Chapter three

Definition3.1: Let V be a vector space over F and $T\in L(V,V)$. Let $A\in V\setminus\{O\}$ and $\lambda\in F$, if $T(A)=\lambda A$, then A is called eigenvector associate with eigenvalue λ and λ is called eigenvalue associate with eigenvector A.

Associate with=with respect to

Eigenvalue=propervalue=characteristicvalue=latentvalue

Example: Let $V=R^2$ and $T\in L(R^2,R^2)$ defined by $T(x,y)=(x+y,2y)$. Determine some eigenvalue.

$T(1,1)=(2,2)=2(1,1) \Rightarrow (1,1)$ is an eigenvector associate with eigenvalue $\lambda=2$.

$T(5,5)=(10,10)=2(5,5)\Rightarrow (5,5)$ is an eigenvector associate with eigenvalue $\lambda=2$

Example: Let $V=\{f: f:R\to R$ is a differentiable function$\}$ and $F=R$. Let $T\in L(V,V)$ defined by $T(f)=f'$. Determine some eigenvalue.

Solution: Let $\lambda\in R$ and $A=f$. Since $T(A)=\lambda A \Rightarrow f'=\lambda f \Rightarrow f'(x)=\lambda f(x)\ \forall x\in R \Rightarrow \frac{df}{f}=\lambda dx$ $\Rightarrow f(x)=Ce^{\lambda x}$. If $C=1\Rightarrow f(x)=e^{\lambda x}$. The function f defined by $f(x)=e^{\lambda x}\ \forall x\in R$ is an eigenvector associate with eigenvalue λ.

Theorem3.1: Let V be n-dimensional vector space over F and $T\in L(V,V)$. Let M,M^* be a matrix of a linear transformation associate with basis $G=\{A_1,\dots,A_n\}$, $G^*=\{A_1^*,\dots,A_n^*\}$ respectively then for each $\lambda\in F$, $|M-\lambda I|=|M^*-\lambda I|$ where I is n-by-n identity matrix.

Proof: Since M,M^* are matrix of a linear transformation T, by Remark2.7, there exist a non-singular matrix P such that $M^*=PMP^{-1}\Rightarrow |M^*-\lambda I|=|PMP^{-1}-\lambda I|=|PMP^{-1}-P\lambda IP^{-1}|=|P(M-\lambda I)P^{-1}|=$ $|P|\,|M-\lambda I|\,|P^{-1}|=|M-\lambda I|\,|P|\,|P^{-1}|=|M-\lambda I|\,|PP^{-1}|=|M-\lambda I|\,|I|=|M-\lambda I|\,1=|M-\lambda I|$.

Let $M=(a_{ij})$ be a matrix of a linear transformation associated with any basis of V. $|M-\lambda I|=$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$ by expanding this determinant

$|M-\lambda I|=(-1)^n\lambda^n+b_1\lambda^{n-1}+\dots+b_{n-1}\lambda+b_n$ by Theorem3.1, if we replace M by M* associated with any basis of V then the result $(-1)^n\lambda^n+b_1\lambda^{n-1}+\dots+b_{n-1}\lambda+b_n$ not change.

Definition3.2: Let V be n-dimensional vector space over F and $T\in L(V,V)$. Let M be a matrix of a linear transformation for T associate with any basis for V. Let $\Delta(t)=|M-tI|$ $=(-1)^n t^n+b_1 t^{n-1}+\ldots+b_{n-1}t+b_n$ is called characteristic polynomial for T.

Example3.1: Let $V=R^2$ and $T\in L(R^2,R^2)$ defined by $T(x,y)=(y,-x)$. Find characteristic polynomial for T.

Solution: The standard basis of R^2 is {(1,0),(0,1)}

T(1,0)=(0,-1)=0(1,0)+(-1)(0,1)

T(0,1)=(1,0)=1(1,0)+0(0,1)

then $M=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$, $\Delta(t)=|M-\lambda I|=\begin{vmatrix}-t & -1\\ 1 & -t\end{vmatrix}=t^2+1$

Theorem3.2: Let V be n-dimensional vector space over F, $T\in L(V,V)$ and $\Delta(t)$ be a characteristic polynomial for T. λ is an eigenvalue for T iff λ is a root of $\Delta(t)$ [$\Delta(\lambda)=0$].

Proof: Let λ be eigenvalue of T $\Rightarrow$ there exist a non-zero vector $A\in V$ such that $T(A)=\lambda A \Rightarrow$ $T(A)=\lambda I(A)$ when I is the identity transformation$\Rightarrow(T-\lambda I)(A)=O\Rightarrow A\in Ker(T-\lambda I)\Rightarrow Ker(T-\lambda I)$ $\neq\{O\}$, by Theorem2.5 and Theorem2.11, $T-\lambda I$ is not isomorphism. Let M be a matrix of linear transformation for T associate with the basis $S=\{A_1,\ldots,A_n\}$, by Remark2.5, $M-\lambda I$ a matrix of linear transformation for $T-\lambda I$, since $T-\lambda I$ is not isomorphism, by Remark2.6, $M-\lambda I$ is not invertable $\Rightarrow |M-\lambda I|=0\Rightarrow\Delta(\lambda)=0\Rightarrow\lambda$ is a root of $\Delta(t)$.

Conversely suppose that $\Delta(\lambda)=0\Rightarrow |M-\lambda I|=0\Rightarrow$ $M-\lambda I$ is not invertable , by Remark2.6, $T-\lambda I$ is not isomorphism by Theorem2.5, $Ker(T-\lambda I)\neq\{O\}\Rightarrow$ there exist a non-zero vector $A\in V\setminus\{O\}$ such that $(T-\lambda I)(A)=O\Rightarrow T(A)=\lambda I(A)\Rightarrow T(A)=\lambda A$ and $A\neq O\Rightarrow\lambda$ be eigenvalue of T.

Definition3.3: Let V be n-dimensional vector space over F and $T\in L(V,V)$. Let M be a matrix of a linear transformation associate with any basis of V. The equation $\Delta(t)=|M-\lambda I|=0$ is called characteristic equation for T.

Example: In example3.1, $\Delta(t)=t^2+1$, $\Delta(t)\neq 0\ \forall t\in R\Rightarrow$ T has no eigenvalue.

Example3.2: Let $V=R^2$ and $T\in L(R^2,R^2)$ defined by $T(x,y)=(x+y,2y)$. Find eigenvalues and eigenvectors for T.

Solution: The standard basis of R^2 is {(1,0),(0,1)}

T(1,0)=(1,0)=1(1,0)+0(0,1)

T(0,1)=(1,2)=1(1,0)+2(0,1)

then $M=\begin{bmatrix}1 & 0\\ 1 & 2\end{bmatrix}$, $\Delta(t)=|M-\lambda I|=\begin{vmatrix}1-t & 0\\ 1 & 2-t\end{vmatrix}$=(1-t)(2-t)⇒ (1-t)(2-t)=0 is a characteristic equation for T⇒t=1 or t=2 are eigenvalue for T

To find eigenvectors,let (x,y)(M-λI)=(0,0)⇒(x,y) $\begin{bmatrix}1-\lambda & 0\\ 1 & 2-\lambda\end{bmatrix}$=(0,0)⇒(1-λ)x+y=0, (2-λ)y=0

Let λ=1, by substitution ⇒ y=0 and y=0⇒y=0. Put x=1, Therefore, (1,0) is an eigenvector associate with eigenvalue λ=1.

Let λ=2, by substitution ⇒ -x+y=0 and 0=0⇒x=y. Put x=1⇒y=1. Therefore, (1,1) is an eigenvector associate with eigenvalue λ=2.

Example: Let $V=C^2$, F=C and $T\in L(C^2,C^2)$ defined by T(x,y)=(y,-x). Find eigenvalues and eigenvectors for T.

Solution: The standard basis of C^2 is {(1,0),(0,1)}

T(1,0)=(0,-1)=0(1,0)+(-1)(0,1)

T(0,1)=(1,0)=1(1,0)+0(0,1)

then $M=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$, $\Delta(t)=|M-\lambda I|=\begin{vmatrix}-t & -1\\ 1 & -t\end{vmatrix}=t^2+1$ is a characteristic polynomial for T. $t^2+1=0$ is a characteristic equation⇒t=i and t=-i are eigenvalue for T.

To find eigenvectors, let (x,y)(M-λI)=(0,0)⇒(x,y) $\begin{bmatrix}-\lambda & -1\\ 1 & -\lambda\end{bmatrix}$=(0,0)⇒ -λx+y=0, -x-λy=0

Let λ=i, by substitution ⇒ -ix+y=0 and –x-iy=0. Put x=1⇒y=i, Therefore, (1,i) is an eigenvector associate with eigenvalue λ=i.

Let λ=-i, by substitution ⇒ ix+y=0 and –x+iy=0. Put x=1⇒y=-i. Therefore, (1,-i) is an eigenvector associate with eigenvalue λ=-i.

Example: Let $V=C^2$, F=R and $T\in L(C^2,C^2)$ defined by T(x,y)=(y,-x). Find eigenvalues and eigenvectors for T.

Solution: The standard basis of C^2 over R is {(1,0),(i,0),(0,1),(0,i)}

T(1,0)=(0,-1)=0(1,0)+0(i,0)+(-1)(0,1)+0(0,i)

T(i,0)=(0,-i)=0(1,0)+0(i,0)+0(0,1)+(-1)(0,i)

T(0,1)=(1,0)=1(1,0)+0(i,0)+0(0,1)+0(0,i)

T(0,i)=(i,0)=0(1,0)+1(i,0)+0(0,1)+0(0,i)

Then $M=\begin{bmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{bmatrix}$, $\Delta(t)=|M-\lambda I|=\begin{vmatrix}-t&0&-1&0\\0&-t&0&-1\\1&0&-t&0\\0&1&0&-t\end{vmatrix}=(-t)\begin{vmatrix}-t&0&-1\\0&-t&0\\1&0&-t\end{vmatrix}+(-1)\begin{vmatrix}0&-t&-1\\1&0&0\\0&1&-t\end{vmatrix}$

$=(-t)(-t)\begin{vmatrix}-t&-1\\1&-t\end{vmatrix}+(-1)(-1)\begin{vmatrix}-t&-1\\1&-t\end{vmatrix}=t^2(t^2+1)+(t^2+1)=(t^2+1)(t^2+1)=(t^2+1)^2$. $\Delta(t)\neq 0$ $\forall t\in R \Rightarrow$ T has no eigenvalue.

Remark3.1: Let V be n-dimensional vector space over F and $T\in L(V,V)$. T has at most n distinct eigenvalues.

Theorem3.3: Let V be a vector space over F, $T\in L(V,V)$ and λ be eigenvalue for T. Let $V_\lambda=\{A\in V: T(A)=\lambda A\}$, then V_λ is a subspace of V.

Proof: Let $A,B\in V_\lambda$ and $r\in F\Rightarrow$ $T(A)=\lambda A$ and $T(B)=\lambda B$ $\Rightarrow$ $T(A+B)=T(A)+T(B)=\lambda A+\lambda B=\lambda(A+B)\Rightarrow A+B\in V_\lambda$. Since $T(rA)=rT(A)=r(\lambda A)=\lambda(rA)$ $\Rightarrow$ $rA\in V_\lambda$. Therefore, V_λ is a subspace of V.

Definition3.4: Let V be vector space over F, $T\in L(V,V)$ and λ be eigenvalue for T. The subspace V_λ is called eigenspace associate with eigenvalue λ and the dimension of V_λ is called geometric multiplicity of λ and denoted by $GM(\lambda)$.

Definition3.5: Let V be n-dimensional vector space over F, $T\in L(V,V)$ and λ be eigenvalue for T. Algebraic multiplicity of λ is defined to be its multiplicity as a root of characteristic equation and denoted by $AM(\lambda)$.

Remark3.2: $AM(\lambda)=n$ iff $\Delta(t)=(t-\lambda)^n g(t)$ when g(t) is a polynomial of t and $g(\lambda)\neq 0$.

Example3.3: Let $V=R^2$ and $T\in L(R^2,R^2)$ defined by $T(x,y)=(x,x+y)$. Find Algebraic multiplicity and geometric multiplicity of each eigenvalues for T.

Solution: The standard basis of R^2 is $\{(1,0),(0,1)\}$

$T(1,0)=(1,1)=1(1,0)+1(0,1)$

$T(0,1)=(0,1)=0(1,0)+1(0,1)$

then $M=\begin{bmatrix}1&1\\0&1\end{bmatrix}$, $\Delta(t)=|M-\lambda I|=\begin{vmatrix}1-t&1\\0&1-t\end{vmatrix}=(1-t)(1-t)\Rightarrow$ $(1-t)(1-t)=0$ is a characteristic equation for T $\Rightarrow$ t=1 is only eigenvalue for T. Since $\Delta(t)=(1-t)^2\Rightarrow AM(1)=2$.

Let $(x,y)(M-\lambda I)=(0,0)\Rightarrow(x,y)\begin{bmatrix}1-\lambda&1\\0&1-\lambda\end{bmatrix}=(0,0)\Rightarrow(1-\lambda)x=0,\ x+(1-\lambda)y=0$

Let $\lambda=1$, by substitution $\Rightarrow$ 0=0 and $x=0 \Rightarrow x=0$. Put y=1, Therefore, (0,1) is an eigenvector associate with eigenvalue $\lambda=1$.

$V_1=\{(x,y)\in R^2: T(x,y)=1.(x,y)\}=\{(x,y)\in R^2: (x,x+y)=(x,y)\}=\{(x,y)\in R^2: x=x \text{ and } x+y=y\} =\{(x,y)\in R^2: x=0\} \Rightarrow \dim V_1=1 \Rightarrow GM(1)=1$. $GM(1)\neq AM(1)$.

Theorem3.4: Let V be n-dimensional vector space over F, $T\in L(V,V)$ and λ is an eigenvalue for T, then $GM(\lambda)\leq AM(\lambda)$.

Proof: Let $GM(\lambda)=\dim V_\lambda=k$ and $\{A_1,\ldots,A_k\}$ be any basis for V_λ. $T(A_1)=\lambda A_1,\ldots, T(A_k)=\lambda A_k$ by extending $\{A_1,\ldots,A_k\}$ to $S=\{A_1,\ldots,A_k,A_{k+1},\ldots,A_n\}$ be a basis for V. The matrix for T with respect to S is

$$\begin{bmatrix} \lambda & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda & 0 & \cdots & 0 \\ a_{k+11} & a_{k+12} & \cdots & a_{k+1k} & a_{k+1k+1} & \cdots & a_{k+1n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} & a_{nk+1} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \lambda I_k & O \\ B & C \end{bmatrix}$$

Where I_k is the k-by-k identity matrix, O is k-by-(n-k) zero matrix, B is the (n-k)-by-k matrix, C is the (n-k)-by-(n-k) matrix.

$\Delta(t)=|M-\lambda I|=\begin{vmatrix} (\lambda-t)I_k & 0 \\ B & C-tI_{n-k} \end{vmatrix} = |(\lambda-t)I_k|\,|C-tI_{n-k}|=(\lambda\text{-}t)^k|C-tI_{n-k}| \Rightarrow(\lambda\text{-}t)^k$ is one factor of $\Delta(t)\Rightarrow k\leq AM(\lambda)\Rightarrow GM(\lambda)\leq AM(\lambda)$.

Theorem3.5: Let V be n-dimensional vector space over F, $T\in L(V,V)$ and λ is an eigenvalue for T, if $AM(\lambda)=1$, then $GM(\lambda)=1$.

Proof: By Theorem3.4, $GM(\lambda)\leq AM(\lambda)=1$, since λ is an eigenvalue for T, then there exist $A\in V\backslash\{O\}$, such that $T(A)=\lambda A$. $A\in V_\lambda$ and $A\neq O \Rightarrow \dim V_\lambda\neq 0 \Rightarrow GM(\lambda)\neq 0 \Rightarrow 0<GM(\lambda)\leq 1\Rightarrow GM(\lambda)=1$.

Theorem3.6: Let V be a vector space over F and $T\in L(V,V)$. Let $A_1,\ldots,A_m$ be eigenvectors of T associate with distinct eigenvalues $\lambda_1,\ldots,\lambda_m$. Then $\{A_1,\ldots,A_m\}$ is linearly independent.

Proof: By induction on m. For m=1, $A_1\neq O$, $\{A_1\}$ is LI set. Suppose that the Theorem is true for m-1. We must prove for m. Let $c_1A_1+c_2A_2+\ldots+c_mA_m=O$ …(1) we must prove that $c_1=\ldots=c_m=o$. Multiplying (1) by λ_m to obtain $c_1\lambda_mA_1+c_2\lambda_mA_2+\ldots+c_m\lambda_mA_m=O$…(2). Also apply T to (1) by linearity of T we get $c_1T(A_1)+c_2T(A_2)+\ldots+c_mT(A_m)=O\Rightarrow$ $c_1\lambda_1A_1+c_2\lambda_2A_2+\ldots+c_m\lambda_mA_m=O$…(3). By subtract(3) from (2) we obtain $c_1(\lambda_1-\lambda_m)A_1+c_2(\lambda_2-\lambda_m)A_2+\ldots+c_{m-1}(\lambda_{m-1}-\lambda_m)A_{m-1}=O$. By induction hypothesis $c_i(\lambda_i-\lambda_m)=o$ for $i=1,\ldots,m-1$. Since $\lambda_i-\lambda_m\neq o$ for $i=1,\ldots,m-1\Rightarrow c_i=o$ for $i=1,\ldots,m-1$. By substitute value of c_i in (1), we get $c_mA_m=O$, but $A_m\neq O\Rightarrow c_m=o\Rightarrow c_i=o$ for $i=1,\ldots,m$.

Theorem3.7: Let V be a vector space over F and $T\in L(V,V)$. Let λ,σ be distinct eigenvalues for T. Then $V_\lambda\cap V_\sigma=\{O\}$.

Proof: Let $A\in V_\lambda\cap V_\sigma\Rightarrow A\in V_\lambda$ and $A\in V_\sigma\Rightarrow T(A)=\lambda A$ and $T(A)=\sigma A\Rightarrow\lambda A=\sigma A\Rightarrow(\lambda-\sigma)A=O$. Since $\lambda\neq\sigma\Rightarrow\lambda-\sigma\neq o\Rightarrow A=O\Rightarrow V_\lambda\cap V_\sigma\subset\{O\}$. Since every subspace containing O $\Rightarrow\{O\}\subset V_\lambda\cap V_\sigma$. Therefore, $V_\lambda\cap V_\sigma=\{O\}$.

Definition3.6: Let V be finite dimensional vector space over F and $T\in L(V,V)$. A basis S of V is said to be diagonalize T if the matrix associate with T relative to S is diagonal matrix.

Definition3.7: Let V be finite dimensional vector space over F and $T\in L(V,V)$, if there exist a basis S of V which diagonalize T, then T is said to be diagonalizable.

Example: Let $V=R^2$ and $T\in L(R^2,R^2)$ defined by $T(x,y)=(y,x)$. Let $S_1=\{(1,0),(0,1)\}$, $S_2=\{(0,1),(1,0)\}$, $S_3=\{(1,1),(2,3)\}$ and $S_4=\{(1,1),(1,-1)\}$ which of these basis diagonalize T.

Solution:

$T(1,0)=(0,1)=0(1,0)+1(0,1)$

$T(0,1)=(1,0)=1(1,0)+0(0,1)\Rightarrow M_T=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ which is not diagonal matrix.

$T(0,1)=(1,0)=0(0,1)+1(1,0)$

$T(1,0)=(0,1)=1(0,1)+0(1,0)\Rightarrow M_T=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ which is not diagonal matrix.

$T(1,1)=(1,1)=1(1,1)+0(2,3)$

$T(2,3)=(3,2)=5(1,1)+(-1)(2,3)\Rightarrow M_T=\begin{bmatrix}1&0\\5&-1\end{bmatrix}$ which is not diagonal matrix.

$T(1,1)=(1,1)=1(1,1)+0(1,-1)$

$T(1,-1)=(-1,1)=0(1,1)+(-1)(1,-1)\Rightarrow M_T=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ which is diagonal matrix$\Rightarrow S_4$ diagonalize T.

Remark3.3: Let V be n-dimensional vector space over F and $T\in L(V,V)$. T is diagonalizable iff

1- The characteristic polynomial of T $\Delta(t)$ must be of the form $\Delta(t)=\alpha(t-\lambda_1)^{r1}\ldots(t-\lambda_k)^{rk}$ such that $r_1+\ldots+r_k=n$, $\lambda_i\in F$. $\lambda_i\neq\lambda_j$ whenever $i\neq j$, $\alpha\in F\backslash\{o\}$.

2- For each eigenvalue λ_i of T, $AM(\lambda_i)=GM(\lambda_i)$ for i=1,…,k.

Example: In Example3.2, T is diagnalizable since satisfies the conditions of Remark3.3.

Example: In Example3.3, T is not diagnalizable since $AM(1)\neq GM(1)$.

Example: In Example3.1, T is not diagnalizable since $\Delta(t)=t^2+1\neq 0\ \forall t\in R$.

Remark3.4: Let V be n-dimensional vector space over F and $T\in L(V,V)$. T is diagonalizable iff there exist a basis for V consisting of eigenvectors of T.

Theorem3.8: Let V be n-dimensional vector space over F and $T\in L(V,V)$. If T has n distinct eigenvalues, then T diagonalizable.

Proof: Let $\lambda_1,\ldots,\lambda_n$ be n-distinct eigenvalues of T, there exist $A_i\neq O$ such that $T(A_i)=\lambda_i A_i$ for i=1,…,n. by Theorem3.6, $S=\{A_1,\ldots,A_n\}$ is linearly independent , by Theorem1.12, S is a basis for V. Therefore, V has a basis for eigenvector, by Remark3.4, T is diagonalizable.

Example: Let $V=R^2$ and $T\in L(R^2,R^2)$ defined by $T(x,y)=(y,x)$. Find a basis which diagonalizes T.

Solution: The standard basis of R^2 is {(1,0),(0,1)}

T(1,0)=(0,1)=0(1,0)+1(0,1)

T(0,1)=(1,0)=1(1,0)+0(0,1)

then $M=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $\Delta(t)=|M-\lambda I|=\begin{vmatrix}-t & 1\\ 1 & -t\end{vmatrix}=t^2-1$ is a characteristic polynomial for T. $t^2-1=0$ is a characteristic equation$\Rightarrow$ t=1 and t=-1 are eigenvalue for T.

To find eigenvectors, let $(x,y)(M-\lambda I)=(0,0)\Rightarrow(x,y)\begin{bmatrix}-\lambda & 1\\ 1 & -\lambda\end{bmatrix}=(0,0)\Rightarrow-\lambda x+y=0,\ x-\lambda y=0$

Let $\lambda=1$, by substitution $\Rightarrow$ -x+y=0 and x-y=0$\Rightarrow$x=y. Put x=1$\Rightarrow$y=1, Therefore, (1,1) is an eigenvector associate with eigenvalue $\lambda=1$.

Let $\lambda=-1$, by substitution $\Rightarrow$ x+y=0 and x+y=0$\Rightarrow$x=-y. Put x=1$\Rightarrow$y=-1, Therefore, (1,-1) is an eigenvector associate with eigenvalue $\lambda=1$. By Theorem3.6, {(1,1),(1,-1)} is a LI set. Since $\dim R^2=2$. By Theorem1.12, {(1,1),(1,-1)} is a basis for R^2 $\Rightarrow$ the basis {(1,1),(1,-1)} diagonalize T.

Definition3.8: A matrix B is said to be similar to a matrix A if there exist a non-singular matrix P such that $B=P^{-1}AP$.

Example: Let $A=\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Choose $P=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow P^{-1}=\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow P^{-1}AP=B$.

Theorem3.9:

1- A is similar to A.

2- If B is similar to A then A is similar to B.

3- If A is similar to B and B is similar to C, then A is similar to C.

Proof of (1): Since $A=I^{-1}AI$.

Proof of (2): Since B is similar to A, there exist a non-singular matrix P such that $B=P^{-1}AP$ $\Rightarrow A=PBP^{-1}=(P^{-1})^{-1}BP^{-1} \Rightarrow$ A is similar to B

Proof of (3): Since A is similar to B, there exist a non-singular matrix P such that $A=P^{-1}BP$. Since B is similar to C, there exist a non-singular matrix Q such that $B=Q^{-1}CQ \Rightarrow A=P^{-1}BP= P^{-1}Q^{-1}CQP=(QP)^{-1}C(QP) \Rightarrow$ A is similar to C.

From property (2) we replaced statement " A is similar to B" or " B is similar to A" by A and B are similar.

Definition3.9: A matrix A is said to be diagonalizable if it is similar to a diagonal matrix.

Example: In the previous example A, B are similar, then A is diagonalizable.

Definition3.10: Let A be n-by-n matrix in F. $\lambda \in F$ is called eigenvalue of A if there exist a non-zero $X=(x_1,\ldots,x_n)\in F^n$ such that $XA=\lambda X$. X is called eigenvector associate with eigenvalue λ.

Example: Let $A=\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$, $\lambda=2$ and $X=(2,-1) \Rightarrow XA=2X \Rightarrow \lambda=2$ is an eigenvalue for A.

Remark3.5: An n-by-n matrix A is diagonalizale iff has n linearly independent eigenvectors.

Example: Find eigenvalues and eigenvectors of $A=\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$. Test whether A is diagonalizable

Solution: $\Delta(t)=|A-\lambda I|=\begin{vmatrix} 1-t & 1 \\ -2 & 4-t \end{vmatrix}=(1-t)(4-t)+2=t^2+-5t+6=(t-2)(t-3)$, $\Rightarrow (t-2)(t-3)=0$ is a characteristic equation for $A \Rightarrow t=2$ or $t=3$ are eigenvalue for A

To find eigenvectors of A, let $(x,y)(A-\lambda I)=(0,0) \Rightarrow (x,y)\begin{bmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{bmatrix}=(0,0) \Rightarrow (1-\lambda)x-2y=0$, $x+(4-\lambda)y=0$.

Let $\lambda=2$, by substitution $-x-2y=0$ and $x+2y=0 \Rightarrow x=-2y$. Put y=1, then x=-2. Therefore, (-2,1) is an eigenvector associate with eigenvalue $\lambda=2$.

Let $\lambda=3$, by substitution $\Rightarrow$ $-2x-2y=0$ and $x+y=0 \Rightarrow x=-y$. Put $y=1 \Rightarrow x=-1$. Therefore, (-1,1) is an eigenvector associate with eigenvalue $\lambda=3$. Let $P=\begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow P^{-1}=\begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} \Rightarrow PAP^{-1}=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$\Rightarrow$ A is diagonalizable.

Remark3.6:"Cayley-Hamilton Theorem": Every matrix satisfies its characteristic equation.

If A is n-by-n matrix and it is a characteristic equation $t^n+a_1t^{n-1}+\ldots+a_{n-1}t+a_n=0$, then $A^n+a_1A^{n-1}+\ldots+a_{n-1}A+a_nI=O$.

Example: Let $A=\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$, then $|A-\lambda I|=\begin{vmatrix} 3-t & 2 \\ 4 & 3-t \end{vmatrix}=(3-t)(3-t)-8=t^2-6t+1 \Rightarrow$ $t^2-6t+1=0$ is a characteristic equation for A, by Cayley-Hamilton Theorem $A^2-6A+I=O$.

Note: In characteristic equation if $a_n=0$, then $|A|=0 \Rightarrow$ A is singular. If $a_n \neq 0$, then we can find A^{-1} directly as $A^n+a_1A^{n-1}+\ldots+a_{n-1}A+a_nI=O \Rightarrow A^n+a_1A^{n-1}+\ldots+a_{n-1}A=-a_nI=O \Rightarrow$

$A(-\frac{1}{a_n}(A^{n-1}+a_1A^{n-2}+\ldots+a_{n-1}I))=I \Rightarrow A^{-1}=-\frac{1}{a_n}(A^{n-1}+a_1A^{n-2}+\ldots+a_{n-1}I)$.

Example: Let $A=\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$, by using Cayley-Hamilton Theorem find A^{-1}.

Solution: In the previous example $t^2-6t+1=0$ is a characteristic equation for A, By Cayley-Hamilton Theorem $A^2-6A+I=O \Rightarrow A^{-1}=6I-A \Rightarrow A^{-1}=\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}-\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}=\begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$.

It is possible to express A^k $(k \geq n)$ by a polynomial in A of degree not exceeding k-1. Since $A^n+a_1A^{n-1}+\ldots+a_{n-1}A+a_nI=O$, by multiplication $A^{k-n} \Rightarrow A^k+a_1A^{k-1}+\ldots+a_{n-1}A^{k-n+1}+a_nA^{k-n}=O \Rightarrow$ $A^k=-(a_1A^{k-1}+\ldots+a_{n-1}A^{k-n+1}+a_nA^{k-n})$. By applying this process we can express A^k.

Example: Let $A=\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, by using Cayley-Hamilton Theorem find A^4.

Solution: $|A-tI|=\begin{vmatrix}-t & 1\\ 1 & 1-t\end{vmatrix}$=(-t)(1-t)-1=$t^2$-t-1$\Rightarrow$ t^2-t-1=0 is a characteristic equation for A, by Cayley-Hamilton Theorem A^2-A-I=O $\Rightarrow$ A^2=A+I $\Rightarrow$ $A^4=A^2A^2$=(A+I)(A+I)=A^2+2A+I=A+I+ 2A+I=3A+2I=$\begin{bmatrix}0 & 3\\ 3 & 3\end{bmatrix}+\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}=\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}$.

Example: Let A=$\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$, by using Cayley-Hamilton Theorem show that $A^5=2A^2$+A-2I.

Solution: $|A-tI|=\begin{vmatrix}-t & 0 & 1\\ 0 & 1-t & 0\\ 1 & 0 & -t\end{vmatrix}$=(1-t)$\begin{vmatrix}-t & 1\\ 1 & -t\end{vmatrix}$=(1-t)($t^2$-1)=$-t^3+t^2$+t-1$\Rightarrow$ $-t^3+t^2$+t-1=0 is a characteristic equation for A, by Cayley-Hamilton Theorem $-A^3+A^2$+A-I=O $\Rightarrow$ $A^3=A^2$+A-I $\Rightarrow$ $A^4=AA^3$=A(A^2+A-I)=A^3+A^2-A=A^2+A-I+A^2-A=$2A^2$-I. $A^5=AA^4$=A($2A^2$-I)=$2A^3$-A =2(A^2+A-I)-A=$2A^2$+2A-2I-A=$2A^2$+A-2I.