Chapter one

Definition1.1: A vector space V over the field F is a set of objects which can be added and multiplied by elements of F in such way that the sum of elements of V is again an element of V, the product of an element of F by an element of V is in V and the following properties are satisfied

V1: For all $A,B,C\in V$, $A+(B+C)=(A+B)+C$

V2: For all $A,B\in V$, $A+B=B+A$

V3: There is an element in V denoted by O such that for all $A\in V$, $A+O=O+A=A$

V4: For all $A\in V$, there exist $-A\in V$ such that $A+(-A)=(-A)+A=O$

V5: For all $a,b\in F$ and $A\in V$, $a(bA)=(ab)A$

V6: For all $a\in F$ and $A,B\in V$, $a(A+B)=aA+aB$

V7: For all $a,b\in F$ and $A\in V$, $(a+b)A=aA+bA$

V8: For all $A\in V$, $1A=A$ (1 is identity of F)

The elements of V are called vectors. The elements of F are called scalars. The operation + is called vector addition. The operation **.** is called scalar multiplication. The vector O is called zero vector.

Example: Let R be the field of real numbers, $V=R^2=\{(x,y): x\in R \text{ and } y\in R\}$

$(x_1,y_1)=(x_2,y_2)$ iff $x_1=x_2$ and $y_1=y_2$

We define vector addition by $(x_1,y_1)+(x_2,y_2)= (x_1+x_2,y_1+y_2)$ [coordinate wise addition]

and scalar multiplication by $a(x_1,y_1)=(ax_1,ay_1)$ [coordinate wise scalar multiplication]

To show that $V=R^2$ is a vector space over R. Let $A,B,C\in R^2$, then $A=(x,y)$, $B=(w,z)$, $C=(g,h)$,

$A+(B+C)=(x,y)+((w,z)+(g,h))=(x,y)+(w+g,z+h)=(x+(w+g),y+(z+h))=((x+w)+g),(y+z)+h))=$
$(x+w,y+z)+(g,h)=((x,y)+(w,z))+(g,h)=(A+B)+C$.

$A+B=(x,y)+(w,z)=(x+w,y+z)=(w+x,z+y)=(w,z)+(x,y)=B+A$

Take $O=(0,0)$, then $A+O=(x,y)+(0,0)=(x+0,y+0)=(x,y)=A$

Take $-A=(-x,-y)\in R^2$, $A+(-A)=(x,y)+(-x,-y)=(0,0)=O$

$a[A+B]=a[(x,y)+(w,z)]=a(x+w,y+z)=(a(x+w),a(y+z))=(ax+aw,ay+az)=(ax,ay)+(aw,az)=$
$a(x,y)+a(w,z)=aA+aB$

$(a+b)A=(a+b)(x,y)=((a+b)x,(a+b)y)=(ax+bx,ay+by)=(ax,ay)+(bx,by)=a(x,y)+b(x,y)=aA+bA$

$(ab)A=(ab)(x,y)=((ab)x,(ab)y)=(abx,aby)=a(bx,by)=a[b(x,y)]=a[bA]$

$1A=1(x,y)=(1x,1y)=(x,y)=A$

Example1.1: Let F be any field and $n\in Z^+$, $F^n=\{(x_1,\dots,x_n): x_i\in F \text{ for } i=1,\dots,n\}$

$(x_1,\dots,x_n)=(y_1,\dots,y_n)$ iff $x_i=y_i$ for $i=1,\dots,n$

We define vector addition by $(x_1,\dots,x_n)+(y_1,\dots,y_n)= (x_1+y_1,\dots,x_i+y_i)$ [coordinate wise addition] and scalar multiplication by $a(x_1,\dots,x_n)= (ax_1,\dots,ax_n)$ [coordinate wise scalar multiplication]. F^n is a vector space over F.

Example1.2: Let F be any field and $m,n\in Z^+$, $M_{mn}(F)=\{A: A$ is m-by-n matrix in $F\}$.

Let + be addition of two matrix and **.** be a multiplication of matrix by constant.

$M_{mn}(F)$ is a vector space over F.

Example1.3: Let F be any field and $n\in Z^+$, $P_n(F)=\{a_0+a_1x+\dots+a_nx^n: a_i\in F \text{ for } i=0,1,\dots,n\}$. $a_0+a_1x+\dots+a_nx^n= b_0+b_1x+\dots+b_nx^n$ iff $a_i=b_i$ for $i=0,1,\dots,n$.

We define vector addition by $(a_0+a_1x+\dots+a_nx^n)+(b_0+b_1x+\dots+b_nx^n)= a_0+b_0+(a_1+b_1)x+\dots+(a_n+b_n)x^n$ [coordinate wise addition]

and scalar multiplication by $a(a_0+a_1x+\dots+a_nx^n)=aa_0+aa_1x+\dots+aa_nx^n$ [coordinate wise scalar multiplication]. $P_n(F)$ is a vector space over F.

Example1.4: Let $X\neq\varphi$ and $V=\{f: f:X\to R$ is a function$\}$. The vector addition is define as sum of two function $(f+g)(x)=f(x)+g(x)$ $\forall x\in X$. Scalar multiplication defined by $(af)(x)=af(x)$ $\forall x\in X$. Then V is a vector space over R.

Example1.5: Let I be any interval subset of R and $C(I)=\{f: f:I\to R$ is a continuous function$\}$. C(I) is a vector space over R.

Example1.6: Let I be any interval subset of R, $n\in N$ and $C^n(I)=\{f: f:I\to R$ is a differentiable n-times whose n-derivatives is continuous$\}$. Then $C^n(I)$ is a vector space over R.

Example1.7: Let R be the field of real numbers, $V=R^2=\{(x,y): x\in R \text{ and } y\in R\}$

$(x_1,y_1)=(x_2,y_2)$ iff $x_1=x_2$ and $y_1=y_2$

We define vector addition by $(x_1,y_1)+(x_2,y_2)= (x_1+x_2,y_1+y_2)$ [coordinate wise addition]

and scalar multiplication by $a(x_1,y_1)=(ax_1,a^2y_1)$. Is V a vector space over R? Explain.

Solution: $<R^2,+>$ is abelian group. Let a=1,b=2 and A=(1,1). Then (a+b)A=3(1,1)=(3,9) but aA+bA=1(1,1)+2(1,1)=(1,1)+(2,4)=(3,5), therefore $(a+b)A\neq aA+bA$. Then V is not a vector space over R.

Theorem1.1: Let V be a vector space over F.

1- Zero vector is unique.

2- For each $A \in V$, $-A$ is unique.

Proof of (1): Let O_1, O_2 be two zero vectors. By V2, $O_1+O_2=O_1$. By V2, $O_2+O_1=O_2$. By V4, $O_1+O_2=O_2+O_1$. Therefore, $O_1=O_2$ contradiction. Therefore, Zero vector is unique.

Proof of (2): Let B,C be two inverse of A. By V3, A+B=B+A=O and A+C=C+A=O. B=O+B=(C+A)+B=C+(A+B)=C+O=C. Therefore, B=C contradiction. Therefore, inverse of A is unique.

Theorem1.2: Let V be a vector space over F.

1- For $a \in F$, aO=O
2- For $A \in V$, oA=O
3- For $A \in V$, (-1)A=-A
4- If aA=O, then either a=o or A=O.

Proof of (1): By V2, aO=a(O+O), by V5, aO=aO+aO.

Adding –(aO) to both sides, we get aO+(–aO)=(aO+aO)+(–aO). By V1, aO+(–aO)=aO+(aO+(–aO)). By V3, O=aO+O. By V2, aO=O.

Proof of (2): Since, oA=(o+o)A, by V6, oA=oA+oA.

Adding –(oA) to both sides, we get oA+(–oA)=(oA+oA)+(–oA). By V1, oA+(–oA)=oA+(oA+(–oA)). By V3, O=oA+O. By V2, oA=O.

Proof of (3): By V8, (–1)A+A=(–1)A+1A

by V6, =((-1)+1)A=oA

by Theorem1.2(2) =O. Therefore, (-1)A=-A.

Proof of (4): If a=o, there is nothing for prove. Otherwise $a \neq o$, then there exist, a^{-1} such that $aa^{-1}=a^{-1}a=1$. Since aA=O, by scalar multiplication to both side a^{-1} we get $a^{-1}(aA)= a^{-1}O=O$, but $a^{-1}(aA)=(a^{-1}a)A=1A=A$. Therefore, A=O.

Definition1.2: Let M be a subset of a vector space V over F. M is said to be subspace of V if M is a vector space under the same operation addition of vector and scalar multiplication.

Theorem1.3: A non empty subset M of a vector space V over F is a subspace of V iff the following conditions are satisfied

a- If $A,B \in M$, then $A+B \in M$.
b- If $A \in M$ and $a \in F$, then $aA \in M$.

Proof: Let M be a subspace of V. Then M is a vector space under the same operations as those of V. Hence M is satisfies (a) and (b).

Conversely if M is satisfies (a) and (b), we have to prove that M is satisfies all axioms of vector space. Properties V1, V4, V5, V6, V7, and V8 hold in M because they hold in V. Since M is not empty then there exist $A\in M$, by (b) $oA\in M$, by Theorem1.2(2), $oA=O$, therefore $O\in M$, M satisfies V2. For each $A\in M$, by (b)

$(-1)A\in M$, by Theorem1.2(3), $(-1)A=-A$, therefore $-A\in M$, M satisfies V3. Hence, M is a subspace of V.

Example: Show that $M=\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}: a,b,c,d\in R \text{ and } d=b+c\}$ is a subspace of $M_{22}(R)$.

Solution: Let $A=\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B=\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ be any two vector in M and $r\in R$. then $d_1=b_1+c_1$ and $d_2=b_2+c_2\Rightarrow d_1+d_2=b_1+c_1+b_2+c_2=b_1+b_2+c_1+c_2\Rightarrow A+B=\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\in M$.

Then $rA=\begin{bmatrix} ra_1 & rb_1 \\ rc_1 & rd_1 \end{bmatrix}$. Since $d_1=b_1+c_1$, by multiplying this equation by r we get $rd_1=rb_1+rc_1\Rightarrow rA\in M$. By Theorem1.3, M is a subspace of $M_{22}(R)$.

Example: Let $M=\{(x,y,z,w)\in R^4: xw=0\}$. Is M is a subspace of R^4.Explain.

Solution: Let $A=(1,0,0,0)$ and $B=(0,0,0,1)\Rightarrow A,B\in M$. But $A+B=(1,0,0,0)+(0,0,0,1)=(1,0,0,1)$, but $1.1=1\neq 0\Rightarrow A+B\notin M$. By Theorem1.3, M is not a subspace of R^4.

Theorem1.4: Let M and N be two subspaces of a vector space V over F, then $M\cap N$ is a subspace of V.

Proof: Let $A,B\in M\cap N\Rightarrow A,B\in M$ and $A,B\in N$, by Theorem1.3, $A+B\in M$ and $A+B\in N\Rightarrow A+B\in M\cap N$. Let $A\in M\cap N$ and $r\in F\Rightarrow A\in M$ and $A\in N$, by Theorem1.3, $rA\in M$ and $rA\in N\Rightarrow rA\in M\cap N$. By Theorem1.3, M is a subspace of V.

Example: In $P_2(R)$, let $M=\{a+bx+cx^2: a+2b-c=0\}$ and $N=\{a+bx+cx^2: a+b+2c=0\}$. Find $M\cap N$.

Solution: Let $A=a+bx+cx^2\in M\cap N\Rightarrow A\in M$ and $A\in N\Rightarrow a+2b-c=0\ldots(1)$ and $a+b+2c=0\ldots(2)$ By subtract (2) from (1), we get $b-3c=0\Rightarrow b=3c$ by substituting in (1) and (2) we get $a+5c=0\Rightarrow a=-5c$. $M\cap N=\{a+bx+cx^2: a=-5c \text{ and } b=3c, c\in R\}$

Example: Let $M=\{(x,y)\in R^2: y=x\}$ and $N=\{(x,y)\in R^2: y=2x\}$. Is $M\cup N$ a subspace of R^2? Explain.

Solution: Let $A=(1,1)$ and $B=(1,2)\Rightarrow A\in M$ and $B\in N\Rightarrow A,B\in M\cup N$ But $A+B=(2,3)$ since $(2,3)\notin M$ and $(2,3)\notin N\Rightarrow(2,3)\notin M\cup N$. By Theorem1.3, $M\cup N$ is not a subspace of R^2.

Theorem1.5: Let M and N be two subspaces of a vector space V over F, then $M\cup N$ is a subspace of V iff $M\subset N$ or $N\subset M$.

Proof: Let $M\cup N$ be a subspace of V. suppose that $M\not\subset N$ and $N\not\subset M\Rightarrow$ there exist $A\in M$, $A\notin N$ and $B\in N$, $B\notin M\Rightarrow A\in M\cup N$ and $B\in M\cup N$. Since $M\cup N$ is a subspace of V, $A+B\in M\cup N\Rightarrow$ either $A+B\in M$ or $A+B\in N$. If $A+B\in M$, then $A+B+(-A)\in M\Rightarrow B\in M$ contradiction. If $A+B\in N$, then $A+B+(-B)\in N\Rightarrow A\in N$ contradiction. Which is impossible$\Rightarrow$ either $M\subset N$ or $N\subset M$.

Conversely let $M\subset N$ or $N\subset M\Rightarrow M\cup N=N$ or $M\cup N=M\Rightarrow M\cup N$ is a subspace of V.

Definition1.3: A subset S of a vector space V over F is said to be linearly dependent if there exist scalars $x_1,x_2,\ldots,x_n$ not all zero, and $A_1,A_2,\ldots,A_n$ such that $x_1A_1+x_2A_2+\ldots+x_nA_n=O$

Example: In R^2, let S={(2,5), (1,3), (-3,-7)}. Test whether S LD or not?

Solution: Since 2(2,5)+(-1)(1,3)+(-3,-7)=(0,0), then S is LD.

Definition1.4: A subset S of a vector space V over F is said to be linearly independent if S is not linearly dependent.

A finite set $\{A_1, A_2,\ldots, A_n\}$ is LI if the unique solution of $x_1A_1+x_2A_2+\ldots+x_nA_n=O$ is $x_1=x_2=\ldots=x_n=o$.

Example: In R^2, let S={(2,0), (1,3)}. Test whether S LD or not?

Solution: Let $x(2,0)+y(1,3)=(0,0) \Rightarrow (2x+y,3y)=(0,0) \Rightarrow 2x+y=0$ and $3y=0 \Rightarrow 2x+y=0$ and $y=0 \Rightarrow x=y=0 \Rightarrow$ S is LI.

Definition1.5: Let V be a vector space over F. A vector A in V is called a linear combination of vectors $A_1,A_2,\ldots,A_n$ if there exist scalars $x_1,x_2,\ldots,x_n$ such that $A=x_1A_1+x_2A_2+\ldots+x_nA_n$.

Example: In R^3, let $A_1=(1,0,1)$, $A_2=(1,1,0)$. Test whether A=(4,2,2), B=(2,3,2) is a LC of A_1,A_2? Explain.

Solution: Let $A=xA_1+yA_2\Rightarrow (4,2,2)=x(1,0,1)+y(1,1,0)$, by solving this linear system we get x=2 and y=2. A=(4,2,2) is a LC of A_1,A_2.

Let $B=xA_1+yA_2\Rightarrow(2,3,2)=x(1,0,1)+y(1,1,0) \Rightarrow (2,3,2)=(x+y,y,x) \Rightarrow x+y=2,\ y=3,\ x=2 \Rightarrow x+y=5$ and $x+y=2\Rightarrow 2=5$ contradiction. The system has no solution. B=(2,3,2) is not a LC of A_1,A_2.

Definition1.6: The span of a subset S of a vector space V over F is the set of all finite linear combination of S and denoted by [S]. $[S]=\{\sum_{i=1}^{n} a_iA_i : n\in N, a_i\in F \text{ and } A_i\in S, \text{ for } i=1,\ldots n\}$.

Theorem1.6: Let S be non empty subset of a vector space V over F, then [S] is a smallest subspace of V containing S.

Proof: Let $A,B\in[S]$ and $r\in F$, then $A=x_1A_1+\ldots+x_nA_n$ and $B= y_1B_1+\ldots+y_mB_m$ for some $A_i,B_i\in S$, $x_i,y_i\in F\Rightarrow A+B=x_1A_1+\ldots+x_nA_n+y_1B_1+\ldots+y_mB_m$ which is a finite LC on $S\Rightarrow A+B\in[S]$. Since $rA=rx_1A_1+\ldots+rx_nA_n$ which is a finite LC on $S\Rightarrow rA\in[S]$. By Theorem1.3, [S] is a subspace of V. For each $A\in S$, by V8, $A=1A\Rightarrow A\in[S]\Rightarrow S\subset[S]$. Let T be subspace of V containing S, we have to prove that $[S]\subset T$, let $A\in[S]$, $A=x_1A_1+\ldots+x_nA_n$ for some $A_i\in S$, $x_i\in F$, for $i=1,\ldots n\Rightarrow A_i\in T$, since T is a subspace, by Theorem1.3, $x_1A_1+\ldots+x_nA_n\in T\Rightarrow A\in T$, therefore $[S]\subset T$. Hence [S] is a smallest subspace of V containing S.

Example: In R^2, let $S=\{(2,0), (0,4)\}$. Find [S].

Solution: $[S]=\{x(2,0)+y(0,4):\ x,y\in R\}=\{(2x,4y):\ x,y\in R\}$. For each $A=(x,y)\in R^2\Rightarrow$ $A=\frac{x}{2}(2,0)+\frac{y}{4}(0,4)\Rightarrow A\in[S]\Rightarrow[S]=R^2$.

Theorem1.7: Let S be non empty subset of a vector space V over F, then [S]=S iff S is a subspace of V.

Proof: If [S]=S by Theorem1.6, [S] is a subspace of $V\Rightarrow S$ is a subspace of V. Conversely if S is a subspace of V. Let $A\in[S]$, then $A= x_1A_1+\ldots+x_nA_n$ for some $A_i\in S$, $x_i\in F\Rightarrow$ Since S is a subspace of $V\Rightarrow x_1A_1+\ldots+x_nA_n\in S\Rightarrow A\in S\Rightarrow[S]\subset S$, by Theorem1.6, $S\subset[S]$. Therefore [S]=S.

Definition1.7: Let M and N be any two subset of a vector space of V. The sum of M and N written as M+N is the set of all vectors of the form A+B where $A\in M$ and $B\in N$.

$M+N=\{ A+B: A\in M \text{ and } B\in N\}$.

Example: In R^2, let $M=\{(1,2), (0,1)\}$ and $N=\{(1,1), (-1,2)\}$ then $M+N=\{(2,3),(0,4),(1,2), (-1,3)\}$.

Theorem1.8: Let M and N be two subspaces of a vector space V over F, then M+N is a subspace of V.

Proof: Let $A_1,A_2\in M+N$ and $r\in F\Rightarrow A_1=B_1+C_1$ for some $B_1\in M$ and $C_1\in N$, $A_2=B_2+C_2$ for some $B_2\in M$ and $C_2\in N$. But $A_1+A_2=B_1+B_2+C_1+C_2$, since M,N are subspace of $V\Rightarrow$ $B_1+B_2\in M$, $C_1+C_2\in N\Rightarrow$ $A_1+A_2\in M+N$. Since $rA_1=r(B_1+C_1)=rB_1+rC_1$ and M,N are subspace of $V\Rightarrow rB_1\in M$, $rC_1\in N\Rightarrow rA_1\in M+N$. By Theorem1.3, M+N is a subspace.

Example: In R^3, let $M=\{(x,y,0)\in R^3: x,y\in R\}$, $N=\{(0,y,z)\in R^3: y,z\in R\}$. Find M+N.

Solution: For any $A=(x,y,z)\in R^3 \Rightarrow (x,y,z)=(x,\frac{y}{2},0)+(0,\frac{y}{2},z) \Rightarrow A\in M+N \Rightarrow M+N=R^3$.

In above example $(1,1,1)=(1,\frac{1}{2},0)+(0,\frac{1}{2},1)=(1,0,0)+(0,1,1)=(1,1,0)+(0,0,1)$

The vector (1,1,1) can be written as the sum of vectors in M and N in more than way.

Definition1.8: A vector space V over F is said to be direct sum of subspaces M and N and denoted by $V=M\oplus N$, if for each $A\in V$, can be expressed uniquely as $A=B+C$ such that $B\in M$ and $C\in N$.

Theorem1.9: A vector space V over F is a direct sum of subspaces M and N iff $V=M+N$ and $M\cap N=\{O\}$

Proof: Let $V=M\oplus N$. For each $A\in V$, can be expressed uniquely as $A=B+C$ such that $B\in M$ and $C\in N \Rightarrow V=M+N$. Suppose that $M\cap N\neq\{O\} \Rightarrow$ there exists $A\in M\cap N$ and $A\neq O \Rightarrow A\in M$ and $A\in N$. Since $A=A+O$ and $A=O+A$, these two of expressing a vector A in M+N contradiction since $V=M\oplus N \Rightarrow M\cap N=\{O\}$.

Conversely suppose that $V=M+N$ and $M\cap N=\{O\}$. Let $A\in V \Rightarrow A=B+C$ for some $B\in M$ and $C\in N$, suppose that it is possible to have another representation $A=X+Y$ for some $X\in M$ and $Y\in N \Rightarrow B+C=X+Y \Rightarrow B-X=Y-C$, but $B-X\in M$ and $Y-C\in N$, therefore $B-X=Y-C\in M\cap N=\{O\} \Rightarrow B-X=Y-C=O$ which is implies that $B=X$ and $C=Y$. So no such second representation for A is possible $\Rightarrow V=M\oplus N$.

Example: In R^2, let $M=\{(x,y)\in R^2: x=y\}$, $N=\{(x,y)\in R^2: x=-y\}$. Show that $R^2=M\oplus N$.

Solution: For any $A=(x,y)\in R^2 \Rightarrow (x,y)=(\frac{x+y}{2},\frac{x+y}{2})+(\frac{x-y}{2},\frac{y-x}{2}) \Rightarrow A\in M+N \Rightarrow M+N=R^2$.

Let $(x,y)\in M\cap N \Rightarrow x=y$ and $x=-y \Rightarrow y=-y \Rightarrow y=0$ by substituting we get $x=0 \Rightarrow (x,y)=(0,0)$ $\Rightarrow M\cap N=\{(0,0)\}$. By Theorem1.9, $R^2=M\oplus N$.

Definition1.9: Let V be vector space over F. V is said to be finite dimensional vector space if there exist a finite subset S of V such that [S]=V.

Example: The space R^2 is fdvs since $S=\{(2,0),(0,4)\}$ is finite set and $[S]=R^2$.

Example1.8: Let $R^\infty=\{(x_1,x_2,\ldots,x_n,\ldots)$: there exist $k\in N$ such that for all $n>k$, $x_n=0\}$. It is clear that R^∞ is a vector space over R.

Suppose that R^∞ is fdvs $\Rightarrow$ there exist a finite set $S\subset R^\infty$ such that $[S]=R^\infty$. Let $S=\{A_1, A_2,\ldots,A_n\}$, for each $A_i\in S$, there exist $k_i\in N$ such that for all $n>k_i$, $x_n=0$. Let

$m=Max\{k_1,\ldots k_n\}$. Let B be a vector all whose coordinate is zero except its m+1 coordinate is 1 $\Rightarrow$ $B=(0,\ldots,0,1,0,\ldots)\Rightarrow B\notin[S]$ which is contradiction. R^∞ is not fdvs.

Note: If V is not fdvs it is called infinite dimensional vector space.

Definition1.10: A subset B of a vector space over F is said to be a basis for V if [B]=V and B is LI.

Example: Let $S=\{(2,0),(0,4)\}$, then S is a basis for R^2 since from the previous examples $[S]=R^2$ is and S is LI.

Example: Find a basis for a subspace $M=\{a+bx+cx^2\in P_2(R): a+b=0\}$ of $P_2(R)$.

Solution: $M=\{-b+bx+cx^2:b,c\in R\}=\{b(-1+x)+cx^2:b,c\in R\}=[\{-1+x,x^2\}]$. Let $c_1(-1+x)+c_2x^2=0\Rightarrow -c_1+c_1x+c_2x^2=0\Rightarrow c_1=c_2=0\Rightarrow\{-1+x,x^2\}$ is LI. $\{-1+x,x^2\}$ is a basis for M.

Remark1.1: Every vector space has a basis.

Remark1.2: Let M be s subspace for a vector space V over F. Let $A=\{A_1,\ldots,A_n\}$ be generate for M and $B=\{B_1,\ldots,B_m\}$ be a LI set vectors in M, then $m\leq n$.

Remark1.3: Let $S=\{A_1,\ldots,A_n\}$ be a set of non-zero vectors in a vector space V over F. Then S is LD iff there exist a vector A_k in S which is LC of the preceding vectors in S ($A_k=x_1A_1+\ldots+x_{k-1}A_{k-1}$ for some $x_1,\ldots,x_{k-1}$ in F).

Theorem1.10: If $A=\{A_1,\ldots,A_n\}$ and $B=\{B_1,\ldots,B_m\}$ are bases for vector space V over F, then m=n.

Proof: Since A and B are bases for V$\Rightarrow$ A and B are LI and [A]=[B]=V. Since A generate V and B is LI, by Remark1.2, $m\leq n$. Since B generate V and A is LI, by Remark1.2, $n\leq m\Rightarrow m=n$.

Definition1.11: Let V be finite dimensional vector space over F. The number of vectors in a basis of V is called dimension of V and denoted by $\dim_F V$ or dimV.

Note: If dimV=n, then V is said to be n-dimensional vector space**.**

Example1.11: Let F be any field$\Rightarrow$ F^n is a vector space over F. Consider the elements $E_1=(1,0,\ldots,0)$, $E_2=(0,1,0,\ldots,0)$, …, $E_n=(0,\ldots,0,1)$. E_i is the vector whose coordinates are zero except i-th which is one. The set $B=\{E_1,\ldots,E_n\}$ is a basis for F^n and it is called standard basis for $F^n\Rightarrow \dim F^n=n$.

Example: $\dim M_{22}(R)=4$, $\dim P_2(R)=3$, $\dim C^2=2$, $\dim_R C^2=4$.

Theorem1.11: Let $S=\{A_1,\ldots,A_n\}$ be a basis for vector space V over F, then for any A in V can be written in one and only one way as LC of vectors in S.

Proof: Every vector A in V can written as a LC of vector in S since [S]=V. Let $A=x_1A_1+\ldots+x_nA_n$ and $A=y_1A_1+\ldots+y_nA_n$. we must show that $x_i=y_i$ for $i=1,\ldots,n$. We have $O=A-A=(x_1A_1+\ldots+x_nA_n)-(y_1A_1+\ldots+y_nA_n)=(x_1-y_1)A_1+\ldots+(x_n-y_n)A_n$. Since S is LI$\Rightarrow x_i-y_i=0\Rightarrow x_i=y_i$ for $i=1,\ldots,n$

Theorem1.12: In n-dimensional vector space V over F any set of n LI vector is a basis.

Proof: Let $S=\{A_1,\ldots,A_n\}$ be a LI set of vectors, to prove that S is a basis to V we must show that [S]=V. For any A in V, the set $\{A_1,\ldots,A_n,A\}$ is LD because V is n-dvs, by Remark1.3, one of the vectors $A_1,\ldots,A_n$, A is LC of the of all predecessors, obviously this one can not be any of $A_1,\ldots,A_n$ because $\{A_1,\ldots,A_n\}$ is a LI set. Then $A\in[\{A_1,\ldots,A_n\}]\Rightarrow A\in[S]\Rightarrow[S]=V$, therefore S is a basis of V.

Example: Show that S={(1,1,1), (1,-1,1),(0,1,1)} is a basis for R^3.

Solution: By simple calculation we obtain S is LI. Since $\dim R^3=3$, by Theorem1.12, S is a basis for R^3.

Remark1.4: Let $\{A_1,\ldots,A_k\}$ be a LI subset of an n-dvs V. then there exist vector $B_{k+1},\ldots,B_n$ such that $\{A_1,\ldots,A_k, B_{k+1},\ldots,B_n\}$ is a basis for V.

Example: Find a basis for R^3 containing the vectors (1,0,1) and (0,-1,1).

Solution: Let $(1,0,0)=x(1,0,1)+y(0,-1,1)=(x,-y,x+y)\Rightarrow x=1$ and $y=0$ and $x+y=0\Rightarrow x+y=1$ and $x+y=0$ contradiction. The set S={(1,0,0),(1,0,1),(0,-1,1)} is LI set, by Theorem1.12, S is basis for R^3.

Theorem1.13: Let M be a subspace of a fdvs V over F

1- $\dim M\leq\dim V$.
2- If $\dim M=\dim V$, then M=V.

Proof of (1): Let $S=\{A_1,\ldots,A_n\}$ be a basis for V. Let $T=\{B_1,\ldots,B_k\}$ be a any basis for M$\Rightarrow$T is LI in M$\Rightarrow$ T is LI in V$\Rightarrow$by Remark1.2, $k\leq n\Rightarrow \dim M\leq\dim V$.

Proof of (2): If $\dim M=\dim V$, let T be basis for M$\Rightarrow$ T contains n-linearly independent vectors in V, by Theorem1.12, T is a basis for V. [T]=V. But [T]=M $\Rightarrow$ M=V.

Remark1.5: Let M and N be two subspaces of a fdvs V over F, then $\dim(M+N)=\dim M+\dim N-\dim(M\cap N)$.

Example: In R^3, let $M=\{(x,y,z)\in R^3: z=x+2y\}$, $N=\{(x,y,z)\in R^3: x=-y\}$. Find dim(M+N) and M+N.

Solution: $\dim M=2$ and $\dim N=2$. $M\cap N=\{(x,y,z)\in R^3: z=x+2y \text{ and } x=-y\}=\{(x,y,z)\in R^3: z=y \text{ and } x=-y\}\Rightarrow \dim(M\cap N)=1$. Remark1.5,

$\dim(M+N)=\dim M+\dim N-\dim(M\cap N)=2+2-1=3\Rightarrow \dim(M+N)=3$. Since $\dim R^3=3\Rightarrow \dim R^3=\dim(M+N)$, by Theorem1.11(2), $M+N=R^3$.

Definition1.12: Let $S=\{A_1,\ldots,A_n\}$ be an ordered basis for a vector space V over F. Then each vector A in V can be written as $A=x_1A_1+\ldots+x_nA_n$. The vector $X=(x_1,\ldots,x_n)$ is called coordinate vector of A relative to the ordered basis S (some time denoted by $[A]_S$). $x_1,\ldots,x_n$ are called coordinates of the vector of A relative to the ordered basis S.

Example: In $P_2(R)$, find coordinate vector to $3-x^2$ relative to the ordered basis $\{3,-1+x, x^2\}$.

Solution: Let $3-x^2=a(3)+b(-1+x)+c(x^2)$ by simple calculation we obtain, a=1, b=0, c=-1. X=(1,0,-1).

Let $S=\{A_1, A_2,\ldots,A_n\}$ and $S^*=\{A_1^*, A_2^*,\ldots,A_n^*\}$ be two ordered bases for n-dvs V over $F\Rightarrow$each vector $A_i\in V$ can be written as a LC of vectors in S*.

$A_1=p_{11}A_1^*+p_{12}A_2^*+\ldots+p_{1n}A_n^*$

$A_2=p_{21}A_1^*+p_{22}A_2^*+\ldots+p_{2n}A_n^*$

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$A_n=p_{n1}A_1^*+p_{n2}A_2^*+\ldots+p_{nn}A_n^*$

Let$P=\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$. P is called transition matrix from the basis S to the basis S*.

Theorem1.14: Let S and S* be two ordered basis for fdvs V over F. Let P be a transition matrix from S to S*. If X is coordinate vector of $A\in V$ with respect to S, then $X^*=XP$ is coordinate vector of $A\in V$ with respect to S*.

Proof: Let $\dim V=n$, $S=\{A_1, A_2,\ldots,A_n\}$ and $S^*=\{A_1^*, A_2^*,\ldots,A_n^*\}$. Since $X=(x_1,x_2,\ldots,x_n)$ is a coordinate vector of A with respect to $S\Rightarrow A=x_1A_1+x_2A_2+\ldots+x_nA_n$. Since P is transition matrix from S to $S^*\Rightarrow A_1=p_{11}A_1^*+p_{12}A_2^*+\ldots+p_{1n}A_n^*$,$A_2=p_{21}A_1^*+p_{22}A_2^*+\ldots+p_{2n}A_n^*$,..., $A_n=p_{n1}A_1^*+p_{n2}A_2^*+\ldots+p_{nn}A_n^*$. By substituting $A=x_1(p_{11}A_1^*+p_{12}A_2^*+\ldots+p_{1n}A_n^*)+x_2(p_{21}A_1^*+p_{22}A_2^*+\ldots+p_{2n}A_n^*)+\ldots+x_n(p_{n1}A_1^*+p_{n2}A_2^*+\ldots+p_{nn}A_n^*)=(x_1p_{11}+x_2p_{21}+\ldots+x_np_{n1})A_1^*+(x_1p_{12}+x_2p_{22}+\ldots+x_np_{n2})A_2^*+\ldots(x_1p_{1n}+x_2p_{2n}+\ldots+x_np_{nn}$

)A_n*. If X*=(x_1*,x_2*,…,x_n*) is a coordinate vector of A with respect to S*, by Theorem1.11,

$x_1^* = x_1p_{11}+x_2p_{21}+\ldots+x_np_{n1}$

$x_2^* = x_1p_{12}+x_2p_{22}+\ldots+x_np_{n2}$

…

$x_n^* = x_1p_{1n}+x_2p_{2n}+\ldots+x_np_{nn}$

we can write in matrix form as $X^*=(x_1^*, x_2^*,\ldots,x_n^*)= (x_1, x_2,\ldots,x_n)\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \Rightarrow X^*=XP.$

Example: Find transition matrix from S={(2,3),(0,3)} to S*={(-1,0),(3,3)}.

Solution: $(2,3)=p_{11}(-1,0)+p_{12}(3,3)=(-p_{11}+3p_{12}, 3p_{12})$, by solving linear system we get $p_{11}=1$, $p_{12}=1$. $(0,3)=p_{21}(-1,0)+p_{22}(3,3)=(-p_{21}+3p_{22}, 3p_{22})$, by solving linear system we get $p_{21}=3$, $p_{22}=1 \Rightarrow P=\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$.

Example: Let $P=\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$ be a transition matrix from S={(2,1),(0,3)} to S*={A_1*, A_2*}. Find A_1*, A_2***.**

Solution: $(2,1)=\frac{1}{\sqrt{5}}A_1^* + \frac{2}{\sqrt{5}}A_2^*$ …(1) $(0,3)=\frac{-2}{\sqrt{5}}A_1^* + \frac{1}{\sqrt{5}}A_2^*$ …(2)

By multiply (1) by 2 and adding to (2) we get $(4,5)=\frac{5}{\sqrt{5}}A_2^* \Rightarrow A_2^*=(\frac{4}{\sqrt{5}},\frac{5}{\sqrt{5}})$, by substituting in (1) we get $A_1^*=(\frac{2}{\sqrt{5}},-\frac{5}{\sqrt{5}})$

Theorem1.15: Let S={A_1,…,A_n} and S*={A_1*,…,A_n*} be ordered bases for a vector space V. If there exist matrix P=(p_{ij}) such that for any $A\in V$ the vector coordinate of A is X and X* relative to S and S* respectively and X*=XP, then P is a transition matrix from S to S*.

Proof: Let Q=(q_{ij}) be a transition matrix from S to S*. By Theorem1.14, X*=XQ, since $X^*=XP \Rightarrow XQ=XP$. The vector coordinate of A_i relative to ordered basis S is X_i=(o,…,o,1,o,...,o) since $A_i=1A_i$, for i=1,…,n$\Rightarrow X_iQ=X_iP \Rightarrow$ i-th row of a matrix Q= i-th row of a matrix P, for i=1,…,n.$\Rightarrow q_{ij}=p_{ij}$ for j=1,…,n for i=1,…,n$\Rightarrow$ Q=P$\Rightarrow$ P is a transition matrix from S to S*.

Theorem1.16: Let S={A_1,…,A_n}, S*={A_1*,…,A_n*} and S**={A_1**,…,A_n**}be ordered bases for a vector space V. Let P be the transition matrix from S to S* and Q be the transition matrix from S* to S**, then PQ is a transition matrix from S to S**.

Proof: Let X, X* and X** be vector coordinate to A relative to the bases S,S* and S**. By Theorem1.14, X*=XP and X**=X*Q$\Rightarrow$X**=X*Q=(XP)Q=X(PQ). By Theorem1.15, PQ is a transition matrix from S to S**.

Theorem1.17: Let P be a transition matrix from S to S*, then P^{-1} is a transition matrix from S* to S.

Proof: The transition matrix from S to S is I since $A_i=1A_i$. Let Q be a transition matrix from S* to S, by Theorem1.16, PQ is a transition matrix from S to S, by Theorem1.11, transition matrix is unique$\Rightarrow$PQ=I. By similar statement QP=I$\Rightarrow$Q=P^{-1} is a transition matrix from S* to S.