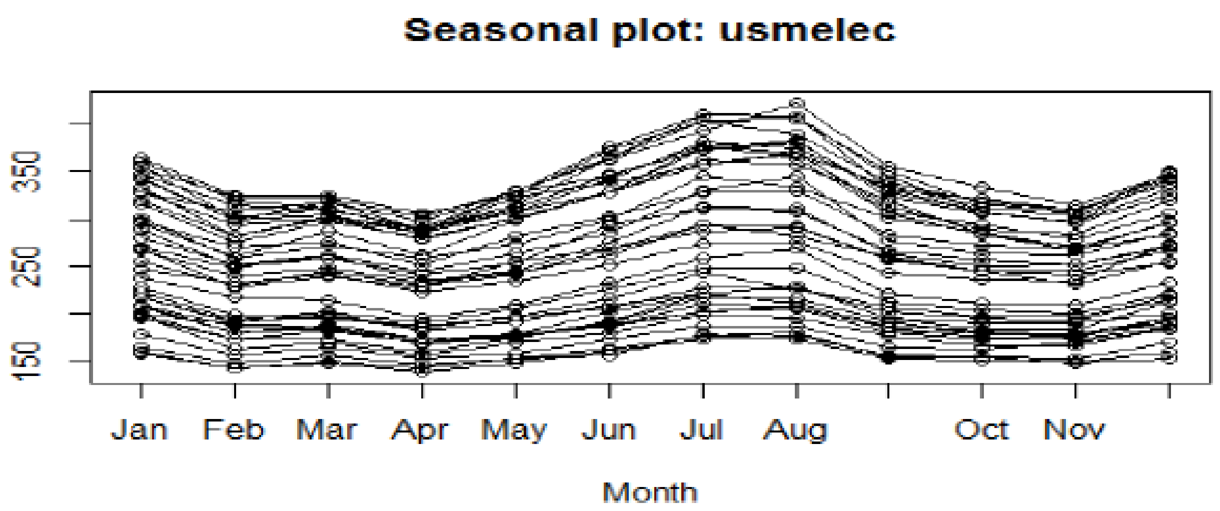
In statistics, time series data is data collected at regular intervals. When there are patterns that repeat over known, fixed periods of time within the data set it is considered to be seasonality, seasonal variation, periodic variation, or periodic fluctuations. This variation can be either regular or semi-regular. Seasonality may be caused by various factors, such as weather, vacation, and holidays and usually consists of periodic, repetitive, and generally regular and predictable patterns in the levels of a time series.

Seasonality can repeat on a weekly, monthly or quarterly basis, these periods of time are structured and occur in a length of time less than a year. Seasonal fluctuations in a time series can be contrasted with cyclical patterns.

Detecting seasonality

The following graphical techniques can be used to detect seasonality:

- A run sequence plot will often show seasonality.
- A seasonal plot will show the data from each season overlapped .
- A seasonal subseries plot is a specialized technique for showing seasonality
- Multiple box plots can be used as an alternative to the seasonal subseries plot to detect seasonality
- An autocorrelation plot (ACF) can help identify seasonality.
- Seasonal Index measures how much the average for a particular period tends to be above (or below) the expected value



seasonality plot of US Electricity Usage

Measuring seasonality:

Seasonal variation is measured in terms of an index, called a **seasonal index**. It is an average that can be used to compare an actual observation relative to what it would be if there were no seasonal variation . An index value is attached to each period of the time series within a year. The following methods use seasonal indices to measure seasonal variations of a time series data.

- Method of simple averages.
- Ratio to trend method
- Ratio-to-moving average method
- Link relatives method

Method of simple averages:

The idea behind the calculation of seasonal variations consists in the possibility and usefulness of determining that part of the annual total that is due to each of the twelve months of year . The random factor that can arise in a given year is considered independently of what might occur next year .

Step-I- We calculate the arithmetic average per month (or quarter) , then the random influences were eliminated among the years .

Step-II- To find out the effect of seasonality therefore will eliminate the trend. The trend calculated by the method of least squares . Monthly (or quarterly) averages are needed over several years .

Ex-3-1: The table below is monthly data series – over two years . Use the method of simple average to calculate the index (coefficient) of seasonality:

year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
I	560	500	450	420	420	480	590	750	860	900	900	850
II	780	720	670	660	630	660	730	860	970	980	950	870

Averages for the previous three years before the given two years are : 520 , 580 , 540 .

Sol. :

months	Year I	Year II	average	Trend	Average - Trend	Seasonality
Jan.	560	780	670	0	670	97.5
Feb.	500	720	610	5	605	88.1
Mar.	450	670	560	10	550	80.1
Apr.	420	660	540	15	525	76.4
May	420	630	525	20	505	73.5
Jun.	480	660	570	25	545	79.3
Jul.	590	730	660	30	630	91.7
Aug.	750	860	805	35	770	112.1
Sep.	860	970	915	40	875	127.4
Oct.	900	980	940	45	895	130.3
Nov.	900	950	925	50	875	127.4
Dec.	850	870	860	55	805	117.2
Total	7680	9480	---	---	8250	1200
Average	640	790	---	---	687	100

The least square method to find the increasing in the trend :

year	X	Y	XY	X^2
I	-2	520	-1040	4
II	-1	580	- 580	1
III	0	540	0	0
IV	1	640	640	1
V	2	790	1580	4
Total	0	3070	600	10

$$a = \frac{\sum Y}{n} = \frac{3070}{5} = 614 \quad ; \qquad b = \frac{\sum XY}{\sum X^2} = \frac{600}{10} = 60$$

Then the linear regression of time series is : $\hat{y} = 614 + 60x$

b = 60 indicate the annual rise of trend , on average of the 12 months . The rise on a single month will be . There is an increasing in the trend equivalent with 5 per month .

To calculate the column of index (coefficient) seasonality , each monthly data of column (average – trend) are divided at the respective average **687**.

Ratio-to-moving average method:

The measurement of seasonal variation by using the ratio-to-moving average method provides an index to measure the degree of the seasonal variation in a time series. The index is based on a mean of 100, with the degree of seasonality measured by variations away from the base .

1. Find the centred 12 monthly (or 4 quarterly) moving averages of the original data values in the time-series.
2. Express each original data value of the time-series as a percentage of the corresponding centred moving average values obtained in step(1).In other words, in a multiplicative time-series model, we get(Original data values)/(Trend values) *100 = (T*C*S*I)/(T*C)*100 = (S*I) *100. This implies that the ratio–to-moving average represents the seasonal and irregular components.
3. Arrange these percentages according to months or quarter of given years. Find the averages over all months or quarters of the given years.
4. If the sum of these indices is not 1200(or 400 for quarterly figures), multiply then by a correction factor = 1200/ (sum of monthly indices). Otherwise, the 12 monthly averages will be considered as seasonal indices.

Ex.-3-2 : calculate the seasonal index by the ratio-to-moving average method from the following data:

	Quarters			
Year	I	II	III	IV
1996	75	60	54	59
1997	86	65	63	80
1998	90	72	66	85
1999	100	78	72	93

Sol. :

year	Quarter	Y	MA(4)	MA(2) [T]	(Y/[T])*100%
1996	1	75			
	2	60			
			62		
	3	54		63.375	85.21
			64.75		
	4	59		65.375	90.25
			66		
1997	1	86		67.125	128.12
			68.25		
	2	65		70.875	91.71
			73.5		
	3	63		74	85.13
			74.5		
	4	80		75.375	106.14
			76.25		
1998	1	90		76.625	117.45
			77		
	2	72		77.625	92.75
			78.25		
	3	66		79.5	83.02
			80.75		
	4	85		81.5	104.29
			82.25		
1999	1	100		83	120.48
			83.75		
	2	78		84.75	92.03
			85.75		
	3	72			
	4	93			

Calculation of seasonal index

	Quarter			
Year	*1*	*2*	*3*	*4*
1996	----	----	*85.21*	*90.25*
1997	*128.12*	*91.71*	*85.13*	*106.14*
1998	*117.45*	*92.75*	*83.02*	*104.29*
1999	*120.48*	*92.04*	----	----
Total	*366.05*	*276.50*	*253.36*	*300.68*
Seasonal Average	*122.01*	*92.16*	*84.45*	*100.23*
Adjusted Seasonal Average	*122.36*	*92.43*	*84.69*	*100.52*

Now the total of seasonal averages is *398.85*. Therefore the corresponding correction factor would be *400/398.85 = 1.00288*. Each seasonal average is multiplied by the correction factor *1.00288* to get the adjusted seasonal indices as shown in the above table. ■

Link relatives method:

This method is slightly more complicated and uses data more completely than other methods. This method is also known as Pearson's method. This method consists in the following steps.

1. The link relatives for each period are calculated by using the below formula :

$$link\ relative\ for\ any\ period = \frac{current\ period\ figure}{previous\ periods\ figure}$$

2. Calculate the average of the link relatives for each period for all the years using mean or median.

3. Convert the average link relatives into chain relatives on the basis of the first season. Chain relative for any period can be obtained by:

$$\frac{\boldsymbol{any\ link\ relative\ for\ that\ period * chain\ relative\ for\ previous\ periods}}{\mathbf{100}}$$

the chain relative for the first period is assumed to be 100.

4. Now the adjusted chain relatives are calculated by subtracting correction factor 'kd' from (k+1)th chain relative respectively.

5. Finally calculate the average of the corrected chain relatives and convert the corrected chain relatives as the percentages of this average. These percentages are seasonal indices calculated by the link relative method.

Ex.-3-3 : Apply the method of link relatives to the following data and calculate seasonal indices :

Quarter	year				
	2003	2004	2005	2006	2007
I	6.0	5.4	6.8	7.2	6.6
II	6.5	7.9	6.5	5.8	7.3
III	7.8	8.4	9.3	7.5	8.0
IV	8.7	7.3	6.4	8.5	7.1

Sol. :

year	Quarter			
	I	II	III	IV
2003	---	108.3	120.0	111.5
2004	62.1	146.3	106.3	86.9
2005	93.2	95.6	143.1	68.8
2006	112.5	80.6	129.3	113.3
2007	77.6	110.6	109.6	88.8
Average	86.35	108.28	121.66	93.86
Chain relatives	100	108.28	131.73	123.64
Corrected chain relatives	100	106.59	128.35	118.57
Seasonal indices	88.20	94.01	113.21	104.58

The calculations in the above table are explained as below:

$$\frac{6.5}{6} * 100 = 108.3 \quad ; \quad \frac{7.8}{6.5} * 100 = 120 \; ; \; \frac{8.7}{7.8} * 100 = 111.5 \; ; \ldots ; etc$$

Chain relatives' row:

100, $\frac{108.28*100}{100} = 108.28 \; ; \; \frac{121.66*108.28}{100} = 131.73 \; and \; \frac{93.86*131.73}{100} = 123.64$

Chain relative of the first quarter (on the basis of first quarter) = 100

Chain relative of the first quarter (on the basis of the last quarter) = $\frac{86.35*123.64}{100} = 106.76$

The difference between these chain relatives = 106.76 – 100 = 6.76.

Difference per quarter =.$\frac{6.76}{4} = 1.69$

Corrected chain relative is: 100; 108.28 – 1.69 = 106.59; 131.73 – 2*1.69 = 128.35; 123.64 – 3*1.69 = 118.57

Average of corrected chain relatives = $\frac{100+106.59+128.35+118.57}{4}$

$= 113.38$

Seasonal variation index are :

$$\frac{100}{113.38} * 100 = 88.20\,; \frac{106.59}{113.38} * 100 = 94.01$$

$$\frac{128.35}{113.38} * 100 = 113.21\,; \frac{118.57}{113.38} * 100 = 10$$

Chapter Four

Stationary

A stationary process has the property that the mean, variance and autocorrelation structure do not change over time.

1. **Strict stationary or strong stationary process:**

Time series Zt is strict stationary if its probability density function does not depend on time i.e pdf of (z_k , z_{k+1}, z_{k+2} , … , z_{k+t}) does not dependent on k.

- **E(z) does not dependent on t.**
- **Var(z) does not dependent on t.**
- **Cov(z_t , z_{t+1}) dependent on k and does not dependent on t.**

2.Weakly stationary:

- **E(z) does not dependent on t.**
- **Cov(z_t , z_{t+1}) dependent on k and does not dependent on t.**

Time series z_t is weakly stationary.

Non-stationary time series:

is the time series properties that change over time and may not be stationary around mean or around variance or both.

A// Non-stationary time series around mean we call this series non stationary around mean if it is dependent up on (t) time and we will take the difference to achieve stationary.

$\Delta z_t = z_t - z_{t-1}$

$= (1 - \beta)\ z_t$

β = back shift operator

$\Delta^2 z_t = \Delta(\Delta z_t)$

$= \Delta(z_t - z_{t-1})$

$= \Delta z_t - \Delta z_{t-1}$

$= z_t - z_{t-1} - (z_{t-1} - z_{t-2})$

$= z_t - 2z_{t-1} + z_{t-2}$

$= (1 - 2\beta + \beta^2)\ z_t$

$= (1 - \beta)^2 z_t$

In general

$$\Delta^j z_t = (1 - \beta)^j z_t$$

Ex// discuss the stationary of

$$z_t = \alpha_0 + \alpha_1 t + a_t$$

Where a_t are uncorrelated random variable with mean zero and variance δ^2.

Solution:

$E(z_t) = \alpha_0 + \alpha_1 E(t) + E(a_t)$

$= \alpha_0 + \alpha_1 \mu_t + 0$

$E(z_t) = \mu_t$

It is not stationary because is dependent up on t

$$wt = \Delta(z_t) = z_t - z_{t-1}$$

$$= \alpha_0 + \alpha_1 t + a_t - (\alpha_0 + \alpha_1(t-1) + a_{t-1})$$

$$= \alpha_0 + \alpha_1 t + a_t - \alpha_0 - \alpha_1 t + \alpha_1 - a_{t-1}$$

$$wt = a_t + \alpha_1 - a_{t-1}$$

$$E(wt) = E(a_t) + \alpha_1 - E(a_{t-1})$$

$= 0 + \alpha_1 + 0$

$E(wt) = \alpha_1$

Is not depend upon t its stationary

Non-stationary time series around variance:

If the series non-stationary around variance we will use the transformation method such that:

1. Square root transformation.
2. Log transformation.
3. Reciprocal Transformation.
4. Standard deviation transformation.

Ex// Discuss the stationary of

$z_t = a_1 + a_2 + a_3 + \cdots + a_t = \sum_{j=1}^{t} a_j$

Where (a_j) is a sequence of uncorrelated random variable with mean zero and variance δ^2

Solution:

$$z_t = a_1 + a_2 + a_3 + \cdots + a_t = \sum_{j=1}^{t} a_j$$

$$E(z_t) = zero$$

It is not depend upon t

It is stationary around mean

$$Var(z_t) = Var(a_1 + a_2 + a_3 + \cdots + a_t) = Var(\sum_{j=1}^{t} a_j)$$

$$= \delta^2 + \delta^2 + \delta^2 + \cdots + \delta^2 = t\delta^2$$

It is depend upon t

It is not stationary

$$\mathrm{Wt} = \frac{1}{\sqrt{t}} z_t$$

$$\mathrm{E(Wt)} = E\left(\frac{1}{\sqrt{t}} z_t\right)$$

$$= \frac{1}{\sqrt{t}} E(z_t)$$

$$= \frac{1}{\sqrt{t}} E(a_1 + a_2 + a_3 + \cdots + a_t)$$

$$= \frac{1}{\sqrt{t}} (0)$$

It is stationary around mean

$$\mathrm{Var(Wt)} = Var\left(\frac{1}{\sqrt{t}} z_t\right)$$

$$= \left(\frac{1}{\sqrt{t}}\right)^2 Var(z_t)$$

$$= \frac{1}{t} t\delta^2 = \delta^2$$

It is not depend upon t it is stationary around variance.

Autocorrelation function (correlogram) :

An important guide to the persistence in a time series is given by the series of quantities called the sample autocorrelation coefficients, which measure the correlation between observations at different times.

$$r_1 = \frac{\sum_{t=1}^{N-1}\left(Y_t - \bar{Y}_{(1)}\right)\left(Y_{t+1} - \bar{Y}_{(2)}\right)}{\sqrt{\sum_{t=1}^{N-1}\left(Y_t - \bar{Y}_{(1)}\right)^2 . \sum_{t=2}^{N}\left(Y_t - \bar{Y}_{(2)}\right)^2}}$$

where ($\overline{Y1}$) is the mean of the first N-1 observations and ($\bar{Y}2$) is the mean of the last N-1 observations.

The quantity r_k is called the autocorrelation coefficient at lag k. The plot of the autocorrelation function as a function of lag is also called the correlogram.

$$r_k = \frac{\sum_{t=1}^{N-k}(Y_t - \bar{Y})(Y_{t+k} - \bar{Y})}{\sum_{t=1}^{N}(Y_t - \bar{Y})^2}$$

The covariance between *Yt* and its value *Yt+k* separated by *k* intervals of time is called the autocovariance at lag *k* and is defined by :

$$\gamma_k = cov(Y_t, Y_{t+k}) = E[(Y_t - \mu)(Y_{t+k} - \mu)]$$

Thus , the autocorrelation at lag k is :

$$\rho_k = \frac{\gamma_k}{\gamma_0}$$

This implies that:ρ_0=1

The most satisfactory estimate of the k^{th} lag autocorrelation is:

$$r_k = \frac{C_k}{C_0}$$

Where $C_k = \frac{1}{N}\sum_{i=1}^{N-k}(Y_t - \overline{Y})(Y_{t+k} - \overline{Y})$, $k = 0, 1, 2, \ldots, K$

Is the estimate of the autocovariance γ_k .

and $\overline{Y}$ is the mean of the time series .

Ex.4-1 : A series of 10 consecutive Yields from a batch chemical process . Find r1 for the first 10 values of batch data :

Y_t	$Y_t - \bar{Y}$	$Y_{t+1} - \bar{Y}$	$(Y_t - \bar{Y})(Y_{t+1} - \bar{Y})$	$(Y_t - \bar{Y})^2$
47	-4	13	-52	16
64	13	-28	-364	169
23	-28	20	-560	784
71	20	-13	-260	400
38	-13	13	-169	169
64	13	4	52	169
55	4	-10	-40	16
41	-10	8	-80	100
59	8	-3	-24	64
48	-3	….	….	9
510	0	….	-1497	1896

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The above calculation is made for illustration only. In practice to obtain a useful estimate of autocorrelation function , we would need at least fifty observations and the estimated autocorrelations rk would be calculated for k = 0 , 1 , 2 , …… , K ; where K was not larger than say N/4 .

Partial AutoCorrelation Function (PACF)

Let $r(x,y|z) = corr(x,y|z)$ denote the partial correlation coefficient between x and y, adjusted for z (or with z held constant).

Denote: $\varphi 2 = corr(xt , xt+2|xt+1)$

$\varphi 3 = corr(xt , xt+3|xt+1 , xt+2)$

$\varphi k = corr(xt , xt+k|xt+1 ,... xt+k-1)$

= partial autocorrelation coefficient at lag k.

- (PACF): $\{ \varphi 1 , \varphi 2, ...\} = \{ \varphi k , k > 1\}$

$\varphi 1 = corr(Xt , Xt+1) = \rho 1$

Applying this here, using $x = X_t$, $y = X_{t+2}$, $z = X_{t+1}$, $\varphi_2 = corr(x_t , x_{t+2}|x_{t+1}) = r(x,y|z)$, along with $\rho_1 = r(x,z)$ and $\rho_2 = r(x,y)$, yields:

$$\varphi_2 = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}$$

Recall that the partial autocorrelation coefficients φ_k are calculated as follows:

$$\varphi_1 = \rho_1 \quad ; \quad \varphi_2 = \frac{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & \rho_2 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix}} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}$$

In general, φk is given as a ratio of determinants involving $\rho_1, \rho_2, \ldots, \rho_k$. The sample partial autocorrelation coefficients are given by these formulae, but with the ρk replaced by their estimates rk:

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Ex.4-3: **Find the estimated partial autocorrelation form the following information:**

r_1 **=-0.39 ,** r_2 **= 0.30 .**

Solution :

$\varphi_1 = r_1$ **= -0.39**

$$\hat{\varphi}_2 = \frac{r_2 - r_1^2}{1 - r_1^2} = \frac{0.30 - (-0.39)^2}{1 - (-0.39)^2} = 0.17$$

The random model which using in the time series:

Autoregressive Moving Averages (ARMA)

Backshift notation:

The backward shift operator B is a useful notational device when working with time series lags:

$$BZ_t = Z_t - 1$$

Some references use L for "lag" instead of B for "backshift".) In other words, B, operating on Zt , has the effect of shifting the data back one period. Two applications of B to Zt shifts the data back two periods:

$$B(BZ_t) = B^2 Z_t = Z_t - 2$$

4-1 Autoregressive model AR(p)

Consider the first order autoregressive model, defined below:AR(1)- Markov process:

$$Z_t = \emptyset_0 + Z_{t-1}\emptyset_1 + u_t$$

with ut ~ WN(0, σ2) and $|\emptyset_1| < 1$.

Note that the AR(1) process reduces to white noise in the special case that . we assume that the process is stationary, such that among others E(Zt) = E(Zt−1) and var(Zt) = var(Zt−1) and use this to derive the expectation and variance of the process together with its first order autocorrelation coefficient.

4-1 Autoregressive model AR(p)

$$z_t = \emptyset_1 z_{t-1} + \emptyset_2 z_{t-2} + \emptyset_3 z_{t-3} + \cdots + \emptyset_p z_{t-p} + u_t$$

$$u_t \sim N\,(0, \delta^2)$$

We can write this model by use back-shift operator.

$$z_t = (\emptyset_1 z_{t-1} + \emptyset_2 z_{t-2} + \emptyset_3 z_{t-3} + \cdots + \emptyset_p z_{t-p}) + u_t$$

$$z_t - \emptyset_1 z_{t-1} - \emptyset_2 z_{t-2} - \emptyset_3 z_{t-3} - \cdots - \emptyset_p z_{t-p} = u_t$$

$$z_t(1 - \emptyset_1\beta - \emptyset_2\beta^2 - \emptyset_3\beta^3 - \cdots - \emptyset_p\beta^p) = u_t$$

Let $\left(1 - \emptyset_1\beta - \emptyset_2\beta^2 - \emptyset_3\beta^3 - \cdots - \emptyset_p\beta^p\right) = \emptyset\beta$

$$\emptyset(\beta) z_t = u_t$$

Markov model AR(1)

$$z_t = \emptyset_1 z_{t-1} + u_t$$

$$z_t - \emptyset_1 z_{t-1} = u_t$$

$$z_t(1 - \emptyset_1 \beta) = u_t$$

Let $(1 - \emptyset_1 \beta) = \emptyset\beta$, Then $(\emptyset\beta)_{zt} = u_t$

Discuss the stationary of Markov model AR(1):

$z_t = \emptyset_1 z_{t-1} + u_t \qquad u_t \sim N(0\,, \delta^2)$

$$E(z_t) = E(\emptyset_1 z_{t-1}) + E(u_t)$$

$$\mu_Z = \emptyset_1 \mu_Z + 0$$

$$\mu_Z - \emptyset_1 \mu_Z = 0$$

$$\mu_Z(1 - \emptyset_1) = 0$$

$$\mu_Z = \frac{0}{(1 - \emptyset_1)} = 0$$

It is stationary around mean

$$z_t = \emptyset_1 z_{t-1} + u_t$$

$$Var(\,z_t) = Var(\emptyset_1 z_{t-1}) + Var(u_t)$$

$$\delta_z^2 = \ \emptyset_1^2 \delta_z^2 + \delta_u^2$$

$$\delta_z^2 - \emptyset_1^2 \delta_z^2 = \delta_u^2$$

$$\delta_z^2 \left(1 - \emptyset_1^2\right) = \delta_u^2$$

$\delta_z^2 = \dfrac{\delta_u^2}{\left(1 - \emptyset^2\right)}$ **it is stationary around variance**

$|\emptyset| < 1 \qquad\qquad -1 < \emptyset < 1$

Auto covariance function and Auto correlation function

$z_t = \emptyset_1 z_{t-1} + u_t$ **We will multiply by** z_{t+k}

$$z_t * z_{t+k} = \emptyset_1 z_{t-1} * z_{t+k} + u_t * z_{t+k}$$

$$E(z_t * z_{t+k}) = E(\emptyset_1 z_{t-1} * z_{t+k}) + E(u_t * z_{t+k})$$

Cov$(z_t * z_{t+k}) = \emptyset_1 cov(z_{t-1} * z_{t+k})$ **+0**

Auto correlation function for AR(1)

$$\{\gamma_k = \emptyset_1 \gamma_{k-1}\} \div \gamma_0$$

$$\rho_k = \emptyset_1 \rho_{k-1}$$

$$\gamma = covariance$$

$$\gamma_0 = variance$$

Moving Average model MA(q):

The moving average model is a common approach for modelling univariate time series. The notation MA(q) refers to the moving average model of order q , then MA(q)-process is:

A moving average is commonly used with time series data to smooth out short-term fluctuations and highlight longer-term trends or cycles.

$$Y_t = \theta_0 + u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2} - \cdots - \theta_q u_{t-q}$$

Where ut ~ WN(0, σ2) are white noise error terms and $\theta_0, \theta_1, \ldots, \theta_q$ are the parameters of the model .

This can be written of the backshift operator B as:

$$Y_t = \theta_0 + \left(1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q\right) u_t$$

The moving average model of order q is defined as.

$$Y_t = u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2} - \theta_3 u_{t-3} - \cdots - \theta_q u_{t-q}$$

$$Y_t = u_t - \theta_1 u_{t-1} \qquad , \ldots \ldots , \qquad u_t \sim WN(0, \sigma^2)$$

We can written this model by use back shift operator

$$Y_t = (1 - \theta_1 B) u_t$$

$$let\,(1 - \theta_1 \beta) = \theta\beta$$

$$Y_t = \theta\beta u_t$$

Auto covariance for MA(1)

$$(Y_t = u_t - \theta_1 u_{t-1}\} * Y_{t+k}$$

$$Y_t Y_{t+k} = u_t Y_{t+k} - \theta_1 u_{t-1} Y_{t+k}$$

$$E(Y_t Y_{t+k}) = E(u_t Y_{t+k}) - \theta_1 E(u_{t-1} Y_{t+k})$$

$$\gamma_k = \gamma_{uz(k)} - \theta_1 \gamma_{uz(-k)}$$

Ex. **What are the properties (mean, variance, covariance and autocorrelation) of *MA(2)?***

***Sol.*:** **Assuming, without loss of generality, that $\theta_0 = 0$; then *MA(2)* process is written as:**

$$Y_t = u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2}$$

As it is a combination of a zero mean white noise, then:

$$E(Y_t) = E(u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2}) = 0 = mean$$

The variance of Y_t is: $\gamma_0 = var(Y_t) = var(u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2})$

$$= \left(1 + \theta_1^2 + \theta_2^2\right)\sigma^2$$

It is easy to calculate the covariance_of Y_t and Y_{t+k} . We get:

$$\gamma_k = cov(Y_t, Y_{t+k}) = E(Y_t Y_{t+k})$$

$$= E[(u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2}).(u_{t+k} - \theta_1 u_{t+k-1} - \theta_2 u_{t+k-2})]$$

$$\gamma_k = (1 + \theta_1^2 + \theta_2^2)\sigma^2 \qquad \text{for } k = 0,$$

$$= (-\theta_1 + \theta_1 \theta_2)\sigma^2 \qquad \text{for } k = \pm 1,$$

$$= -\theta_2 \sigma^2 \qquad \text{for } k = \pm 2,$$

$$= 0 \qquad \text{for } |k| > 2,$$

This shows that the autocovariances depend on lag, but not on time. Dividing γ_k by γ_0 we obtain the autocorrelation function:

$$\rho_k = 1 \qquad \text{for } k = 0,$$

$$= \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2} \qquad \text{for } k = \pm 1,$$

$$= \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2} \qquad \text{for } k = \pm 2,$$

$$= 0 \qquad \text{for } |k| > 2.$$

Autoregressive Moving Average (ARMA) model :

The process {Yt ; t ∈ Z} is an autoregressive moving average process of order (p, q), denoted with Yt ~ ARMA(p, q), if

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Ex.6-4: The process $\{Y_t ; t \in Z\}$ defined by

$$Y_t = 0.7Y_{t-1} - 0.5Y_{t-2} + u_t \;\; ; u_t \sim WN(0, \sigma_u^2) \;\; ; \forall t \in Z$$

Is it an *AR(2)* process?

Sol. : Since $\phi(z) = 1 - 0.7z + 0.5z^2$

Hence the process is a realization of the *AR(2)* process. ∎

Ex.6-5 : The process $\{Y_t\,; t \in Z\}$ defined by

$$Y_t = u_t + 0.7u_{t-1} \quad ; \quad u_t \sim WN(0, \sigma_u^2) \quad ; \quad \forall t \in Z$$

Is an *MA(1)* process?

Sol.: Since $\theta(z) = 1 + 0.7z$.

Ex.6-9: Find the autocorrelation of *ARMA(1,1)* assuming $\phi_o = 0$.

Sol.: The ARMA(1,1) where $\phi_o = 0$ is:

$$Y_t = \phi_1 Y_{t-1} + u_t - \theta_1 u_{t-1}$$

The mean is :

$$E(Y_t) = E(\phi_1 Y_{t-1} + u_t - \theta_1 u_{t-1}) \rightarrow E(Y_t) = \phi_1 E(Y_{t-1}) \rightarrow E(Y_t) = 0$$

The Variance is :

$$\gamma_0 = E[(\phi_1 Y_{t-1} + u_t - \theta_1 u_{t-1})(\phi_1 Y_{t-1} + u_t - \theta_1 u_{t-1})]$$

$$= \phi_1^2 \gamma_0 + \sigma^2 + \theta_1^2 \sigma^2 - 2\phi_1 \theta_1 \sigma^2 \quad ; \quad \text{where } E(u_{t-1}, Y_{t-1}) = \sigma^2$$

$$= \frac{(1+\theta_1^2 - 2\phi_1\theta_1)\sigma^2}{1-\phi_1^2}$$

The covariance is :

$$\gamma_1 = E[Y_{t-1}(\phi_1 Y_{t-1} + u_t - \theta_1 u_{t-1})] = \phi_1 \gamma_0 - \theta_1 \sigma^2$$

$$\gamma_2 = E[Y_{t-2}(\phi_1 Y_{t-1} + u_t - \theta_1 u_{t-1})] = \phi_1 \gamma_1$$

The autocorrelation is:

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{(1 - \phi_1\theta_1)(\phi_1 - \theta_1)}{1 + \theta_1^2 - 2\phi_1\theta_1}$$

$$\rho_k = \phi_1 \rho_{k-1} \qquad ; \qquad \text{For } k \geq 2 \qquad \blacksquare$$

EX/6.10/ write the formula of ARMA(2,3) model by use back shift operator .

Solution:

$$Y_t = \emptyset_1 Y_{t-1} + \emptyset_2 Y_{t-2} + u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2} - \theta_3 u_{t-3}$$
$$Y_t(1 - \emptyset_1 \beta - \emptyset_2 \beta^2) = u_t(1 - \theta_1 \beta - \theta_2 \beta^2 - \theta_3 \beta^3)$$
$$Let\ (1 - \emptyset_1 \beta - \emptyset_2 \beta^2) = \emptyset\beta$$
$$(1 - \theta_1 \beta - \theta_2 \beta^2 - \theta_3 \beta^3) = \theta\beta$$

$$Y_t \emptyset\beta = u_t \theta\beta$$