زانكۆی سەلاحەدین - هەولێر
Salahaddin University-Erbil

# The Sombor index of the conjugacy class graph of some finite groups

**Research Project**

**Submitted to the department of (Mathematics) in partial fulfillment of the requirements for the degree of B.Sc. in (MATHEMATICS)**

***By:***

***Helin Ayub Abdulrahim***

***Supervised by:***

***Dr. Sanhan Khasraw***

**April– 2024**

# Certification of the Supervisors

Mathematics / College of Education / Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

Signature:

Supervisor: Dr. Sanhan Khasraw

Scientific grade: Assist. Professor

Date:

In view of the available recommendations, I forward this work for debate by the examining committee.

Signature:

Name: Dr. Rashad Rashid Haji

Scientific grade: Assist. Professor

Chairman of the Mathematics Department

Date:

# ACKNOWLEDGEMENTS

I would like to thank my supervisor, Dr. Sanhan Khasraw, for bringing the weight of his considerable experience and knowledge to this project. His high standards have made me better at what I do.

I sincerely thank Dr. Rashad Rashid haji, the head of the department of Mathematics, College of Education, for all the facilities provided to us in the future pursuit of this project.

I acknowledge our deep sense of gratitude to my loving parents for being a constant Source of inspiration and motivation.

# ABSTRACT

This project is concerned with a recently introduced graph invariant, namely the Sombor index, which is defined by $\boldsymbol{SO(\Gamma)} = \sum_{u,v \in E(\Gamma)} \sqrt{d_\Gamma(u)^2 + d_\Gamma(v)^2}$, where $\boldsymbol{d_\Gamma(u)}$ is the degree of vertex $u$ in the graph $\Gamma$. We determine the Sombor index of the conjugacy class graph of some finite groups such as $\boldsymbol{D_{2n}}$, $\boldsymbol{SD_{8n}}$ and $\boldsymbol{T_{4n}}$.

# TABLE OF CONTENTS

# LIST OF FIGURES

# INTRODUCTION

Given a finite group G, let Γ(G) be the simple undirected graph whose vertices are the distinct sizes of noncentral conjugacy classes of G, two of them being adjacent if and only if they are not coprime numbers. The interplay between certain properties of this graph and the group structure of G has been widely studied in the past decades, and it is nowadays a classical topic in finite group theory. The present note is a contribution in this direction.

The study of the mathematical aspects of the degree-based graph invariants (also known as topological indices) is considered to be one of the very active research areas within the field of chemical graph theory. Recently, the mathematical chemist Ivan Gutman, one of the pioneers of chemical graph theory, proposed a geometric approach to interpret degree-based graph invariants and based on this approach, he devised three new graph invariants; namely the Sombor index, the reduced Sombor index and the average Sombor index. The Sombor index, being the simplest one among the aforementioned three invariants, has attracted a significant attention from researchers within a very short time (e.g[2,7,8,9,15]) (Igor Milovanovic 2021). The Sombor index has received a lot of attention within mathematics and chemistry. For example, the chemical applicability of the Sombor index, especially the predictive and discriminative potentials.

This project consists of four chapters. In the first chapter, we give some necessary backgrounds about groups and graphs. In the second chapter, we find the Sombor index of the conjugacy class graph of the dihedral group $\boldsymbol{D_{2n}}$.In the third chapter, we find the Sombor index of the conjugacy class graph of the generalized quaternion group $\mathbf{SD}_{\boldsymbol{8n}}$ in the fourth chapter, we find the Sombor index of the conjugacy class graph of the generalized quaternion group $\mathbf{T_{4n}}$.

# CHAPTER ONE

## Background

**Definition 1.1 (Foote, 2003):** A group is a set G paired with a binary operation ($*$), that satisfies:

I: closure: If $\boldsymbol{a, b \in G \implies a * b \in G}$.

II: Associativity: $\boldsymbol{a, b, c \in G \implies (a * b) * c = a * (b * c)}$.

III: Identity: $\exists\ \boldsymbol{e \in G}$ such that $\boldsymbol{e * g = g * e, \forall g \in G}$.

IV: Inverse: $\exists \boldsymbol{g^{-1} \in G}$ such that $\boldsymbol{g * g^{-1} = g^{-1} * g = e\ \forall g \in G}$.

**Definition 1.2 (LIEBECK, 2001) :** Let $\boldsymbol{x, y \in G}$.We say that *x* is conjugate to *y* in *G* if $\boldsymbol{y = g^{-1}xg}$ for some $\boldsymbol{g \in G}$.

**Definition 1.3 (LIEBECK, 2001)**: Let $\boldsymbol{G = D_{2n}}$ the dihedral group of order 2n. Then $\boldsymbol{G = \langle r, s : r^n = s^2 = 1, s^{-1}rs = r^{-1} \rangle}$.

**Definition 1.4 (Ashrafi, 2020):** Let $\boldsymbol{G}$ =$SD_{8n}$, the semi-dihedral group of order 8n. Then $\boldsymbol{G = \langle r, s : r^{4n} = s^2 = 1, srs = r^{2n-1} \rangle}$.

**Definition 1.5 (LIEBECK, 2001):** Let $\boldsymbol{G = T_{4n}}$, the dicyclic group of order 4n. Then $\boldsymbol{G = \langle r, s : r^{2n} = 1, r^n = s^2, s^{-1}r\, s = r^{-1} \rangle}$.

**Definition 1.6 (GOSSETT, 2009):** Let a and b integers that are not both 0. The greatest common divisor (gcd) of a and b is positive integer d such that

**.** d | a and d | b .

**.** If c divides both a and b, then c | d.

The greatest common divisor of a and b is denoted by gcd(a,b). An alternative notation is (a,b).

**Definition 1.7 (Nuwairan, 2023):** The center of a group $(\boldsymbol{G},*)$, denoted by $\boldsymbol{cent}(\boldsymbol{G})$ or $\boldsymbol{Z}(\boldsymbol{G})$ is the set $\boldsymbol{cent}(\boldsymbol{G}) = \{\boldsymbol{a} \in \boldsymbol{G} : \boldsymbol{a} * \boldsymbol{g} = \boldsymbol{g} * \boldsymbol{a}\ \boldsymbol{for\ all}\ \boldsymbol{g} \in \boldsymbol{G}\}$.

**Definition 1.8 (Deo, 1974):** A Graph $\Gamma = (V, E)$ consists of a set of objects $V = \{v1, v2, \dots\}$ called vertices, and another set $E = \{e1, e2, \dots\}$, whose elements are called edges, such that each edge $\boldsymbol{e_k}$ is identified with an unordered pair $(\boldsymbol{v_i}, \boldsymbol{v_j})$ of vertices.

**Definition 1.9 (Ray, 2013):** A complete graph is a simple graph in which each pair of distinct vertices is joined by an edge. In other words, a simple graph in which there exists an edge between every pair of vertices is called a complete graph.

**Definition 1.10 (Foote, 2003):** A group (G, *) is said to be finite if G is a finite set.

**Definition 1.11 (Gutman, 2021):** Let **Γ** be a graph. Then the sombor index of **Γ**, denoted by SO (**Γ**), is defined by

$$\boldsymbol{SO}(\boldsymbol{\Gamma}) = \sum_{\boldsymbol{u},\boldsymbol{v} \in \boldsymbol{E}(\boldsymbol{\Gamma})} \sqrt{\boldsymbol{d}_\Gamma(\boldsymbol{u})^2 + \boldsymbol{d}_\Gamma(\boldsymbol{v})^2}\ .$$

**Theorem 1.12 (LIEBECK, 2001):** The dihedral group $\boldsymbol{D_{2n}}$ (n odd) has precisely $\frac{1}{2}(\boldsymbol{n}+\boldsymbol{3})$ conjugacy classes: $\{\boldsymbol{1}\}, \{\boldsymbol{r}, \boldsymbol{r}^{-1}\}, \dots, \left\{\boldsymbol{r}^{(\boldsymbol{n}-1)/2}, \boldsymbol{r}^{-(\boldsymbol{n}-1)/2}\right\}, \{\boldsymbol{s}, \boldsymbol{rs}, \dots, \boldsymbol{r}^{\boldsymbol{n}-1}\boldsymbol{b}\}$.

**Theorem 1.13 (LIEBECK, 2001):** The dihedral group $\boldsymbol{D_{2n}}$ (n even, n=2m) has precisely $(\boldsymbol{m}+\boldsymbol{3})$ conjugacy classes: $\{\boldsymbol{1}\}, \{\boldsymbol{r}^{\boldsymbol{m}}\}, \{\boldsymbol{r}, \boldsymbol{r}^{-1}\}, \dots, \{\boldsymbol{r}^{\boldsymbol{m}-1}, \boldsymbol{r}^{-\boldsymbol{m}+1}\}$.

# CHAPTHER TWO

## The sombor index of the conjugacy class graph of the dihedral group $D_{2n}$

In this chapter we compute the sombor index graph of conjugacy class graph of dihedral group $D_{2n}$.

Recall that the dihedral group $D_{2n}$ of order 2n is defined by the presentation

$D_{2n} = \langle r, s{:}r^n = s^2 = 1, s^{-1}rs = r^{-1}\rangle$ and

$$Z(D_{2n}) = \begin{cases} 1 & \text{if } \text{ n is odd} \\ \{1, r^{\frac{n}{2}}\} & \text{if } \text{ n is even} \end{cases}$$

**Theorem 2.1:** Let $\Gamma = \Gamma(D_{2n})$be a conjugacy class of $D_{2n}$. If n is odd. Then

$\Gamma = K_{\frac{1}{2}(n-1)} \cup K_1$.

**Proof:** By Theorem 1.12, $D_{2n}$ has $\frac{n+1}{2}$ noncentral conjugacy classes such that $\frac{n-1}{2}$ conjugacy classes have size 2 and only one conjugacy class is of size $n$, then

$\Gamma = K_{\frac{(n-1)}{2}} \cup K_1$.

**Theorem 2.2:** Let $\Gamma = \Gamma(D_{2n})$ be a conjugacy class graph of $D_{2n}$. If n is odd. Then,

$SO(\Gamma) = \frac{1}{16}(n-1)(n-3)^2\sqrt{2}$.

**Proof:** By Theorem2.1, $\Gamma = K_{\frac{n-1}{2}} \cup K_1$.

There are $\frac{\frac{(n-1)}{2}\cdot\left(\frac{(n-1)}{2}-1\right)}{2} = \frac{(n-1)(n-3)}{2}$ edges each has both end−vertices of degree $\frac{n-1}{2} - 1 = \frac{n-3}{2}$. Thus,

$$SO(\Gamma) = \frac{(n-1)(n-3)}{8}\sqrt{(\frac{n-3}{2})^2 + (\frac{n-3}{2})^2}$$

$$= \frac{(n-1)(n-3)^2}{16}\sqrt{2}$$

**Example 2.3**: If $\boldsymbol{n} = \mathbf{7}$, then the vertex set is

$$\boldsymbol{c_1} = \{\boldsymbol{b}, \boldsymbol{ab}, \boldsymbol{a^2b}, \boldsymbol{a^4b}, \boldsymbol{a^5b}, \boldsymbol{a^6b}\} \qquad \boldsymbol{c_2} = \{\boldsymbol{a}, \boldsymbol{a^6}\}$$

$$\boldsymbol{c_3} = \{\boldsymbol{a^2}, \boldsymbol{a^5}\} \qquad \boldsymbol{c_2} = \{\boldsymbol{a^3}, \boldsymbol{a^4}\}$$

**Solution:** $\boldsymbol{SO}\ (\boldsymbol{D_{2\cdot 7}})= \frac{\mathbf{1}}{\mathbf{16}}\ (\mathbf{7}-\mathbf{1})(\mathbf{7}-\mathbf{3})^2\sqrt{\mathbf{2}}.$

$=6\sqrt{\mathbf{2}}$ .

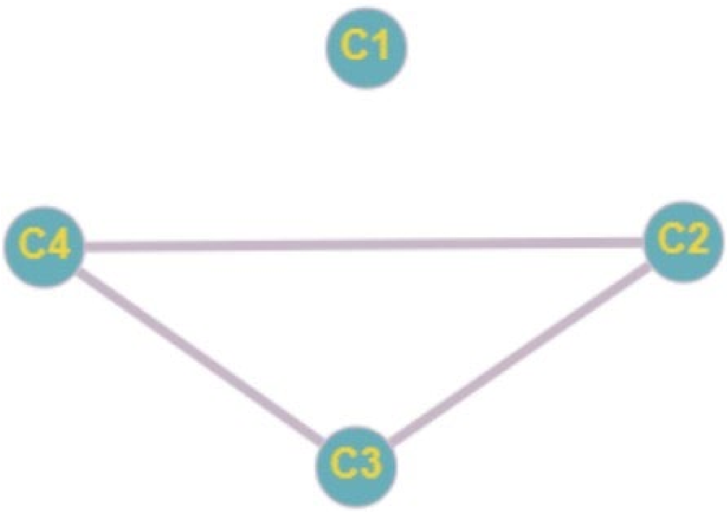


**Figure 1: conjugacy class graph $\boldsymbol{D_{2\cdot 7}}$**

**Theorem 2.4:** Let $\boldsymbol{\Gamma} = \Gamma(\boldsymbol{D_{2n}})$ be a conjugacy class graph of $D_{2n}$. If $\boldsymbol{n = 2m}$, then

$$\Gamma = \begin{cases} K_{m+1} & if\ 2|m \\ K_{m-1} \cup K_2 & if\ 2 \nmid m \end{cases}$$

**Proof:** There are two cases for $\boldsymbol{m}$ to consider.

**Case1**: If $\boldsymbol{n = 2m}$, $\boldsymbol{m}$ is even has $(\boldsymbol{m+1})$ noncentral conjugacy classes such that $(\boldsymbol{m-1})$ conjugacy classes have size and only two conjugacy class has size m. Then $\Gamma = \boldsymbol{K_{m+1}}$.

**Case2:** If $\boldsymbol{n = 2m}$, $\boldsymbol{m}$ is odd, there are $(\boldsymbol{m+1})$ noncentral conjugacy classes of $\boldsymbol{D_{2n}}$ for which $\boldsymbol{m-1}$ classes size 2 and two classes have size $\boldsymbol{m}$.therefore,

$\Gamma = \boldsymbol{K_{m-1}} \cup \boldsymbol{K_2}$.

**Theorem 2.5:** Let $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}(\boldsymbol{D_{2m}})$ be a conjugacy class graph of $D_{2n}$. If $\boldsymbol{n = 2m}$, then

$$SO\ (\Gamma) = \begin{cases} \frac{m^2(m+1)}{2}\sqrt{2} & if\quad 2|m \\ (1+\frac{(m-1)(m-2)^2}{2})\sqrt{2} & if\quad 2 \nmid m \end{cases}$$

**Proof:** There are two cases for m to consider.

**Case1:** If 2|m, then there are $\frac{(m+1)(m+1-1)}{2} = \frac{m(m+1)}{2}$ edges each has both end−vertices of degree m. Thus,

$$SO\ (\Gamma) = \frac{(m+1)(m)}{2}\sqrt{m^2+m^2}$$
$$= \frac{m^2(m+1)}{2}\sqrt{2}\ .$$

**Case2:** If $2 \nmid \boldsymbol{m}$, then there are $\frac{(m-1)(m-1-1)}{2} = \frac{(m-1)(m-2)}{2}$ edges each

has both **end − vertices** of degree $\boldsymbol{m-2}$ and one edge with both end− vertices

of degree one. Thus,

$$SO\ (\Gamma) = \frac{(m-1)(m-2)}{2}\sqrt{(m-2)^2+(m-2)^2} + \sqrt{1^2+1^2}$$
$$= \sqrt{2} + \frac{(m-1)(m-2)}{2}(m-2)\sqrt{2}$$
$$= (1+\frac{(m-1)(m-2)^2}{2})\sqrt{2}\ .$$

**Example 2.6:** If $\boldsymbol{n} = \mathbf{6}$, then $\boldsymbol{D_{2\cdot 6}} = \{\mathbf{1}, \boldsymbol{r}, \boldsymbol{r^2}, \boldsymbol{r^3}, \boldsymbol{r^4}, \boldsymbol{r^5}, \boldsymbol{s}, \boldsymbol{sr}, \boldsymbol{sr^2}, \boldsymbol{sr^3}, \boldsymbol{sr^4}, \boldsymbol{sr^5}\}$

$\boldsymbol{c_1}=\{\boldsymbol{r}, \boldsymbol{r^5}\}$ $\boldsymbol{c_2}=\{\boldsymbol{r^2}, \boldsymbol{r^4}\}$ $\boldsymbol{c_3}= \{\boldsymbol{s}, \boldsymbol{sr^2}, \boldsymbol{sr^4}\}$ $\boldsymbol{c_4}= \{\boldsymbol{sr}, \boldsymbol{sr^3}, \boldsymbol{sr^5}\}$

**Solution:** $\boldsymbol{SO}\ (\boldsymbol{D_{2\cdot 6}}) = \left(\mathbf{1} + \frac{(\mathbf{3}-\mathbf{1})(\mathbf{3}-\mathbf{2})^2}{\mathbf{2}}\right)\sqrt{\mathbf{2}}$

$=1+\sqrt{\mathbf{2}}$

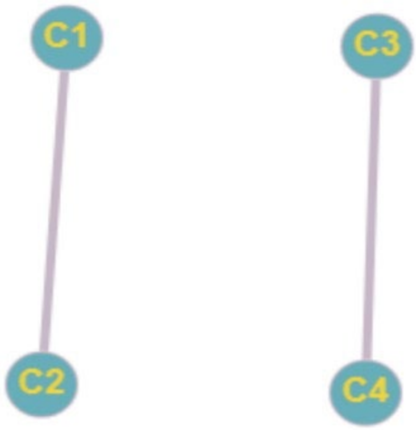


Figure 2: conjugacy class graph $\boldsymbol{D_{2\cdot 6}}$

**Example 2.7:** If $n = 8$, then.

$D_{2\cdot 8}$=$\{1, r, r^2, r^3, r^4, r^5, r^6, r^7, s, sr, sr^2, sr^3, sr^4, sr^5, sr^6, sr^7\}$

$c_1 = \{r, r^7\}$ $c_2 = \{r^2, r^6\}$ $c_3 = \{r^3, r^5\}$ $c_4$ =$\{s, sr^2, sr^4, sr^6\}$
$c_5 = \{sr, sr^3, sr^5, sr^7\}$

**Solution:** $SO(D_{2\cdot 8}) = (\frac{(4)^2(4+1)}{2})\sqrt{2}$

$= 40\sqrt{2}$

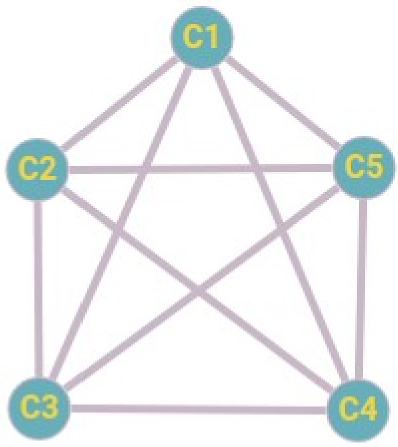


Figure 3:conjugacy class graph $D_{2\cdot 10}$

# CHAPTER THREE

## The Sombor index of the conjugacy class graph of quasi dihedral group $SD_{8n}$

In this chapter we compute the Sombor index of conjugacy class graph of quasi dihedral group $SD_{8n}$ .

Recall that the quasi dihedral group $SD_{8n}$ of order $8n$ is defined by the presentation $SD_{8n} = \langle r, s: r^{4n} = s^2 = 1, srs = r^{2n-1} \rangle$ and

$$Z(SD_{8n}) = \begin{cases} \{1, r^{2n}\} & if \quad n \ is \ even \\ \{1, r^n, r^{2n}, r^{3n}\} & if \quad n \ is \ odd \end{cases}$$

**Theorem 3.1**: Let $\Gamma = \Gamma(SD_{8n})$ be a conjugacy class graph of $SD_{8n}$. Then

$$\Gamma = \begin{cases} K_{2n+1} & if \quad n \, is \, even \\ K_{2n-2} \cup K_4 & if \quad n \, is \, odd \end{cases}$$

**Proof:** By **(Ashrafi, 2020)**, there are two cases to consider.

**Case1:** When n is even. There are $(2n-1)$ noncentral conjugacy classes of size 2 and two noncentral conjugacy classes of size $2n$. It is clear that size of any two conjugacy Classes are not coprime.

Thus, $\Gamma = K_{2n+1}$.

**Case2:** When n is odd. There are $(2n-2)$ noncentral conjugacy Classes of size 2 and four noncentral conjugacy classes of size $n$.

Thus, $\Gamma = K_{2n-2} \cup K_4$ .

**Theorem 3.2:** Let $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}(\boldsymbol{SD_{8n}})$ be a conjugacy class graph of $\boldsymbol{SD_{8n}}$. Then

$$|\boldsymbol{E}(\boldsymbol{\Gamma})| = \begin{cases} \boldsymbol{n(2n+1)} & \boldsymbol{if} \quad \boldsymbol{n \text{ is even}} \\ \boldsymbol{2n^2 - 5n + 9} & \boldsymbol{if} \quad \boldsymbol{n \text{ is odd}} \end{cases}$$

**Proof:** By Theorem 3.1, there are two cases to consider.

**Case1:** When $\boldsymbol{n}$ is even. There are $(\boldsymbol{2n-1})$ noncentral conjugacy classes of size 2 and two noncentral conjugacy classes of size $\boldsymbol{2n}$. It is clear that size of any two conjugacy Classes are not coprime.

Thus, $\boldsymbol{\Gamma} = \boldsymbol{K_{2n+1}}$.

**Case2:** When $\boldsymbol{n}$ is odd. There are $(\boldsymbol{n-1})$ noncentral conjugacy Classes of size 2 and four noncentral conjugacy classes of size $\boldsymbol{n}$.

Thus, $\boldsymbol{\Gamma} = \boldsymbol{K_{2n-1}} \cup \boldsymbol{K_4}$ .

**Theorem 3.3:** Let $\boldsymbol{\Gamma} = \Gamma(\boldsymbol{SD_{8n}})$ be a conjugacy class graph of $\boldsymbol{SD_{8n}}$. Then

$$\text{SO}(\boldsymbol{\Gamma}) = \begin{cases} \boldsymbol{2n^2(2n+1)\sqrt{2}} & \boldsymbol{if} \quad \boldsymbol{n \text{ is even}} \\ \boldsymbol{[(n-1)(2n-3)^2 + 18]\sqrt{2}} & \boldsymbol{if} \quad \boldsymbol{n \text{ is odd}} \end{cases}$$

**Proof:** There are two cases to consider.

**Case1:** When $\boldsymbol{n}$ is even, then by Theorem 3.1, if $\boldsymbol{n(2n+1)}$ edges with end-vertices order $\boldsymbol{2n}$. Then

$$\boldsymbol{SO(\Gamma)} = \boldsymbol{n(2n+1)\sqrt{(2n)^2 + (2n)^2}}$$
$$= \boldsymbol{n(2n+1) \cdot (2n)\sqrt{2}}$$
$$= \boldsymbol{2n^2(2n+1)\sqrt{2}}$$

**Case2:** When $\boldsymbol{n}$ is odd. There are $(\boldsymbol{n-1})\ (\boldsymbol{2n-3})$ edges with end-vertex of degree $\boldsymbol{2n-3}$ and 6 edges with end-vertex of degree 3. Then

$$\boldsymbol{SO(\Gamma)} = \boldsymbol{(n-1)\ (2n-3) \cdot \sqrt{(2n-3)^2 + (2n-3)^2}} + 6 \cdot \boldsymbol{\sqrt{3^2 + 3^2}}$$
$$= \boldsymbol{(n-1)\ (2n-3) \cdot (2n-3)\sqrt{2} + 18\sqrt{2}}$$
$$= \boldsymbol{[(n-1)(2n-3)^2 + 18]\sqrt{2}}$$

**Example 3.4:** If n=3, then $\boldsymbol{SD_{8\cdot 3}} =$

$\{\boldsymbol{1, r, r^2, r^3, r^4, r^5, r^6, r^7, r^8, r^9, r^{10}, r^{11}, s, sr, sr^2, sr^3, sr^4, sr^5, sr^6, sr^7, sr^8, sr^9, sr^{10}, sr^{11}}\}$.

$\boldsymbol{c_1 = \{r, r^{11}\}}$ $\boldsymbol{c_2 = \{r^2, r^{10}\}}$ $\boldsymbol{c_3 = \{r^4, r^8\}}$ $\boldsymbol{c_4 = \{r^5, r^7\}}$ $\boldsymbol{c_5 = \{s, sr^2, sr^4\}}$ $\boldsymbol{c_6 = \{sr, sr^3, sr^5\}}$ $\boldsymbol{c_7 = \{sr^6, sr^8, sr^{10}\}}$

$\boldsymbol{c_8 = \{sr^7, sr^9, sr^{11}\}}$

**Solution:** $\boldsymbol{SO\ (\Gamma) = [(n-1)(2n-3)^2 + 18]\sqrt{2}}$

$= [(3-1)(\boldsymbol{2 \cdot 3 - 3})^2 + \boldsymbol{18}]\sqrt{\boldsymbol{2}}$

$=12\sqrt{\boldsymbol{2}}$

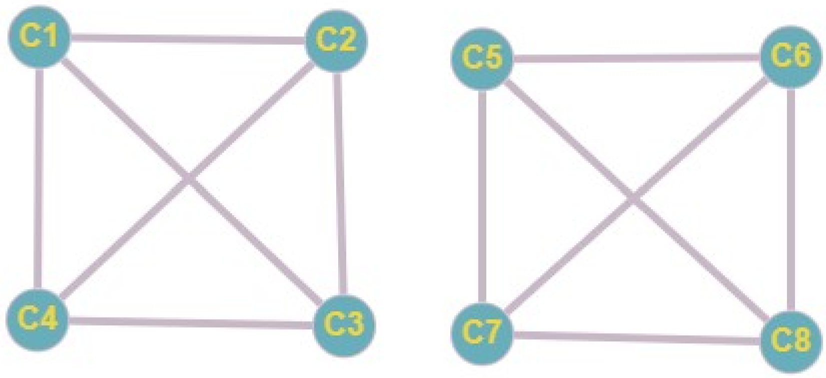


Figure 4: conjugacy class graph $\boldsymbol{SD_{8\cdot 3}}$

**Example 3.5:** If n=4, then. $SD_{8\cdot4} =$

$\{1, r, r^2, r^3, r^4, r^5, r^6, r^7, r^8, r^9, r^{10}, r^{11}, r^{12}, r^{13}, r^{14}, r^{15}, s, sr, sr^2, sr^3, sr^4, sr^5, sr^6, sr^7, sr^8, sr^9, sr^{10}, sr^{11}, sr^{12}, sr^{13}, sr^{14}, sr^{15}\}$.

$c_1 = \{r, r^{15}\}$ $c_2 = \{r^2, r^{14}\}$ $c_3 = \{r^3, r^{13}\}$ $c_4 = \{r^4, r^{12}\}$ $c_5 = \{r^5, r^{11}\}$ $c_6 = \{r^6, r^{10}\}$ $c_7 = \{r^7, r^9\}$

$c_8 = \{s, sr^2, sr^4, sr^6, sr^8, sr^{10}, sr^{12}, r^{14}\}$

$c_9 = \{sr, sr^3, sr^5, sr^7, sr^9, sr^{11}, sr^{13}, r^{15}\}$

**Solution:** $SO\ (\Gamma) = 2n^2(2n+1)\sqrt{2}$

$=2(4)^2(2 \cdot 4 + 1)\sqrt{2}$

$=288\sqrt{2}$

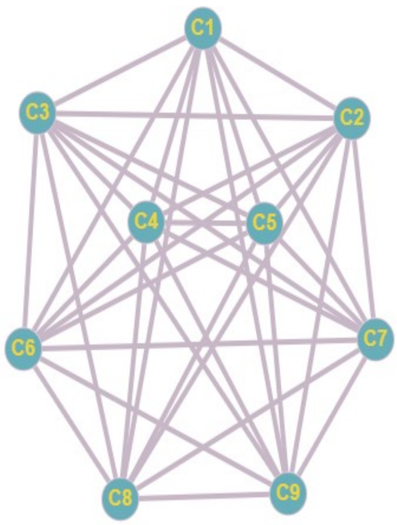


Figure 5: conjugacy class graph $SD_{8\cdot4}$

# CHAPTER FOUR

## The sombor index of the conjugacy class graph of the dicyclic group of $T_{4n}$

In this chapter we compute the Sombor index of conjugacy class graph of dicyclic group $T_{4n}$ .

Recall that the dicyclic group $T_{4n}$ of order $4n$ is defined by the presentation

$T_{4n}=\langle r, s : r^{2n} = 1, r^n = s^2, s^{-1} r\, s = r^{-1} \rangle$ and $Z(T_{4n}) = \{1, r^n\}$.

**Theorem 4.1:** Let $\Gamma = \Gamma(T_{4n})$ be a conjugacy class graph of $T_{4n}$. Then

$$\Gamma = \begin{cases} K_{n+1} & \text{if} \quad n \text{ is even} \\ K_{n-1} \cup K_2 & \text{if} \quad n \text{ is odd} \end{cases}$$

**Proof:** By **(LIEBECK, 2001)**, there are two cases to consider.

**Case1:** When $n$ is even. There are $(n-1)$ noncentral conjugacy

classes of size 2 and two noncentral conjugacy classes of size $n$. Then

the cardinality of any pair of these Classes are not coprime.

Thus, $\Gamma = K_{n+1}$.

**Case2:** When $n$ is odd. There are $(n-1)$ noncentral conjugacy

Classes of size 2 and two noncentral conjugacy classes of size $n$.

Thus, $\Gamma = K_{n-1} \cup K_2$ .

**Theorem 4.2:** Let $\Gamma = \Gamma(T_{4n})$ be a conjugacy class graph of $T_{4n}$. Then

$$SO(\Gamma) = \begin{cases} \frac{n^2(n+1)}{2}\sqrt{2} & if \quad \text{n is even} \\ (\frac{(n-1)(n-2)^2}{2}+1)\sqrt{2} & if \quad n \ is \ odd \end{cases}$$

**Proof:** By Theorem 4.1, there are two cases to consider.

**Case1:** When n is even, $\Gamma = K_{n+1}$. There are $\frac{(n+1)(n+1-1)}{2}$ $= \frac{n(n+1)}{2}$ edges each has both end-vertices of degree $n$. Then

$$SO(\Gamma) = \frac{(n+1)(n)}{2}\sqrt{n^2+n^2}$$

$$= \frac{n^2(n+1)}{2}\sqrt{2}.$$

**Case2:** When $n$ is odd, $\Gamma = k_{n-1} \cup K_2$. There are $\frac{(n-1)(n-1-1)}{2}$ $= \frac{(n-1)(n-2)}{2}$ edges each has both end-vertices of degree $n-2$ and one edge with both end-vertices of degree 1. Thus,

$$SO(\Gamma) = \frac{(n-1)(n-2)}{2}\sqrt{(n-2)^2+(n-2)^2} + \sqrt{1^2+1^2}$$

$$= \frac{(n-1)(n-2)^2}{2}\sqrt{2}+\sqrt{2}$$

$$= (\frac{(n-1)(n-2)^2}{2}+1)\sqrt{2}.$$

**Example 4.3:** If n=5, then. $T_{4\cdot 5}$ =

$\{1, r, r^2, r^3, r^4, r^5, r^6, r^7, r^8, r^9, s, sr, sr^2, sr^3, sr^4, sr^5, sr^6, sr^7, sr^8, sr^9\}$.

$c_1 = \{r, r^9\}$ $c_2 = \{r^2, r^8\}$ $c_3 = \{r^3, r^7\}$ $c_4 = \{r^4, r^6\}$
$c_5 = \{s, sr^2, sr^4, sr^6, sr^8\}$ $c_6 = \{sr, sr^3, sr^5, sr^7, sr^9\}$

**Solution:** $SO(\Gamma) = \frac{(n-1)(n-2)^2}{2}\sqrt{2} + \sqrt{2}$

$=\sqrt{2}\,(\frac{(n-1)(n-2)^2}{2} + 1)$

$=\sqrt{2}\,(\frac{(5-1)(5-2)^2}{2} + 1)$

$=19\sqrt{2}$

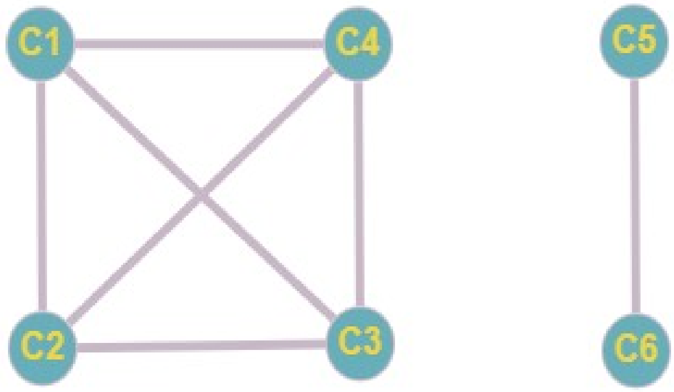


Figure 6: conjugacy class graph $T_{4\cdot 5}$

**Example 4.4:** If n=8, then. $T_{4\cdot 8}$ =

$\{1, r, r^2, r^3, r^4, r^5, r^6, r^7, r^8, r^9, r^{10}, r^{11}, r^{12}, r^{13}, r^{14}, r^{15}, s, sr, sr^2, sr^3, sr^4, sr^5, sr^6, sr^7, sr^8, sr^9, sr^{10}, sr^{11}, sr^{12}, sr^{13}, sr^{14}, sr^{15}\}$.

$c_1 = \{r, r^{15}\}$ $c_2 = \{r^2, r^{14}\}$ $c_3 = \{r^3, r^{13}\}$ $c_4 = \{r^4, r^{12}\}$ $c_5 = \{r^5, r^{11}\}$ $c_6 = \{r^6, r^{10}\}$ $c_7 = \{r^7, r^9\}$

$c_8 = \{s, sr^2, sr^4, sr^6, sr^8, sr^{10}, sr^{12}, r^{14}\}$

$c_9 = \{sr, sr^3, sr^5, sr^7, sr^9, sr^{11}, sr^{13}, r^{15}\}$

**Solution:** $SO(\Gamma) = \frac{n^2(n+1)}{2}\sqrt{2}$

$= \frac{8^2(8+1)}{2}\sqrt{2}$

$=288\sqrt{2}$

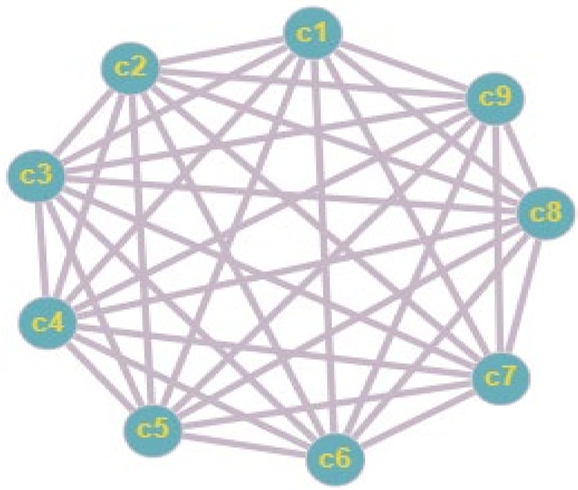


Figure 7: conjugacy class graph $T_{4\cdot 8}$

# References


1. Ashrafi, M. A. S. &. A. R., 2020. Commuting Conjugacy Class Graph of Finite CA-Groups. *khayyam journal of mathematics,* 6(1), pp. 108-118.
2. Deo, N., 1974. *Graph Theory with Applications to Engineering and Computer Science.* s.l.:Prentice-Hall.
3. Foote, D. S. D. &. R. M., 2003. *Abstract Algebra, 3rd Edition.* s.l.:John Wiley & Sons, Inc..
4. GOSSETT, E., 2009. *DISCRETE MATHMATICS WITH PROOF.* SECOND EDITION ed. s.l.:WILEY.
5. Gutman, I., 2021. Geometric Approach to Degree-based Topological Indecies:Sombor index. *MATCH-COMMUNICATIONS IN MATHEMATICAL AND IN COMPUTER CHEMISTRY,* pp. 11-16.
6. LIEBECK, G. J. A. M., 2001. *REPRESENTIONS AND CHAPACTERS OF GROUPS.* s.l.:CAMBRIDGE UNIVERSITY PRESS 1993, 2001.
7. Nuwairan, B. A. S. &. M. A., 2023. *A Gentle Introduction to Group Theory.* s.l.:Springer.
8. Ray, S. S., 2013. *Graph Theory with Algorithms and its Applications.* India: s.n.

# پوختە


ئەم پڕۆژەیە پەیوەندی بە نەگۆڕێکی گرافیەوە هەیە کە بەم دواییە ناسێندرا، ئەویش sumbor index .

ئەویش بەم شێوەیە پێناسە دەکرێت $SO(\Gamma)=\sum_{u,v\in E(\Gamma)}\sqrt{d_\Gamma(u)^2+d_\Gamma(v)^2}$ کە

$d_\Gamma$(u) is the degree of vertex $u$ and in $\Gamma$

ئێمە لێرەدا the Sombor index of conjugacy class graph of some finite groups

دیاری دەکەین. ($D_{2n}$, $SD_{8n}$, $T_{4n}$)


# الخلاصة


يهتم هذا المشروع بالرسم البياني ($\Gamma$) الذي تم إدخاله مؤخرا ، وهو مؤشر sombor.

يتم تعريفه على أنة $\sqrt{d_\Gamma(u)^2+d_\Gamma(v)^2}$ $SO(\Gamma)=\sum_{u,v\in E(\Gamma)}$ حيث

$d_\Gamma$(u) يكون درجة الرأس في $\Gamma$

نحدد مؤشر the Sombor index of conjugacy class graph of some finite groups

($D_{2n}$, $SD_{8n}$, $T_{4n}$)