زانكۆی سەلاحەددین - هەولێر
Salahaddin University-Erbil

The Sombor Index of the Inclusion Graph of some Finite Groups

Research Project

Submitted to the department of (Mathematics) in partial fulfilment of the requirements for the degree of B.S.C in (Mathematics)

By:

Khanda Mohammad Ismail

Supervised by:

Dr. Sanhan Khasraw

April- 2024

CERTIFICATION OF THE SUPERVISOR

I certify that this work was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University-Erbil in partial fulfilment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

Signature:

Supervisor: Dr. Sanhan Khasraw

Scientific grade: Assist. Professor

Date: / 4 / 2024

In view of the available recommendations, I forward this work for debate by the examining committee.

Signature:

Name: Dr. Rashad Rashid Haji

Scientific grade: Assist. Professor

Chairman of the Mathematics Department

Date: / 4/ 2024

ACKNOWLEDGEMENT

I would like to thank my supervisor, Dr. Sanhan Khasraw, for bringing the weight of his considerable experience and knowledge to this project. His high standards have made me better at what I do. I sincerely thank Dr. Rashad Rashid Haji, the Head of the Department of Mathematics, College of Education, for all the facilities provided to us in the pursuit of this project. I acknowledge our deep sense of gratitude to my loving parents for being a constant source of inspiration and motivation.I would like to thank my supervisor, Dr. Sanhan Khasraw, for bringing the weight of his considerable experience and knowledge to this project. His high standards have made me better at what I do. I sincerely thank Dr. Rashad Rashid Haji, the Head of the Department of Mathematics, College of Education, for all the facilities provided to us in the pursuit of this project. I acknowledge our deep sense of gratitude to my loving parents for being a constant source of inspiration and motivation.

ABSTRACT

This project is concerned with a recently introduced graph invariant,namely the sombor index is defined by SO(Γ) $= \sum_{uv\in E(\Gamma)}\sqrt{deg(u)^2 + deg(v)^2}$, where $deg(u)$ is the degree of vertex u in the graph Γ . We determine The Sombor index of the inclusion graph of some finite groups such as $\mathbb{Z}_n$ and D_{2n}.

TABLE OF CONTENT

INTRODUCTION

Graph theory is a helpful tool in studying groups properties Cayley, in 1878, in which a graph represents a finite group. This graph is associated with a group G and a subset A of G. The set of vertices of this graph is the set of elements of G. Let G be a finite group. The order graph of G, Γ_G is the(undirected) graph those whose vertices are non-trivial subgroups of G and two distinct vertices H and G are adjacent. The Sombor index (SO) has a lot of attention within mathematic and chemistry, then the Sombor index of Γ is defined as

SO(Γ)=$\sum_{uv\in E(\Gamma)}\sqrt{\deg(u)^2+\deg(v)^2}$, lately introduced by (IvanGuttman) in 2021.

This project consists of three chapters. In the first chapter, we give some necessary backgrounds about groups and graphs. In Chapter 2, we compute the Sombor index of $\mathbb{Z}_n$. In Chapter 3, we compute Sombor index of D_{2n}.

CHAPTER ONE

Background

Definition 1.1 (FOOTE, 2003): A group $(G,*)$ is a set G, together with a binary operation $*$ on G, such that the following axioms are satisfied:

I - The binary operation $*$ is associative.

II - There is an element e in G such that $x * e = e * x = x$ for all $x \in G$, (the element e is an identity element for $*$ on G).

III - For each a in G, there is an element a' in G with the property that

$a * a' = a' * a = e$ (the element a' is an inverse of a with respect to $*$).

Definition 1.2 (FOOTE, 2003): Let $(G,\star)$ be a group and H be a non-empty subset of G, such that (H, $\star$) is a group then, "H" is called a subgroup of G

That means H also forms a group under a binary operation, i.e., $(H,\star)$ is a group.

Definition 1.3 (FOOTE, 2003) : Let n be a positive integer the collection $\mathbb{Z}_n$is defined as

$\mathbb{Z}_n = \{[1], [2], \dots [n-1]\}$.

Definition 1.4 (Gutman, 2021) : Let $G = (V(G), E(G))$ be a graph. Then, the Sombor index of G is

defined as $SO=\sum_{uv\in E(G)} \sqrt{deg^2(u) + deg^2(v)}$.

Definition 1.5 (Gary Chartrand, et al., 2016) : The degree of vertex v of a graph G is the number of edges of G incident with V and denoted by $deg(v)$.

Definition 1.6 (Devi, 2012) : The dihedral group D_{2n} of order $2n$ is defined by the presentation

$D_{2n} = < r , s \mid r^n = s^2 = 1, srs^{-1} = 1 >$.

Theorem 1.7 (Devi, 2012) : The proper non-trivial subgroups of D_{2n} are:

1. Cyclic groups: $H^k = < r^{\frac{n}{k}} >$ of order k, where k is divisor of n and $k \neq 1$.

2. Cyclic groups: $H_i = < sr^i >$ of order 2, where $i = 1, \dots, n$.

3. Dihedral groups: $H_k^i = < r^{\frac{n}{k}}, sr^i >$ of order 2k, where k is divisor of n, $k \neq 1, n$ and $i = 1, \dots, \frac{n}{k}$.

Definition 1.8 (Rao, 2006) : Let G be a cycle graph of order $(n-1)$. The graph obtained by joining a single new vertex v to each vertex of G called wheel graph of order n. A wheel graph of order n is denoted by W_n .

Definition 1.9 (Ray, 2013) : A complete graph is a simple graph in which each pair of distinct vertices is joined by an edge. In other words, a simple graph in which there exists an edge between every pair vertices is called a complete graph.

CHAPTER TWO

The Sombor Index of the Inclusion Graph of $\mathbb{Z}_n$

In this chapter we compute the Sombor index of the inclusion graph of the group $\mathbb{Z}_n$.

Theorem 2.1: Let $\Gamma = \Gamma(\mathbb{Z}_n)$,where $n = p^r$ and p is prime, be an inclusion graph of $\mathbb{Z}_n$. Then $\Gamma = K_r$.

Proof: The vertices of Γ are of the form $H_i := < p^i >$, where $0 \leq i \leq r-1$.

Suppose H_i and H_j are any two vertices of Γ. Then either $H_i \subseteq H_j$ or $H_j \subseteq H_i$.

Thus, H_i and H_j are adjacent. So, every two vertices of Γ are adjacent. Therefor Γ is a complete graph.

Theorem 2.2: Let $\Gamma = \Gamma(\mathbb{Z}_n)$, be an inclusion graph of $\mathbb{Z}_n$. If $n = p^r$, then

$SO(\Gamma) = \frac{r(r-1)^2}{2}\sqrt{2}$.

Proof: By Theorem2.1, $\Gamma = K_r$. The degree of any vertex in Γ is r-1. Since there are $\frac{r(r-1)}{2}$ edges in Γ, then $SO(\Gamma) = \frac{r(r-1)}{2}\sqrt{(r-1)^2 + (r-2)^2} = \frac{r(r-1)^2}{2}\sqrt{2}$.

Example 2.3: if $n = 9$,then the non-trivial subgroups of $\mathbb{Z}_9$ are

$H_1 = \{0\}$

$H_2 =< 3 >= \{0,3,6\}$ and $\mathbb{Z}_9$. Thus $\Gamma = K_2$

Therefore, $SO(\Gamma) = \frac{2(2-1)^2}{2}\sqrt{2} = \sqrt{2}$.

Theorem 2.4: Let $\Gamma = \Gamma(\mathbb{Z}_n)$, where $n = pq$, p and q are distict primes, be an inclusion graph of $\mathbb{Z}_n$. Then $\Gamma = K_{1,2}$.

Proof: The proper subgroup of $\mathbb{Z}_n$, $n = pq$, are $\{0\}, < p >$ and $< q >$. Since $\{0\} \subseteq < p >, \{0\} \subseteq < q >$ and $< p > \nsubseteq < q >$, then $\Gamma = K_{1,2}$.

Theorem 2.5: Let $\Gamma = \Gamma(\mathbb{Z}_n)$, where $n = pq$, p and q are distinct primes, be an inclusion graph of $\mathbb{Z}_n$. Then $SO(\Gamma)$=$2\sqrt{5}$.

Proof: By Theorem2.4, $\Gamma = K_{1,2}$. There are two edges in Γ both with one end-vertex of degree 2 and the other end- vertex of degree 1. Thus,

$SO(\Gamma) = 2 \cdot \sqrt{2^2 + 1^2} = 2\sqrt{5}$.

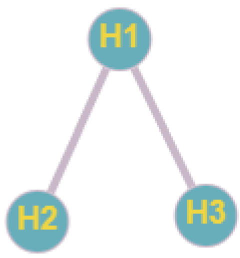
Example 2.6: if $n = 3.2 = 6$,then the non-trivial subgroups of $\mathbb{Z}_6$ are

H_1=$\{0\}$

H_2=$< 2 >$=$\{0,2,4,\}$

H_3=$< 3 >$=$\{0,3,\}$ and $\mathbb{Z}_6$.Thus Γ=$K_{1,2}$

Therefore, $So(\Gamma) = 2\sqrt{5}$.



Theorem 2.7: Let $\Gamma = \Gamma(\mathbb{Z}_n)$, where $n = pqr$, p,q and r are distict primes, be an inclusion graph of $\mathbb{Z}_n$. Then Γ=W_7(a wheel graph).

Proof: The proper subgroup of $\mathbb{Z}_n$, $n = pqr$, are $\{0\}, < p >, < q >, < r >$,

$< pq >, < pr >$ and $< qr >$. It is clear that $\{0\}$ is a subset of each other subgroups. One can see that $< pq > \subseteq < p >, < pr > \subseteq < p >, < qr > \subseteq < q >, < pq > \subseteq < q >, < qr > \subseteq < r >, < pr > \subseteq < r >$. Therefore, Γ is a wheel graph .

Theorem 2.8 : Let $\Gamma = \Gamma(\mathbb{Z}_n)$, where $n = pqr$, p, q and r are distict primes, be an inclusion graph of $\mathbb{Z}_n$. Then $SO(\Gamma)$ $=18\sqrt{2}$ $+18\sqrt{5}$.

Proof: By Theorem2.7, Γ=W_7. There are six edges each with both end-vertices of degree three and there are six edges with one end-vertices of degree three and the other end-vertices of degree six. Thus,

$SO(\Gamma)$ $=6\cdot\sqrt{3^2+3^2}$ $+6\cdot\sqrt{3^2+6^2}$

$=6\cdot 3\cdot\sqrt{2}$ $+6\cdot 3\cdot\sqrt{1+2^2}$

$=18\sqrt{2}$ $+18\sqrt{5}$.

Example 2.9: if $n = 2\cdot 3\cdot 5 = 30$,then the non-trivial subgroups of Z_{30} are

$< p >= \{0,2,4,6,8,10,12,14,16,18,20,22,24,26,28\}$

$< q >= \{0,3,6,9,12,15,18,21,24,27\}$

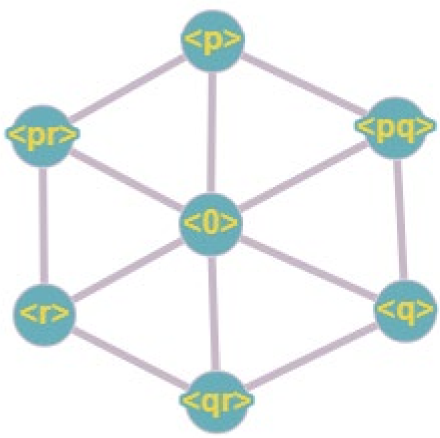
$< r >= \{0,5,10,15,20,25\}$

$< pq >= \{0,6,12,18,24\}$

$< pr >= \{0,10,20\}$

$< qr >= \{0,15\}$ and Z_{30} , Then

$SO(\Gamma)$$=6\cdot\sqrt{3^2+3^2}$ $+6\cdot\sqrt{3^2+6^2}$$=18\sqrt{2}$ $+18\sqrt{5}$.



Theorem 2.10: Let $\Gamma = \Gamma(\mathbb{Z}_n)$, where $n = p^2\, q$, p and q are distict primes, be an inclusion graph of $\mathbb{Z}_n$. Then $SO(\Gamma)$ =$2\sqrt{13}$+$3\sqrt{2}$+$4\sqrt{5}$+10 .

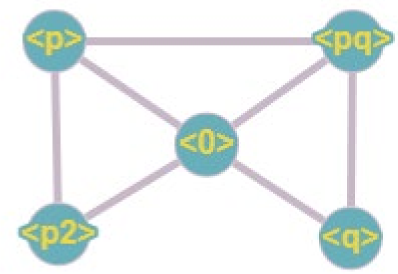
Proof: The proper subgroups of $\mathbb{Z}_n$, where $n = p^2 q$, are $< p >, < q >$,

$< p^2 >$, $< pq >$ and $\{0\}$, one can see that $< p^2 > \subseteq < p >$,

$< pq > \subseteq < p >$, $< pq > \subseteq < q >$ and $\{0\}$ is a subset of each other subgroups. Thus,

SO(Γ)= $2\sqrt{2^2 + 3^2}$ +$\sqrt{3^2 + 3^2}$ +$2\sqrt{2^2 + 4^2}$ +$2\sqrt{3^2 + 4}$

=$2\sqrt{13}$ +$3\sqrt{2}$ +$4\sqrt{5}$ +10



Example 2.11: if $n = \ 2^2 \cdot 3 \ = 12$,then the non-trivial subgroups of Z_{12} are

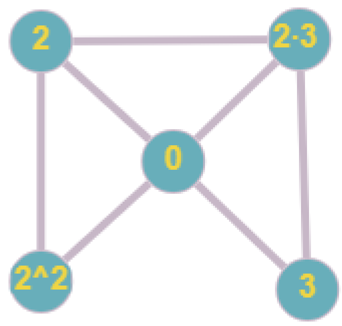
$< \ 2 \ >$={0,2,4,6,8,10}

$< \ 3 \ >$={0,3,6,9}

$< \ 2^2 >$= {0,4,8}

$< \ 2 \cdot 3 \ >$={0,6} and z_{12} , Then

Therefore, $So(\Gamma)$ =$2\sqrt{13}$ +$3\sqrt{2}$ +$4\sqrt{5}$ +10 .



CHAPTER THREE

The Sombor Index of the Inclusion Graph of D_{2n}

In this chapter we compute the Sombor index of the inclusion graph of the dihedral group D_{2n}.

Recall that the dihedral group D_{2n} of order 2n is defined by the presentation

$D_{2n} =< r,s \,|r^n = s^2 =1,srs^{-1} = r^{-1} >$.

Theorem 3.1: Let $\Gamma = \Gamma(D_{2n})$, $n = p, p$ is a prime, be an inclusion graph of D_{2n}. Then $\Gamma = K_{1,n+1}$.

proof: The vertices of Γ are $\{1\}, < r >$ and $< sr^i >, i = 1,2, \dots , n$.

One can see that only $\{1\}$ is a subset of all other subgroup of $\{1\}$ Thus,

$\Gamma = K_{1,n+1}$.

Theorem 3.2: Let $\Gamma = \Gamma(D_{2n}), n = p, p$ is prime, be an inclusion graph of D_{2n}. Then $SO(\Gamma) = (n + 1) \cdot \sqrt{n^2 + 2n + 2}$.

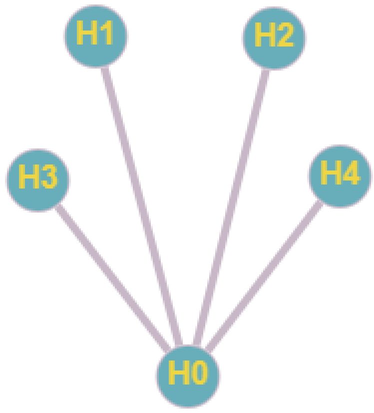
proof: By Theorem 3.1, $\Gamma = K_{1,\ n+1}$. Thus

$SO(\Gamma) = (n + 1)\sqrt{1^2 + (n + 1)^2}$

$= (n + 1)\sqrt{n^2 + 2n + 2}$.

Example 3.3: Let $n = 3$

D_{2n}=D_6 $= \{1,r,r^2,s,sr,sr^2\}$



1- $H^3 =< r^{\frac{3}{3}} >=< r >$=$\{1,r,r^2\}$
2- $H_1 = \{1, s\}$
$H_2 = \{1, sr\}$
H_3=$\{1,sr^2\}$.

Theorrem 3.4: Let $\Gamma = \Gamma(D_{2n})$, $n = p^2$, p is prime, be an inclusion graph of D_{2n}. Then

$$\deg(v)=\begin{cases} p^2+p+2 & if \quad v=\{1\} \\ 2 & if \quad v=H_i, i=1,2,\dots,p^2 \; or \; v=H^{p^2} \\ p+2 & if \quad v=H_p^i, i=1,2,\dots,p \;\; or \;\; v=H^p \end{cases}$$

proof: The proper subgroups of D_{2n}, where $n = p^2$, are $\{1\}$, H^p, H^{p^2};

$i = 1, \dots, p^2$, and H_p^i; $i = 1, \dots, p$. It is clear that $\{1\}$ is a subset 0f each other subgroups, and then $\deg(\{1\}) = p^2 + p + 2$, The subset of H_p^i are $\{1\}$, H^p and H_i, H_{i+p},…,$H_{i+(p-1)p}$. Thus, $\deg(H_p^i) = p + 2$. Since $H^p \subseteq H^{p^2}$, Then $\deg(H^{p^2}) = 2$. Finally, $\deg(H_i) = 2$.

Theorrem 3.5: Let $\Gamma = \Gamma(D_{2n})$, $n = p^2$, p is prime be an inclusion graph of D_{2n}. Then $|E(\Gamma)|$=$2P^2 + 2P + 3$.

Proof:

$$|E(\Gamma)|= \frac{(p^2+p+2)+2(p^2+1)+(p+2)(p+1)}{2}$$

$$= \frac{p^2+p+2+2p^2+2+p^2+p+2p+2}{2}$$

$$= \frac{4p^2+4p+6}{2} = 2p^2+2p+3.$$

Theorrem 3.6: Let $\Gamma = \Gamma(D_{2n})$, $n = p^2$, p is prime, be an inclusion graph of D_{2n}. Then

$$SO(\Gamma) = (p^2+1)\sqrt{p^4+2p^3+5p^2+4p+8} + (p+ 1)\sqrt{p^4+2p^3+6p^2+8p+8} + (p^2+1)\sqrt{p^2+4p+8} + p(p+2)\sqrt{2}\,.$$

Proof: There are p^2+1 edges each has one end-vertex of degree 2 and the other end-vertex of degree p^2+p+2, There are p edges each has one end-vertex of degree p^2+p+2 and the other end-vertex of degree $p+2$, There is one edge with one end-vertex of degree p^2+p+2 and the other end-vertex of degree $p+2$ and the other end other end-vertex of degree $p+2$, There are p^2 edge each has one end-vertex of degree 2 and the other end-vertex of degree $p+2$, There are p edges each has both end-vertex of degree $p+2$, and there is one edge with one end-vertex of degree 2 and the other end-vertex of degree $p+2$. Thus,

$$So(\Gamma) = (p^2+1)\sqrt{2^2+(p^2+p+2)^2} + p\sqrt{p^2+p+2)^2+(p+2)^2} + \sqrt{(p^2+p+2)^2+(p+2)} + p^2\sqrt{2^2+(p+2)^2} + p\sqrt{(p+2)^2+(p+2)^2} + \sqrt{2^2+(p+2)^2}$$

$$= (p^2+1)\sqrt{p^4+2p^3+5p^2+4p+8} \quad +(p+1)\sqrt{p^4+2p^3+6p^2+8p+8} + (p^2+1)\sqrt{p^2+4p+8} + p(p+2)\sqrt{2}\,.$$

Example 3.7: Let $n = 3^3 = 9$

1- $H^k =< r^{\frac{n}{k}} >$, k is divisor of n and k≠1 of order k

   $\{1\}$=H^3=$< r^{\frac{9}{3}} >=< r^3 >$=$\{1,r^3,r^6\}$
   $\{2\}$=$H^9 =< r^{\frac{9}{9}} >=< r >$=$\{1,r,r^2,r^3,r^4,r^5,r^6,r^7,r^8\}$

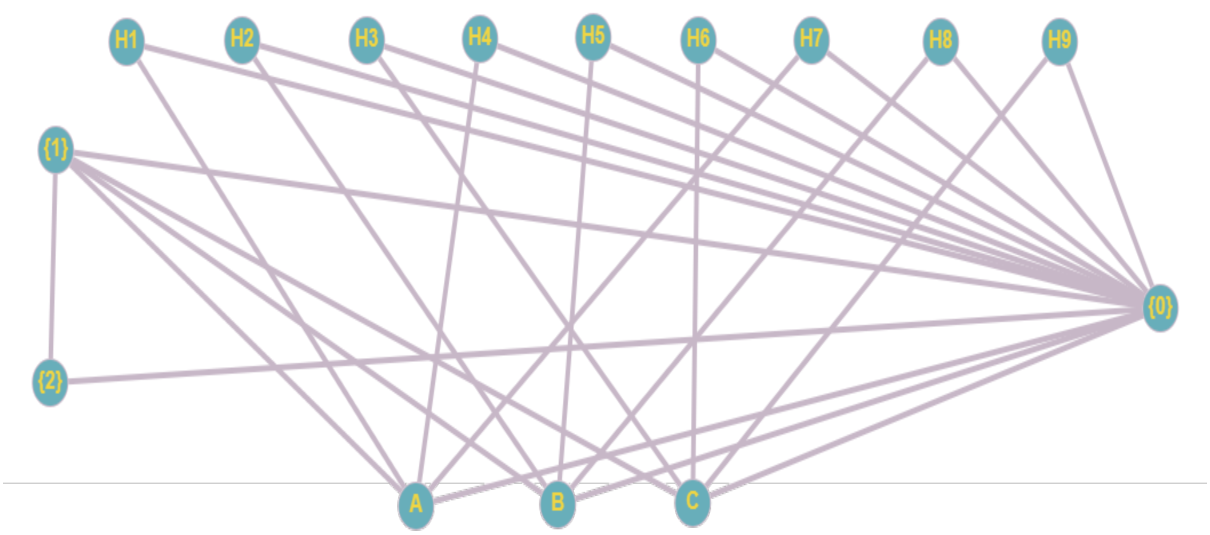
2- $H_i =< sr^i >$; i=1,2, …,n
   H_1=$< sr >$=$\{1, sr\}$
   H_2=$< sr^2 >$=$\{1, sr^2\}$
   $H_3 =< sr^3 >= \{1, sr^3\}$

$H_4 = < sr^4 > = \{1, sr^4\}$
$H_5 = < sr^5 > = \{1, sr^5\}$
$H_6 = < sr^6 > = \{1, sr^6\}$
$H_7 = < sr^7 > = \{1, sr^7\}$
$H_8 = < sr^8 > = \{1, sr^8\}$
$H_9 = < sr^9 > = < s > = \{1, s\}$

3- $H_k^i = < r^{\frac{n}{k}}, sr^i >$ of order $2k$, k a divisor of n ,$k \neq 1, n$,$i = 1,2, \dots, \frac{n}{k}$
$K = 3$,$i = 1,2,3$

A=$H_3^1 = < r^3, sr > = \{1, r^3, r^6, sr, sr^4, sr^7\}$
B=$H_3^2 = < r^3, sr^2 > = \{1, r^3, r^6, sr^2, sr^5, sr^8\}$
C=$H_3^3 = < r^3, sr^3 > = \{1, r^3, r^6, sr^3, sr^6, s\}$

$$So(\Gamma) = (3^2+1)\sqrt{3^4 + 2(3^3) + 5(3^2) + 4(3) + 8}$$
$$+(3+1)\sqrt{3^4 + 2(3^3) + 6(3^2) + 8(3) + 8} + (3^2+1)\sqrt{3^2 + 4(3) + 8}$$
$$+3(3+2)\sqrt{2}$$
$$= 10\sqrt{200} + 4\sqrt{221} + 10\sqrt{29} + 15\sqrt{2}$$



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پوختە

ئەم پرۆژەیە پەیوەندی بە نەگۆڕێکی گرافیەوە هەیە کەبەم دواییە سەلمێندرا،ئەویش sombor index,
کەبەم شێوەیەپێناسە دەکرێت

کە $So(\Gamma)=\sum_{uv\in E(\Gamma)}\sqrt{\deg(u)^2+\deg(v)^2}$

ئێمە لێرەدا d_r(u) is the degree of vertex u in Γ

The Sombor index of inclusion graph of some finite groups

دیاری دەکەین (Z_n,D_{2n})

الخلاصة

يهتم هذا المشرع بالرسم البياتى Γ الذي تم أدخاله مؤخرا ، وهو مؤشرsombor
يتم تعريفه على أنة

$d_r(u)$, حيث $So(\Gamma)=\sum_{uv\in E(\Gamma)}\sqrt{deg(u)^2+deg(v)^2}$

يكون درجة الرأس في Γتحدد مؤشر

The Sombor index of inclusion graph of some finite groups

$(Z_n$ و $D_{2n})$