Chapter two

Definition2.1: Let V and W be vector spaces over F. The function $T:V\rightarrow W$ is said to be linear transformation(linear map, linear operator) if

1- For all $A,B\in V$, $T(A+B)=T(A)+T(B)$.

2- For all $A\in V$ and $r\in F$, $T(rA)=rT(A)$.

Example: Let $V=R^2$, $W=R^3$ and F=R. Let $T:V\rightarrow W$ defined by $T(x,y)=(x+y,-3y,x)$, show that T is LT.

Solution: Let $A=(x,y)$, $B=(z,w)$ and $r\in R$. $T(A)=T(x,y)=(x+y,-3y,x)$, $T(B)=T(z,w)=(z+w,-3w,z)$ and $A+B=(x+z,y+w)\Rightarrow T(A+B)=T(x+z,y+w)=(x+z+y+w,-3(y+w),x+z)=(x+y,-3y,x)+(z+w,-3w,z)=T(x,y)+T(z,w)=T(A)+T(B)\Rightarrow T(A+B)=T(A)+T(B)$.

$T(rA)=T(rx,ry)=(rx+ry,-3ry,rx)=r(x+y,-3y,x)=rT(x,y)= rT(A) \Rightarrow T(rA)=rT(A) \Rightarrow$ T is LT.

Example: Let $V=W=R^2$ and F=R. Let $T:V\rightarrow W$ defined by $T(x,y)=(x+1,x+y)$, show that T is not a LT.

Solution: Let $A=(1,0)$ and $B=(0,1)\Rightarrow T(A)=T(1,0)=(2,1)$, $T(B)=T(0,1)=(1,1)$. $T(A+B)=T(1,1)=(2,2)$. But $T(A)+T(B)=(2,1)+(1,1)=(3,2)\Rightarrow T(A+B)\neq T(A)+T(B)\Rightarrow$ T is not LT.

Example: Define $T:C^1(a,b)\rightarrow C(a,b)$ by $T(f)=f'$. By properties of derivative T is a LT.

Example2.1: Let V and W be any vector spaces over F. The constant function $T:V\rightarrow W$ defined by $T(A)=O$ $\forall A\in V$. T is a LT and it is called zero transformation (Null transformation).

Example2.2: Let V be any vector spaces over F. The identity function $I_V:V\rightarrow V$ defined by $I_V(A)=A$ $\forall A\in V$. T is a LT and it is called identity transformation on V.

Without ambiguity some time we use I instead of I_V.

Theorem2.1: Let V and W be vector spaces over F, and $T:V\rightarrow W$ is LT, then the following are hold

1- $T(O)=O$ 2- $T(-A)=-T(A)$ 3- $T(x_1A_1+x_2A_2+\ldots+x_nA_n)= x_1T(A_1)+x_2T(A_2)+\ldots+x_nT(A_n)$.

Proof: From Theorem1.2 and second condition of LT, $T(O)=T(oO)=oT(O)=O$.

From Theorem1.2 and second condition of LT, $T(-A)=T((-1)A)=(-1)T(A)=-T(A)$.

From first and second condition of LT, $T(x_1A_1+x_2A_2+\ldots+x_nA_n)=T(x_1A_1)+T(x_2A_2)+\ldots+T(x_nA_n)= x_1T(A_1)+x_2T(A_2)+\ldots+x_nT(A_n)$.

Theorem2.2: A linear transformation T is completely determined by its value on the elements of basis precisely. If $S=\{A_1,\ldots,A_n\}$ be a basis for a fdvs V over F. For any set $\{B_1,\ldots,B_n\}$ of n vectors (not necessarily distinct) in W, then there exist a unique linear transformation $T:V\to W$ such that $T(A_i)=B_i$.

Proof: Let $A\in V\Rightarrow A$ can be expressed uniquely in the form $A=x_1A_1+\ldots+x_nA_n$. We defined $T(A)= x_1B_1+\ldots+x_nB_n$. We now claim T is required transformation. Let $A,B\in V$ and $r\in F\Rightarrow$ $A=x_1A_1+\ldots+x_nA_n$ and $B=y_1A_1+\ldots+y_nA_n\Rightarrow A+B=(x_1+y_1)A_1+\ldots+(x_n+y_n)A_n$. Hence by definition of T, we have $T(A+B)=(x_1+y_1)B_1+\ldots+(x_n+y_n)B_n=x_1B_1+\ldots+x_nB_n+y_1B_1+\ldots+y_nB_n$ $=T(A)+T(B)$. But $rA=rx_1A_1+\ldots+rx_nA_n\Rightarrow T(rA)= rx_1B_1+\ldots+rx_nB_n=r(x_1B_1+\ldots+x_nB_n)=rT(A)\Rightarrow$ T is a linear transformation. Since $A_i=oA_1+\ldots+oA_{i-1}+1A_i+oA_{i+1}+\ldots+oA_n$ $\Rightarrow T(A_i)=oB_1+\ldots+oB_{i-1}+1B_i+oB_{i+1}+\ldots+oB_n\Rightarrow T(A_i)=B_i$.

To show that T is unique, suppose that there exist another LT $S:V\to W$ with $S(A_i)=B_i$. Let $A\in V\Rightarrow A$ can be expressed uniquely in the form $A=x_1A_1+\ldots+x_nA_n$ $\Rightarrow$ $S(A)=S(x_1A_1+\ldots+x_nA_n)=x_1S(A_1)+\ldots+x_nS(A_n)=x_1B_1+\ldots+x_nB_n=T(A)$ $\Rightarrow$ $S(A)=T(A)$ $\forall A\in V$. Therefore, S=T.

Example: Let $\{(1,0),(2,1)\}$ be a basis for R^2. Find a LT, $T:R^2\to P_2(R)$ such that $T(1,0)=1+x$ and $T(2,1)=x-x^2$.

Solution: Let $(a,b)\in R^2\Rightarrow(a,b)=c(1,0)+d(2,1)\Rightarrow$ $c+2d=a$ and $d=b\Rightarrow c=a-2b$ and $d=b$. By Theorem2.2, $T(a,b)=(a-2b)(1+x)+b(x-x^2)\Rightarrow$ $T(a,b)=a-2b+(a-b)x-bx^2$.

Definition2.2: Let V and W be vector spaces over F. For any LT, $S,T:V\to W$ and $r\in F$, we define $S+T:V\to W$ by $(S+T)(A)=S(A)+T(A)$ $\forall A\in V$ and $rT:V\to W$ by $(rT)(A)=rT(A)$ $\forall A\in V$.

Example: Let $S,T:R^2\to R^2$ be two LT, defined by $T(x,y)=(x,2y)$ and $S(x,y)=(y,x)$. Find 2T, S+T and 3T-4S.

Solution: $(2T)(x,y)=2T(x,y)=2(x,2y)=(2x,4y)$. $(S+T)(x,y)=S(x,y)+T(x,y)=(y,x)+(x,2y)$ $=(x+y,x+2y)$. $(3T-4S)(x,y)=3T(x,y)-4S(x,y)=3(x,2y)-4(y,x)=(3x-4y,-4x+6y)$.

Theorem2.3: Let V and W be vector spaces over F. For any LT, $S,T:V\to W$ and $r\in F$, S+T, rT are LT.

Proof: $\forall A,B\in V$ and $\forall x\in F$, $(S+T)(A+B)=S(A+B)+T(A+B)=S(A)+S(B)+T(A)+T(B)=$ $S(A)+T(A)+S(B)+T(B)=(S+T)(A)+(S+T)(B)$.

$(S+T)(xA)=S(xA)+T(xA)= xS(A)+xT(A)=x(S+T)(A)\Rightarrow S+T$ is a LT.

$\forall A,B\in V$ and $\forall x\in F$, $(rT)(A+B)=rT(A+B)=r(T(A)+T(B))=rT(A)+rT(B)=(rT)(A)+(rT)(B)$ $(rT)(xA)=rT(xA)=rxT(A)= xrT(A)=x(rT(A))=x(rT)(A) \Rightarrow rT$ is a LT.

Definition2.3: The set of all linear transformation from V to W denoted by L(V,W).

Remark2.1: Let V and W be vector spaces over F. Then L(V,W) is vector space over F.

Definition2.4: Let $T:V\rightarrow W$ be a LT

1-Kernel of T denoted by KerT defined by $KerT=\{A\in V: T(A)=O\}$

2-Image (Range) of T denoted by ImT defined by $ImT=\{B\in W: \exists A\in V \text{ such that } T(A)=B\}$.

Example: Let $T\in L(R^3, R^2)$ defined by $T(x,y,z)=(x+y+z,0)$. Find KerT and ImT.

Solution: $KerT=\{(x,y,z)\in R^3:T(x,y,z)=(0,0)\}=\{(x,y,z)\in R^3: (x+y+z,0)=(0,0)\}=\{(x,y,z)\in R^3: x+y+z=0 \text{ and } 0=0\}=\{(x,y,z)\in R^3: x+y+z=0\}$

$ImT=\{(a,b)\in R^2: \exists(x,y,z)\in R^3 \text{ such that } T(x,y,z)=(a,b)\}=\{(a,b)\in R^2: \exists(x,y,z)\in R^3 \text{ such that } (x+y+z,0)=(a,b)\} =\{(a,b)\in R^2: \exists(x,y,z)\in R^3 \text{ such that } a=x+y+z \text{ and } b=0\}=\{(a,b)\in R^2: b=0\}$.

Theorem2.4: Let V and W be vector spaces over F and $T\in L(V,W)$

1- KerT is a subspace of V.
2- ImT is a subspace of W.

Proof of (1): Let $A,B\in KerT$ and $r\in F\Rightarrow T(A)=O$ and $T(B)=O\Rightarrow T(A+B)=T(A)+T(B) =O+O=O \Rightarrow A+B\in KerT$. Since $T(rA)=rT(A)=rO=O \Rightarrow rA\in KerT$, by Theorem1.3, KerT is a subspace of V.

Proof of (2): Let $B_1,B_2\in ImT$ and $r\in F\Rightarrow\exists A_1,A_2\in V$, $T(A_1)=B_1$ and $T(A_2)=B_2 \Rightarrow B_1+B_2=T(A_1)+T(A_2)=T(A_1+A_2)\Rightarrow B_1+B_2\in ImT$. Since $rB_1=rT(A_1)=T(rA_1)\Rightarrow rB_1\in ImT$, by Theorem1.3, ImT is a subspace of W.

Definition2.4: Let V and W be vector spaces over F and $T\in L(V,W)$

1- If KerT is finite dimensional subspace of V, the dimension of KerT is called nullity of T and denoted by dimKerT or n(T).
2- If ImT is finite dimensional subspace of W, the dimension of ImT is called rank of T and denoted by dimImT or r(T).

Example: In previous example n(T)= dimKerT=2 and r(T)=dimImT=1.

Remark2.2: Let V and W be vector spaces over F**,** V is fdvs and $T\in L(V,W)$, then dimKerT+dimImT=dimV.

Theorem2.5: Let V and W be vector spaces over F and $T\in L(V,W)$, T is one-one iff $KerT=\{O\}$.

Proof: Let T be one-one LT. Let $A \in KerT \Rightarrow T(A)=O=T(O) \Rightarrow A=O \Rightarrow KerT \subset \{O\}$ and $O \in KerT \Rightarrow KerT=\{O\}$. Conversely if $KerT=\{O\}$, let $A,B \in V$ such that $T(A)=T(B) \Rightarrow T(A-B)=T(A)-T(B)=T(A)-T(A)=O \Rightarrow A-B \in KerT \Rightarrow A-B=O \Rightarrow A=B \Rightarrow$ T is one-one LT.

Remark2.3: Let V and W be vector spaces over F and $T \in L(V,W)$, T is onto iff ImT=W.

Example: Find $T \in L(R^2, R^2)$ such that $\{(2,1)\}$ is a basis for ImT.

Solution: Let $\{(1,0),(0,1)\}$ be a standard basis for R^2. Define $T \in L(R^2, R^2)$ by $T(1,0)=(2,1)$ and $T(0,1)=(0,0) \Rightarrow$ For each $(x,y) \in R^2$, $(x,y)=x(1,0)+y(0,1) \Rightarrow T(x,y)=T(x(1,0)+y(0,1))= xT(1,0)+yT(0,1)=x(2,1)+y(0,0)=(2x,x) \Rightarrow T(x,y)=(2x,x)$.

Definition2.6: Let V,U and W be vector spaces over F, $T \in L(V,U)$ and $S \in L(U,W)$, we can define the function $SoT:V \rightarrow W$ by $(SoT)(A)=S(T(A))\ \forall A \in V$ and it is called composition of S,T.

Theorem2.6: Let V,U and W be vector spaces over F, $T \in L(V,U)$ and $S \in L(U,W)$, then $SoT \in L(V,W)$.

Proof: $\forall A,B \in V$ and $\forall r \in F$, $(SoT)(A+B)=S[T(A+B)]=S[T(A)+T(B)]=S(T(A))+S(T(B))= (SoT)(A)+(SoT)(B)$. $(SoT)(rA)=S[T(rA)]=S[rT(A)]=r(S[T(A)]=r(SoT)(A) \Rightarrow$ SoT is a LT.

Example: Let $T \in L(R^3,R^2)$ define by $T(x,y,z)=(x+2y,z+2x)$ and $S \in L(R^2,R^3)$ define by $S(x,y)=(x,2y,x+y)$. Find SoT and ToS.

Solution: $SoT \in L(R^3,R^3)$ and $(SoT)(x,y,z)=S[T(x,y,z)]=S(x+2y,z+2x)=(x+2y,2z+4x, 3x+2y+z)$ and $ToS \in L(R^2,R^2)$ and $(ToS)(x,y)=T[S(x,y)]=T(x,2y,x+y)=(x+4y,3x+y)$.

Definition2.7: Let V and W be vector spaces over F, $T \in L(V,W)$ and $S \in L(W,V)$,

1- If $SoT=I_V$, then S is called left inverse to T.
2- If $ToS=I_W$, then S is called right inverse to T.
3- S is called inverse to T if S is left inverse to T and S is right inverse to T.

Example: Let $T \in L(R^2,R^3)$ defined by $T(x,y)=(x,x+2y,x-y)$.Show that $S \in L(R^3,R^2)$ defined by $S(x,y,z)=(x,-2x+y+z)$ is left inverse to T.

Solution: $(SoT)(x,y)=S[T(x,y)]=S(x,x+2y,x-y)=(x,-2x+x+2y+x-y)=(x,y) \Rightarrow SoT=I$

Example: Let $T \in L(R^2,P_1(R))$ defined by $T(a,b)=a+b+bx$. Show that $S \in L(P_1(R),R^2)$ defined by $S(a+bx)=(a-b,b)$ is inverse to T.

Solution: $(SoT)(a,b)=S[T(a,b)]=S(a+b+bx)=(a+b-b,b)=(a,b) \Rightarrow SoT=I$ and $(ToS)(a+bx)=T[S(a+bx)]=T(a-b,b)= a-b+b+bx = a+bx \Rightarrow ToS=I \Rightarrow$ S is inverse to T.

Theorem2.7: Let V and W be vector spaces over F, $T\in L(V,W)$. If T has left and right inverse, then they must be equal.

Proof: Let $S,S'\in L(W,V)$ such that $SoT=I_V$ and $ToS'=I_W$ $\Rightarrow$ $S=SoI_W=So(ToS')=(SoT)oS'=I_VoS'=S'$. Therefore, $S=S'$.

Definition2.8: Let V and W be vector spaces over F. $T\in L(V,W)$ is called non-singular or invertable or isomorphism if there exist $S\in L(W,V)$ such that $SoT=I_V$ and $ToS=I_W$, and we write $S=T^{-1}$.

Definition2.9: Let V and W be vector spaces over F. V is said to be isomorphic to W if there exist $T\in L(V,W)$ which is isomorphism, we denote by $V\cong W$.

Example: In previous example $R^2\cong P_1(R)$.

Theorem2.8: Let V and W be vector spaces over F, $T\in L(V,W)$. If there exist $S:W\to V$ such that $SoT=I_V$ and $ToS=I_W$, then S must be LT.

Proof: Let $B_1,B_2\in W$ and $r\in F$. Let $A_1=S(B_1)$ and $A_2=S(B_2)\Rightarrow$ $B_1=I_W(B_1)=(ToS)(B_1)=T[S(B_1)]=T(A_1)$ by similar statements $B_2=T(A_2)$. $B_1+B_2=T(A_1)+T(A_2)=T(A_1+A_2)\Rightarrow S(B_1+B_2)=S[T(A_1+A_2)]=(SoT)(A_1+A_2)=I_V(A_1+A_2)=$ $A_1+A_2=S(B_1)+S(B_2)$. $S(rB_1)=S(rT(A_1))=S(T(rA_1))=(SoT)(rA_1)=I_V(rA_1)=rA_1=rS(B_1)$. Therefore, S is LT.

Theorem2.9: Let V and W be fdvs over F, $T\in L(V,W)$. T has right inverse iff T is onto.

Proof: Let $S:W\to V$ be right inverse of $T\Rightarrow ToS=I_W$. For any $B\in W$ let $A=S(B)$ and $T(A)=T[S(B)]=(ToS)(B)=I_W(B)=B\Rightarrow$ $ImT=W$, by Remark2.3, T is onto. Conversely if T is onto, by Remark2.3, $ImT=W$. Let $dimV=m$, $dimW=dimImT=n$ by Remark2.2, $dimKerT=dimV-dimImT=m-n$. Let $\{A_1,\ldots,A_{m-n}\}$ be any basis for KerT, by Remark1.4, there exist vector $B_1,\ldots,B_n$ such that $\{A_1,\ldots,A_{m-n}, B_1,\ldots,B_n\}$ is a basis for V. Let $C_1=T(B_1),\ldots,C_n=T(B_n)\Rightarrow$ $\{C_1,\ldots,C_n\}$ is a basis for $W=ImT$. We define $S\in L(W,V)$ by $S(C_1)=B_1,\ldots,S(C_n)=B_n$. For any $B\in W$, $B= x_1C_1+\ldots+x_nC_n\Rightarrow$ $S(B)=S(x_1C_1+\ldots+x_nC_n)=$ $x_1S(C_1)+\ldots+x_nS(C_n)=x_1B_1+\ldots+x_nB_n\Rightarrow$ $(ToS)(B)=T(x_1B_1+\ldots+x_nB_n)=$ $x_1T(B_1)+\ldots+x_nT(B_n)=$ $x_1C_1+\ldots+x_nC_n=B\Rightarrow$ $ToS=I_W$.

Theorem2.10: Let V and W be fdvs over F, $T\in L(V,W)$. T has left inverse iff T is one-one.

Proof: Let $S:W\to V$ be left inverse of $T\Rightarrow SoT=I_V$. let $A,B\in V$ such that $T(A)=T(B)$ $\Rightarrow S[T(A)]=S[T(B)]\Rightarrow(SoT)(A)=(SoT)(B)\Rightarrow I_V(A)=I_V(B)\Rightarrow A=B$ then T is one-one. Conversely if T is one-one, by Theorem2.5, $KerT=\{O\}$. Let $dimV=m$, $dimW=n$ by

Remark2.2, dimImT=dimV-dimKerT=m-0=m. Let $\{A_1,\dots,A_m\}$ be any basis for V. Let $B_1=T(A_1),\dots,B_m=T(A_m)$. Suppose that $y_1B_1+\dots+y_mB_m=O\Rightarrow y_1T(A_1)+\dots+y_mT(A_m)=O\Rightarrow T(y_1A_1+\dots+y_mA_m)=O$. Since KerT={O}$\Rightarrow y_1A_1+\dots+y_mA_m=O$. Since $\{A_1,\dots,A_m\}$ is LI $\Rightarrow y_1=\dots=y_m=o \Rightarrow \{B_1,\dots,B_m\}$ is LI, by Remark1.4, there exist a vectors $C_1,\dots,C_{n-m}\Rightarrow \{B_1,\dots,B_m,C_1,\dots,C_{n-m}\}$ is a basis for W. We define $S\in L(W,V)$ by $S(B_1)=A_1,\dots,S(B_m)=A_m$ and $S(C_1)=O,\dots,S(C_{n-m})=O$. For any $B\in W$, $B=x_1B_1+\dots+x_mB_m+y_1C_1+\dots+y_{n-m}C_{n-m}$ $\Rightarrow S(B)=S(x_1B_1+\dots+x_mB_m+y_1C_1+\dots+y_{n-m}C_{n-m})\Rightarrow S(B)=x_1A_1+\dots+x_mA_m$. For any $A\in V$, $A=x_1A_1+\dots+x_mA_m$. $(SoT)(A)=S[T(A)]=S(x_1B_1+\dots+x_nB_n)=x_1S(B_1)+\dots+x_nS(B_n)=x_1A_1+\dots+x_nA_n=A \Rightarrow SoT=I_V$.

Example: Let $T\in L(R^2,R^3)$ defined by T(x,y)=(y,x,x+y). Find left and right inverse if exist?

Solution: $KerT=\{(x,y)\in R^2:T(x,y)=(0,0,0)\}=\{(x,y)\in R^2: (y,x,x+y)=(0,0,0)\}=\{(x,y)\in R^2: x=y=x+y=0\}=\{(x,y)\in R^2: x=y=0\}$ by Theorem2.5, T is one-one, by Theorem2.10, T has left inverse. Let {(1,0),(0,1)} be standard basis for R^2. Let $B_1=T(1,0)=(0,1,1)$ and $B_2=T(0,1)=(1,0,1)\Rightarrow\{(0,1,1),(1,0,1),(0,0,1)\}$ is a basis for R^3. define $S\in L(R^3,R^2)$ by S(0,1,1)=(1,0), S(1,0,1)=(0,1) and S(0,0,1)=(0,0). Let $(x,y,z)=c_1(0,1,1)+c_2(1,0,1)+c_3(0,0,1)=(c_2,c_1,c_1+c_2+c_3)\Rightarrow c_2=x,\ c_1=y$ and $c_1+c_2+c_3=z\Rightarrow c_2=x$, $c_1=y$ and $c_3=z-x-y$. S(x,y,z)=yS(0,1,1)+xS(1,0,1)+(z-x-y)S(0,0,1)=y(1,0)+x(0,1)+(z-x-y)(0,0)=(y,x)$\Rightarrow$S(x,y,z)=(y,x) is a left inverse of T. Suppose that T(x,y)=(1,1,0)$\Rightarrow$ (y,x,x+y)=(1,1,0)$\Rightarrow$ x=y=1 and x+y=0$\Rightarrow$x+y=2 and x+y=0 contradiction $\Rightarrow (1,1,0)\notin ImT\Rightarrow$ T is not onto, by Theorem2.9, T has no right inverse.

Theorem2.11: Let V and W be fdvs over F, $T\in L(V,W)$. T is isomorphism iff T is one-one and onto.

Proof: Let T be isomorphism $\Rightarrow$there exist $S\in L(W,V)$ such that $SoT=I_V$ and $ToS=I_W$. Since, $SoT=I_V\Rightarrow$T has left inverse by Theorem2.10, T is one-one. Since, $ToS=I_W\Rightarrow$ T has right inverse by Theorem2.9, T is onto$\Rightarrow$ T is one-one and onto. Conversely if T is one-one and onto $\Rightarrow$ Since T is one-one by Theorem2.10, there exist $S\in L(W,V)$ such that $SoT=I_V$. Since T is onto, by Theorem2.9, there exist $S'\in L(W,V)$ such that $ToS'=I_W$. By Theorem2.7, $S=S'\Rightarrow SoT=I_V$ and $ToS=I_W \Rightarrow$ T is isomorphism.

Theorem2.12: Let V and W be fdvs over F, $T\in L(V,W)$. $V\cong W$ iff dimV=dimW.

Proof: Let $V\cong W\Rightarrow$ there exist an isomorphism LT $T\in L(V,W)$. By Theorem2.11, KerT={O} and ImT=W. by Remark2.2, dimV=dimKerT+dimImT=o+dimW=dimW$\Rightarrow$ dimV=dimW.

Conversely dimV=dimW=n. Let $\{A_1,\ldots,A_n\},\{B_1,\ldots,B_n\}$ be a basis for V,W respectively. By Theorem2.2, there exist a unique LT $T\in L(V,W)$ such that $T(A_1)=B_1,\ldots,T(A_n)=B_n$. For any $A\in V$, $A=x_1A_1+\ldots+x_nA_n \Rightarrow T(A)=x_1B_1+\ldots+x_nB_n$. We can define $S\in L(W,V)$, for any $B\in W$, $B=y_1B_1+\ldots+y_nB_n \Rightarrow S(B)=y_1A_1+\ldots+y_nA_n$. $(SoT)(A)=S[T(A)]=S[T(x_1A_1+\ldots+x_nA_n)]= S(x_1B_1+\ldots+x_nB_n)=x_1A_1+\ldots+x_nA_n=A \Rightarrow SoT=I_V$. $(ToS)(B)=T[S(B)]=T[S(y_1B_1+\ldots+y_nB_n)]= T(y_1A_1+\ldots+y_nA_n)=y_1B_1+\ldots+y_nB_n=B \Rightarrow ToS=I_W \Rightarrow$ T is isomorphism $\Rightarrow V\cong W$.

Example: Let F be any field and V be an n-dimensional vector space over $F \Rightarrow V\cong F^n$.

Example: Let F be any field $\Rightarrow P_n(F)\cong F^{n+1}$ and $M_{mn}(F)\cong F^{mn}$.

Example: $C\cong R^2$ over R.

Definition2.10: Let V and W be fdvs over F such that dimV=m and dimW=n. Let $H=\{A_1,\ldots,A_m\}$ be an ordered basis for V and $S=\{B_1,\ldots,B_n\}$ be an ordered basis for W. Let $T\in L(V,W)$, then $T(A_i)\in W$ for i=1,…,m. $T(A_i)$ can express uniquely as LC of vectors in S.

$T(A_1)=a_{11}B_1+a_{12}B_2+\ldots+a_{1n}B_n$
$T(A_2)=a_{21}B_1+a_{22}B_2+\ldots+a_{2n}B_n$
.
.
.
$T(A_m)=a_{m1}B_1+a_{m2}B_2+\ldots+a_{mn}B_n$

Then $M_T=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ the matrix M_T is a matrix of T relative to the order basis H and S and it is denoted by (M_T,H,S) or M_T.

Example: Let $T\in L(R^3, R^2)$ defined by T(x,y,z)=(x+y,x+z). Find M_T relative to the standard basis for R^3 and R^2.

Solution: Let {(1,0,0),(0,1,0),(0,0,1)} and {(1,0),(0,1)} be standard basis for R^3 and R^2.

T(1,0,0)=(1,1)=1(1,0)+1(0,1)

T(0,1,0)=(1,0)=1(1,0)+0(0,1)

T(0,0,1)=(0,1)=0(1,0)+1(0,1)

$\Rightarrow M_T=\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Theorem2.13: Let $H=\{A_1,\ldots,A_m\}$ be an ordered basis for V and $S=\{B_1,\ldots,B_n\}$ be an ordered basis for W. Let $T\in L(V,W)$ and $A\in V$. Let $X=(x_1,\ldots,x_m)$, $Y=(y_1,\ldots,y_n)$ be a vector coordinate of A ,T(A) with respect to the ordered basis H,S respectively. Let M_T be a matrix of T relative to the ordered basis H,S. Then $Y=XM_T$.

Proof: Let $X=(x_1,\ldots,x_m)$ is a vector coordinate of A with respect to the ordered basis H $\Rightarrow A=x_1A_1+\ldots+x_mA_m=\sum_{k=1}^{m} x_k A_k$ Since T is a linear $T(A)=\sum_{k=1}^{m} x_k T(A_k)$ Since $T(A_k)=\sum_{j=1}^{n} a_{kj}B_j$

$$T(A)=\sum_{k=1}^{m} x_k \sum_{j=1}^{n} a_{kj}B_j = \sum_{k=1}^{m}\sum_{j=1}^{n} x_k a_{kj}B_j = \sum_{j=1}^{n}(\sum_{k=1}^{m} x_k a_{kj})B_j$$

Since $Y=(y_1,\ldots,y_n)$ be a vector coordinate of T(A) with respect to the ordered basis S$\Rightarrow$ $T(A)=y_1B_1+\ldots+y_nB_n=\sum_{j=1}^{n} y_j B_j$. Hence $\sum_{j=1}^{n} y_j B_j = \sum_{j=1}^{n}(\sum_{k=1}^{m} x_k a_{kj})B_j$ by Theorem1.11, $y_j=\sum_{k=1}^{m} x_k a_{kj}$ for j=1,…,n. Thus

$y_1=x_1a_{11}+x_2a_{21}+\ldots+x_ma_{m1}$
$y_2=x_1a_{12}+x_2a_{22}+\ldots+x_ma_{m2}$
.
.
.
$y_n=x_1a_{1n}+x_2a_{2n}+\ldots+x_ma_{mn}$

Therefore, $(y_1,\ldots,y_n)=(x_1,\ldots,x_m)\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \Rightarrow Y=XM_T.$

Remark2.4: Let $G=\{A_1,\ldots,A_m\}$, $H=\{B_1,\ldots,B_n\}$ and $J=\{C_1,\ldots,C_p\}$ be an ordered basis for a vector space U, V and W over F respectively. Let $T\in L(U,V)$ and $S\in L(V,W)$. Let M_T be a matrix of T relative to the ordered basis G,H. Let M_S be a matrix of S relative to the ordered basis H,J. Then matrix of SoT relative to the ordered basis G,J is M_TM_S (i.e. $M_{SoT}=M_TM_S$).

Remark2.5: Let $G=\{A_1,\ldots,A_m\}$ and $H=\{B_1,\ldots,B_n\}$ be an ordered basis for a vector space V and W over F respectively. Let $T,S\in L(V,W)$ and $r\in F$. Let M_T, M_S, M_{S+T}, and M_{rT} be a matrix of T, S, S+T, rT relative to the ordered basis G, H respectively. Then
1- $M_{S+T}=M_S+M_T$ 2- $M_{rT}=rM_T$.

Remark2.6: Let V and W be fdvs over F, $T\in L(V,W)$. T is isomorphism iff the matrix for T relative to any pair of bases of V and W is invertable.

Theorem2.14: Let V be fdvs over F, $T,S\in L(V,V)$. If SoT=I, then ToS=I.

Proof: Let SoT=I, by Theorem2.10, T is one-one, by Theorem2.5, KerT={O} by Remark2.2, dimV=dimKerT+dimImT=o+dimImT=dimImT, by Theorem1.13(2), ImT=V, by Remark2.3, T is onto, by Theorem2.9, there exist $S^{'} \in L(V,V)$ such that $ToS^{'}=I$, by Theorem2.7, $S=S^{'}$. Hence, ToS=I.

Theorem2.15: Let F be any field and $M \in M_{mn}(F)$, then there exist a $T \in L(F^m, F^n)$ such that the matrix for T relative to the standard basis for F^m and F^n is M (i.e. $M_T=M$).

Proof: Let $M=(a_{ij})$, define $T:F^m \rightarrow F^n$ by T(X)=XM. It is obvious that T is a LT. Let $G=\{E_1,\ldots,E_m\}$ and $H=\{E_1^*,\ldots,E_n^*\}$ be standard basis for F^m and F^n. $T(E_i)=E_iM=(a_{i1},a_{i2},\ldots,a_{in})=\sum_{i=1}^{n} a_{ij}E_j^* \Rightarrow M_T=M$.

Theorem2.16: Let M,N be n-by-n matrices over a field F. If MN=I, then NM=I.

Proof: By Theorem2.15, there exist linear transformation $S,T:F^n \rightarrow F^n$ define by T(X)=XM and S(X)=XN and $M_T=M$, $M_S=N$ relative to the standard basis. $(SoT)(X)=S[T(X)]=S(XM)=(XM)N=X(MN)=XI=X \Rightarrow SoT=I$, by Theorem2.14, $ToS=I \Rightarrow M_{SoT}=I$, by Remark2.4, $M_TM_S=I \Rightarrow NM=I$.

Theorem2.17: Let V and W be fdvs over F such that dimV=m and dimW=n. Let $T \in L(V,W)$ such that dimImT=r, then there exist a basis for V and W such that the matrix for T relative to this basis is $M_T=\begin{bmatrix} Ir & O \\ O & O \end{bmatrix}$ " this form is called Normal form".

Proof: Since dimImT=r, by Remark2.2, dimKerT=m-r. Let $\{B_1,\ldots,B_{m-r}\}$ be any basis for KerT. By Remark1.4, there exist vectors $A_1,\ldots,A_r$ such that $\{A_1,\ldots,A_r, B_1,\ldots,B_{m-r}\}$ is a basis for V. Let $C_1=T(A_1),\ldots,C_r=T(A_r)$. Let $y_1C_1+\ldots+y_rC_r=O \Rightarrow y_1T(A_1)+\ldots+y_rT(A_r)=O \Rightarrow T(y_1A_1+\ldots+y_rA_r)=O \Rightarrow y_1A_1+\ldots+y_rA_r \in KerT$, by Theorem1.11, $y_1A_1+\ldots+y_rA_r=x_1B_1+\ldots+x_{m-r}B_{m-r} \Rightarrow y_1A_1+\ldots+y_rA_r-x_1B_1-\ldots-x_{m-r}B_{m-r}=O \Rightarrow y_1=\ldots=y_r=o \Rightarrow \{C_1,\ldots,C_r\}$ is LI set, by Remark1.4, there exist vector $D_1,\ldots,D_{n-r}$ such that $\{C_1,\ldots,C_r, D_1,\ldots,D_{n-r}\}$ is a basis for W.

$T(A_1)=C_1=1.C_1+oC_2+\ldots+oC_r+oD_1+\ldots+oD_{n-r}$

.

.

.

$T(A_r)=C_r=oC_1+oC_2+\ldots+1C_r+oD_1+\ldots+oD_{n-r}$

$T(B_1)=O=oC_1+oC_2+\ldots+oC_r+oD_1+\ldots+oD_{n-r}$

.

.

.

$T(B_{m-r})=O=oC_1+oC_2+\ldots+oC_r+oD_1+\ldots+oD_{n-r}$.

The matrix for T relative to the basis $\{A_1,\dots,A_r, B_1,\dots,B_{m-r}\}$ and $\{C_1,\dots,C_r, D_1,\dots,D_{n-r}\}$ for V and W respectively is $M_T=\begin{bmatrix} Ir & O \\ O & O \end{bmatrix}$.

Example: Let $T\in L(R^3, R^2)$ defined by $T(x,y,z)=(x+y,x+z)$. Find a basis for R3 and R^2 which the matrix for T relative to this basis is normal form.

Solution: $\text{KerT}=\{(x,y,z)\in R^3:T(x,y,z)=(0,0)\}=\{(x,y,z)\in R^3: (x+y,x+z)=(0,0)\}=\{(x,y,z)\in R^3: x+y=0 \text{ and } x+z=0\}=\{(x,y,z)\in R^3: -x=y=z\}\Rightarrow$ basis for KerT is $\{(1,-1,-1)\}$. By extending it to a basis for R^3 $\{(1,0,0),(0,1,0),(1,-1,-1)\}\Rightarrow C_1=T(1,0,0)=(1,1)$ and $C_2=T(0,1,0)=(1,0)$

The matrix for T relative to the basis $\{(1,0,0),(0,1,0),(1,-1,-1)\}$ and $\{(1,0),(0,1)\}$ for R^2 and R^3 respectively is $M_T=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Remark2.7: Let $M,M^*\in M_{mn}(F)$ and dimV=m, dimW=n. M,M^* are matrices for a $T\in L(V,W)$ relative to distinct pair of basis of V,W iff there exist invertable matrices $P\in M_{mm}(F)$ and $Q\in M_{nn}(F)$ such that $M^*=PMQ^{-1}$.