زانكۆی سەلاحەددین - هەولێر
Salahaddin University-Erbil

The Sombor Coindex of the Inclusion Graph of some Finite Groups

Research Project

Submitted to the department of (Mathematics) in partial fulfilment of the requirements for the degree of B.Sc. in (Mathematics)

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CERTIFICATION OF THE SUPERVISOR

I certify that this work was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University-Erbil in partial fulfilment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

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Supervisor: Dr. Sanhan Khasraw

Scientific grade: Assist. Professor

Date: / 4 / 2024

In view of the available recommendations, I forward this work for debate by the examining committee.

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Scientific grade: Assist. Professor

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Date: / 4/ 2024

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ABSTRACT

This project is concerned with a recently introduced graph invariant, namely the Sombor coindex is defined by $SOC(\Gamma) = \sum_{uv \notin E(\Gamma)} \sqrt{deg(u)^2 + deg(v)^2}$, where $deg(u)$ is the degree of vertex u in Γ. We determine The Sombor coindex of the inclusion graph of some finite groups such as $\mathbb{Z}_n$ and D_{2n} .

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INTRODUCTION

Graph theory is a helpful tool in studying groups properties Cayley, in 1878, in which a graph represents a finite group. This graph is associated with a group G and a subset A of G. The set of vertices of this graph is the set of elements of G. Let G be a finite group. The order graph of G, Γ_G is the(undirected) graph those whose vertices are non-trivial subgroups of G and two distinct vertices H and G are adjacent. The Sombor Coindex (SOC) has a lot of attention within mathematic and chemistry, then the Sombor Coindex of Γ is defined as

$SOC(\Gamma)= \sum_{uv\notin E(\Gamma)} \sqrt{\deg(u)^2 + \deg(v)^2}$, lately introduced by (Ivan Guttman) in 2021.

This project consists of three chapters. In the first chapter, we give some necessary backgrounds about groups and graphs. In Chapter 2, we compute the Sombor Coindex of $\mathbb{Z}_n$. In Chapter 3, we compute Sombor Coindex of D_{2n}.

CHAPTER ONE

Background

Definition 1.1 (FOOTE, 2003): A group (G,∗) is a set G, together with a binary operation ∗ on G, such that the following axioms are satisfied:

I - The binary operation ∗ is associative.

II - There is an element e in G such that x ∗ e = e ∗ x = x for all x ∈ G, (the element e is an identity element for ∗ on G).

III - For each a in G, there is an element a′ in G with the property that

a ∗ a′= a′ ∗ a=e (the element a' is an inverse of a with respect to ∗).

Definition 1.2 (FOOTE, 2003): Let (G, ⋆) be a group and H be a non-empty subset of G, such that (H, ⋆) is a group then, "H" is called a subgroup of G

That means H also forms a group under a binary operation, i.e., (H, ⋆) is a group.

Definition 1.3 (FOOTE, 2003) : Let n be a positive integer the collection Zn is defined as

$\mathbb{Z}_n$ = {[1],[2], … [n − 1]}.

Definition 1.4 (Gutman, 2021) : Let G = (V(G), E(G)) be a graph. Then, the Sombor Coindex of G is defined as $SOC=\sum_{uv\notin E(G)}\sqrt{deg^2(u)+deg^2(v)}$.

Definition 1.5 (Gary Chartrand, et al., 2016) : The degree of vertex v of a graph G is the number of edges of G incident with V and denoted by $deg(v)$.

Definition 1.6 (Devi, 2012) : The dihedral group D_{2n} of order 2n is defined by the presentation

$D_{2n} = < r , s \mid r^n = s^2 = 1, srs^{-1} = 1 >$.

Definition1.7 (Devi, 2012) : The proper non-trivial subgroups of D_{2n} are:

1. Cyclic groups: $H^k =< r^{\frac{n}{k}} >$ of order k, where k is divisor of n and k ≠1.

2. Cyclic groups: $H_i =< sr^i >$ of order 2, where i = 1, … , n.

3. Dihedral groups: $H_k^i =< r^{\frac{n}{k}}, sr^i >$of order 2k, where k is divisor of n, k ≠ 1, n and $i = 1, \ldots, \frac{n}{k}$.

Definition 1.8 (Rao, 2006) : Let G be a cycle graph of order (n-1). The graph obtained by joining a single new vertex v to each vertex of G called wheel graph of order n. A wheel graph of order n is denoted by W_n .

Definition 1.9 (Ray, 2013) : A complete graph is a simple graph in which each pair of distinct vertices is joined by an edge. In other words, a simple graph in which there exists an edge between every pair vertices is called a complete graph.

CHAPTER TWO

The Sombor Coindex of the Inclusion Graph of $\mathbb{Z}_n$

In this chapter we compute the Sombor Coindex of the inclusion graph of the group$\mathbb{Z}_n$.

Theorem2.l : Let $\Gamma = \Gamma(\mathbb{Z}_n)$,where $n = p^r$ and p is prime, be an inclusion graph of $\mathbb{Z}_n$. Then $\Gamma = K_r$.

Proof: The vertices of Γ are of the form $H_i :=< p^i >$, where 0≤ $i \leq r-1$.

Suppose H_i and H_j are any two vertices of Γ. Then either $H_i \subseteq H_j$ or $H_j \subseteq H_i$.

Thus, H_i and H_j are adjacent. So, every two vertices of Γ are adjacent.

Therefore, Γ is a complete graph.

Theorem2.2: Let $\Gamma = \Gamma(\mathbb{Z}_n)$ be an inclusion graph of $\mathbb{Z}_n$. If $n = p^r$, then $SOC(\Gamma) = 0$.

Proof: By Theorem 2.1, $\Gamma = K_r$. Since there is no off-edge in Γ, then $SOC(\Gamma) = 0$.

Example2.3: if $n = 3^2$, then the non-trivial subgroups of $\mathbb{Z}_9$ are

H_1={0}

$H_2 =< 3 >$={0,3,6} and Z_9.Thus Γ=K_2

Therefore , $SOC(\Gamma) = 0$

Theorem2.4: Let $\Gamma = \Gamma(\mathbb{Z}_n)$, where $n = pq$ and p,q are distict primes, be an inclusion graph of subgroups of $\mathbb{Z}_n$. Then $\Gamma = K_{1,2}$.

Proof: The proper subgroup of $\mathbb{Z}_n$, $n = pq$, are $\{0\}$, $<\mathrm{p}>$ and $<\mathrm{q}>$. Since $\{0\} \subseteq < p >$, $\{0\} \subseteq < q >$ and $< p > \nsubseteq < q >$, then $\Gamma = k_{1,2}$.

Theorem2.5: Let $\Gamma = \Gamma(\mathbb{Z}_n)$, where $n = pq$ and p, q are distict primes, be an inclusion graph of subgroups of $\mathbb{Z}_n$. Then $SOC(\Gamma) = \sqrt{2}$.

Proof: By Theorem 2.4, $\Gamma = K_{1,2}$. Since there is only one off-edge in Γ with both end-vertices of degree 1, then $(\Gamma) = \sqrt{1^2 + 1^2} = \sqrt{2}$.

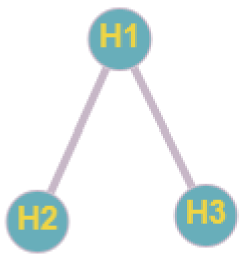
Example2.6: if $n = 3 * 2 = 6$, then the non-trivial subgroups of Z_6 are

H_1=$\{0\}$

H_2=$< 2 >$=$\{0,2,4,\}$

H_3=$< 3 >$=$\{0,3,\}$ and $\mathbb{Z}_6$.Thus Γ=$K_{1,2}$

Therefore, $SOC(\Gamma) = \sqrt{2}$.



Theorem2.7: Let $\Gamma = \Gamma(\mathbb{Z}_n)$, where $n = pqr$, p,q and r are distict primes, be an inclusion graph of Zn. Then Γ=W_7(a wheel graph).

Proof: The proper subgroup of Z_n, $n = pqr$, are $\{0\}$, $< p >$, $< q >$, $< r >$, $< pq >$, $< pr >$ and $< qr >$. It is clear that $\{0\}$ is a subset of each other subgroups.One can see that $< pq > \subseteq < p >$, $< pr > \subseteq < p >$, $< qr > \subseteq < q >$, $< pq > \subseteq < q >$, $< qr > \subseteq < r >$, $< pr > \subseteq < r >$. Therefore, Γ is a wheel graph.

Theorem2.8: Let $\Gamma = \Gamma(\mathbb{Z}_n)$, where $n = pqr$ where p, q and r are distict primes, be an inclusion graph of subgroups of $\mathbb{Z}_n$. Then $SOC(\Gamma) = 27\sqrt{2}$.

Proof: By Theorem 2.7, $\Gamma = W_7$. There are nine off-edges each with both end-vertices of degree three. Thus, $SOC(\Gamma) = 9 * \sqrt{3^2 + 3^2} = 9 * 3 * \sqrt{2} = 27\sqrt{2}$.

Example2.8:if $n = 2 * 3 * 5 = 30$, then the non-trivial subgroups of Z_{30} are

$< p >= \{0,2,4,6,8,10,12,14,16,18,20,22,24,26,28\}$

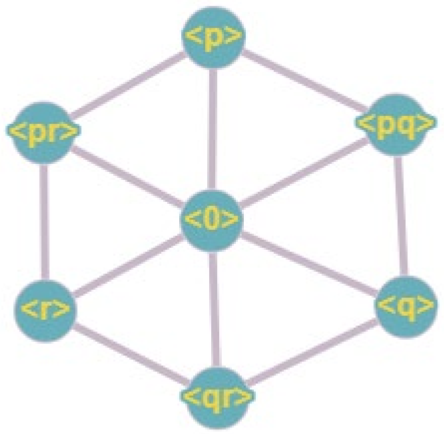
$< q >= \{0,3,6,9,12,15,18,21,24,27\}$

$< r >= \{0,5,10,15,20,25\}$

$< pq >= \{0,6,12,18,24\}$

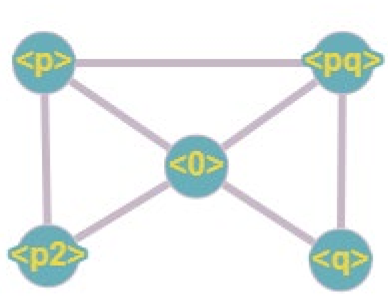
$< pr >= \{0,10,20\}$

$< qr >= \{0,15\}$ and Z_{30} , then $SOC(\Gamma) = 27\sqrt{2}$.



Theorem2.9: Let $\Gamma = \Gamma(\mathbb{Z}_n)$, where $n = p^2\, q$ where p, q are distict primes, be an inclusion graph of subgroups of $\mathbb{Z}_n$. Then $(\Gamma) = 2\sqrt{2} + 2\sqrt{13}$.

Proof: In Γ, the non-adjacent vertices are $< p^2 >$ and $< q >$, $< p >$ and $< q >$, and $< p^2 >$ and $< pq >$. Thus, $SOC(\Gamma) = \sqrt{2^2 + 2^2} + 2\sqrt{2^2 + 3^2} = 2\sqrt{2} + 2\sqrt{13}$.



Example 2.10: if $n = 2^2 * 3 = 12$, then the non-trivial subgroups of $\mathbb{Z}_{12}$ are

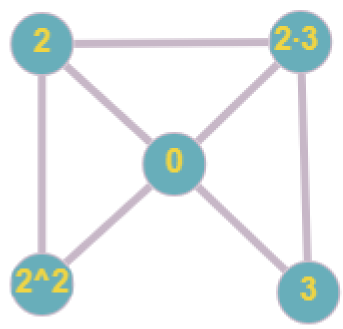
$< 2 >$={0,2,4,6,8,10}

$< 3 >$={0,3,6,9}

$< 2^2 >=$ {0,4,8}

$< 2 * 3 >$={0,6} and $\mathbb{Z}_{12}$.

Therefore, $SOC(\Gamma) = 2\sqrt{2} + 2\sqrt{13}$.



CHAPTER THREE

The Sombor Coindex of the Inclusion Graph of D_{2n}

In this chapter we compute the Sombor Coindex of the inclusion graph of the dihedral group D_{2n}

Recall that the dihedral group D_{2n} of order 2n is defined by the presentation

$D_{2n} =< r,s \mid r^n = s^2 = 1, srs^{-1} = r^{-1} >$.

Theorem 3.1: Let $\Gamma = \Gamma(D_{2n})$, $n = p$, p is a prime, be an inclusion graph of D_{2n}. Then $\Gamma = K_{1,n+1}$.

proof: The vertices of Γ are {1}, $< r >$ and $< sr^i >$, $i = 1,2, \dots, n$.

One can see that only {1} is a subset of all other subgroup of {1} Thus,

$\Gamma = K_{1,n+1}$.

Theorem 3.2: Let $\Gamma = \Gamma(D_{2n}), n = p$, p is prime, be an inclusion graph of D_{2n}. Then $SOC(\Gamma) = \frac{n(n+1)}{2}\sqrt{2}$.

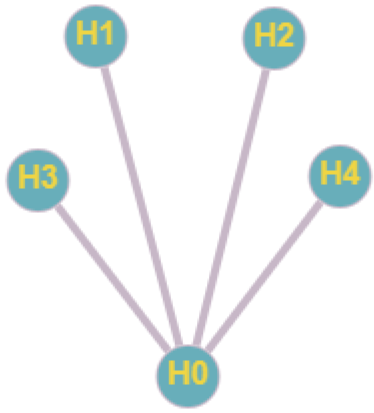
proof: There are $\frac{n(n+1)}{2}$ off-edges each has both end-vertices of degree one.

Thus, $SOC(\Gamma) = \frac{n(n+1)}{2}\sqrt{1^2 + 1^2} = \frac{n(n+1)}{2}\sqrt{2}$.

Example 3.3: Let $n = 3$

$D_{2n} = D_{2*3} = D_6 = \{1, r, r^2, s, sr, sr^2\}$

1- $H^3 =< r^{\frac{3}{3}} >=< r >=\{1, r, r^2\}$
2- $H_1 = \{1, s\}$
   $H_2 = \{1, sr\}$
   $H_3 = \{1, sr^2\}$
   Then $SOC(\Gamma) = 6\sqrt{2}$.



Theorrem 3.4: Let $\Gamma = \Gamma(D_{2n})$, n=p^2, p is a prime, be an inclusion graph of D_{2n}. Then

$$deg(v) = \begin{cases} p^2 + p + 2 & if \quad v = \{1\} \\ 2 & if \quad v = H_i, i = 1,2, \dots, p^2 \; or \; v = H^{p^2} \\ p + 2 & if \quad v = H_p^i, i = 1,2, \dots, p \; \; or \; \; v = H^p \end{cases}$$

proof: The proper subgroups of D_{2n}, where n=p^2 , are $\{1\}$, H^p, H^{p^2}, $H_i\,; i = 1, \dots, p^2$, and $H_p^i\,; i = 1, \dots, p$. H is clear that $\{1\}$ is a subset of each other subgroups, and then $deg(\{1\}) = p^2 + p + 2$. The subsets of H_p^i are $\{1\}$, H^p and H_i, H_{i+p} , ... , $H_{i+(p-1)p}$. Thus, $deg(H_p^i) = p + 2$. Since $H^p \subseteq H^{p^2}$, then $deg(H^{p^2}) = 2$. Finally, $deg(H_i) = 2$.

Theorrem 3.5: Let $\Gamma=\Gamma(D_{2n})$, $n = p^2$, p is a prime be an inclusion graph of D_{2n}. Then $|E(\Gamma)| = 2P^2 + 2P + 3$.

Proof: $|E(\Gamma)| = \frac{(p^2+p+2)+2(p^2+1)+(p+2)(p+1)}{2}$

$= \frac{p^2 + p + 2 + 2p^2 + 2 + p^2 + p + 2p + 2}{2}$

$= \frac{4p^2 + 4p + 6}{2}$

$= 2p^2 + 2p + 3.$

Theorrem 3.6: Let $\Gamma=\Gamma(D_{2n})$, n=p^2, p is a prime, be an inclusion graph of D_{2n}. Then $SOC(\Gamma) = (p^3 + p)\sqrt{p^2 + 2p + 6} + (\frac{2p^4+p^3+3p^2-2p}{2})\sqrt{2}$.

Proof: There are p off-edges each has one end-vertex of degree 2 and the other end-vertex of degree $p + 3$ there are p^2 off-edges each has one end-vertex of degree $p + 2$ and the other end-vertex of degree 2, there are p^2 off-edges each has both end-vertices of degree 2, there are $\frac{p(p-1)}{2}$ off-edges each has both end-vertices of degree $p + 2$, there are $p(p^2 - p)$ off-edges each has one end-vertex of degree $p + 2$ and the other end-vertex of degree 2, and there are $\frac{p^2(p^2-1)}{2}$ off-edges each has both end-vertices of degree 2. Thus,

$SOC(\Gamma) = P\sqrt{2^2 + (P + 2)^2} + p^2\sqrt{(p + 2)^2 + 2^2} + p^2\sqrt{2^2 + 2^2} +$
$\frac{p(p-1)}{2}\sqrt{(p + 2)^2 + (p + 2)^2} + p(p^2 - p)\sqrt{(p + 2)^2 + 2} +$
$\frac{p^2(p^2-1)}{2}\sqrt{2^2 + 2^2}\sqrt{2}$

$$= (p + p^2 + p^3 - p^2)\sqrt{p^2 + 2p + 4 + 2} + \left(p^2 + \frac{p^2(p^2 - 1)}{2}\right) 2 * \sqrt{2} + \frac{p(p-1)(p+2)}{2}\sqrt{2}$$

$$= (p^3 + p)\sqrt{p^2 + 2p + 6} + (p^4 + p^2)\sqrt{2} + \frac{p(p-1)(p+2)}{2}\sqrt{2}$$

$$= (p^3 + p)\sqrt{p^2 + 2p + 6} + (p^4 + p^2 + \frac{p(p-1)(p+2)}{2})\sqrt{2}$$

$$= (p^3 + p)\sqrt{p^2 + 2p + 6} + (\frac{2p^4 + p^3 + 3p^2 - 2p}{2})\sqrt{2}\ .$$

Example 3.7: Let $n = 3^3 = 9$

1- $H^k =< r^{\frac{n}{k}} >$, k is divisor of n and k≠1 of order k
   $\{1\} = H^3 =< r^{\frac{9}{3}} >=< r^3 >= \{1, r^3, r^6\}$
   $\{2\} = H^9 =< r^{\frac{9}{9}} >=< r >= \{1, r, r^2, r^3, r^4, r^5, r^6, r^7, r^8\}$

2- $H_i =< sr^i >$; i=1,2, …,n
   $H_1 =< sr >= \{1, sr\}$
   $H_2 =< sr^2 >= \{1, sr^2\}$
   $H_3 =< sr^3 >= \{1, sr^3\}$
   $H_4 =< sr^4 >= \{1, sr^4\}$
   $H_5 =< sr^5 >= \{1, sr^5\}$
   $H_6 =< sr^6 >= \{1, sr^6\}$
   $H_7 =< sr^7 >= \{1, sr^7\}$
   $H_8 =< sr^8 >= \{1, sr^8\}$
   $H_9 =< sr^9 >=< s >= \{1, s\}$

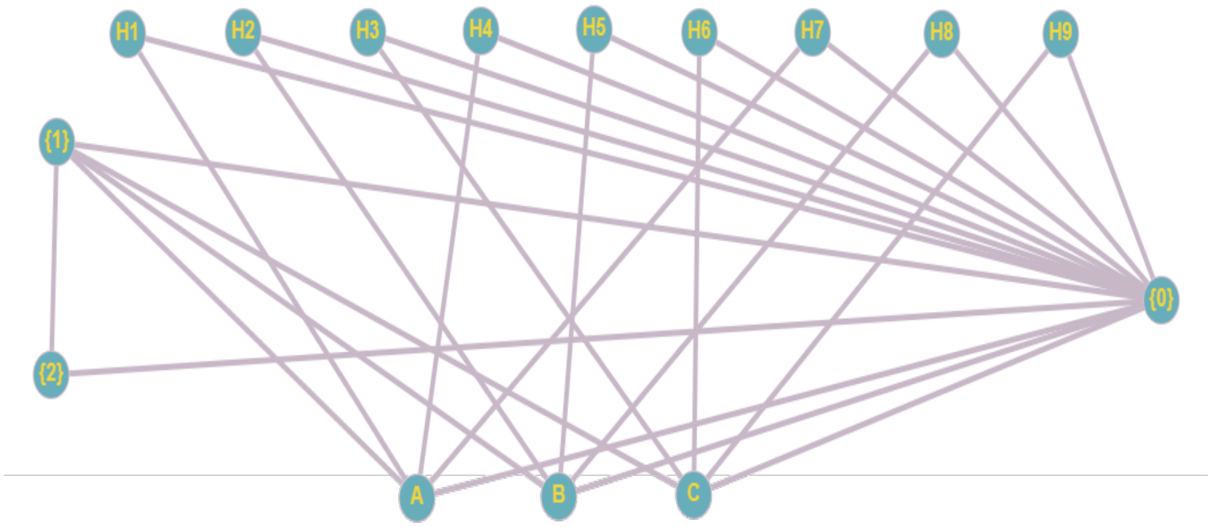
3- $H_k^i =< r^{\frac{n}{k}}, sr^i >$ of order 2k, k a divisor of n ,k≠1,n ,i=1,2,…,$\frac{n}{k}$
   K=3 ,i=1,2,3

   $A = H_3^1 =< r^3, sr >= \{1, r^3, r^6, sr, sr^4, sr^7\}$
   $B = H_3^2 =< r^3, sr^2 >= \{1, r^3, r^6, sr^2, sr^5, sr^8\}$
   $C = H_3^3 =< r^3, sr^3 >= \{1, r^3, r^6, sr^3, sr^6, s\}$

Then,

$$SOC(\Gamma) = (3^3 + 3)\sqrt{3^2 + 2 * 3 + 6} + (\frac{2*3^4+3^3+3*3^2-2*3}{2})\sqrt{2}$$

$$= 30\sqrt{21} + 105\sqrt{2}\ .$$



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پوختە

ئەم پرۆژەیە پەیوەندی بە نەگۆڕێکی گرافیەوە هەیە کەبەم دواییە سەلمێندرا،ئەویش
کە بەم شێوەیە پێناسە دەکرێت SomborCoindex,

کە $SOC(\Gamma) = \sum_{uv \in E(\Gamma)} \sqrt{\deg(\mathrm{u})^2 + \deg(v)^2}$

ئێمە لێرەدا $d_r(u)$ is the degree of vertex u in Γ

The Sombor Coindex of inclusion graph of some finite groups

دیاری دەکەین $(\mathbb{Z}_n, D_{2n})$

الخلاصة

الذي تم أدخاله مؤخرا ، وهو مؤشر Γ يهتم هذا المشرع بالرسم البياتى Sombor Coindex

يتم تعريفه على أنة

$d_r(u)$, حيث $SOC(\Gamma) = \sum_{uv \in E(\Gamma)} \sqrt{deg(u)^2 + deg(v)^2}$

يكون درجة الرأس فيΓتحدد مؤشر

The Sombor Coindex of inclusion graph of some finite groups

$(\mathbb{Z}_n, D_{2n})$