Chapter four

Definition4.1: Let V be any real vector space. An inner product on V is a function $\langle,\rangle: V\times V\to R$ that assigns to each pairs of vectors A,B of V a real number $\langle A,B\rangle$ satisfying

1- For all $A,B\in V$, $\langle A,B\rangle=\langle B,A\rangle$

2- For all $A,B\in V$ and $r\in R$, $\langle rA,B\rangle=r\langle A,B\rangle$

3- For all $A,B,C\in V$, $\langle A+B,C\rangle=\langle A,C\rangle+\langle B,C\rangle$

4- For all $A\in V$, 1-$\langle A,A\rangle\geq 0$ 2-If $\langle A,A\rangle=0$, then A=O.

Example4.1: Let $V=R^2$ and $A=(x_1,x_2)$, $B=(y_1,y_2)\in R^2$, define $\langle,\rangle: R^2\times R^2\to R$ by $\langle A,B\rangle=x_1y_1+x_2y_2$. Show that $\langle,\rangle$ is an inner product on R^2.

(It is called standard inner product on R^2)

Solution: Let $A=(x_1,x_2)$, $B=(y_1,y_2)$, $C=(z_1,z_2)\in R^2$ and $r\in R$

1- $\langle A,B\rangle= x_1y_1+x_2y_2= y_1x_1+y_2x_2=\langle B,A\rangle$

2- $\langle rA,B\rangle=(rx_1)y_1+(rx_2)y_2=r(x_1y_1+x_2y_2)= r\langle A,B\rangle$

3- $\langle A+B,C\rangle= (x_1+y_1)z_1+(x_2+y_2)z_2= x_1z_1+x_2z_2+ y_1z_1+y_2z_2=\langle A,C\rangle+\langle B,C\rangle$

4- $\langle A,A\rangle=x_1^2+x_2^2\geq 0$ and If $\langle A,A\rangle=0$, then $x_1^2+x_2^2=0$, then $x_1=x_2=0$, then A=O.

Example4.2: Let $V=R^2$ and $A=(x_1,x_2)$, $B=(y_1,y_2)\in R^2$, define $\langle,\rangle: R^2\times R^2\to R$ by $\langle A,B\rangle=x_1y_1-x_2y_1-x_1y_2+3x_2y_2$. Show that $\langle,\rangle$ is an inner product on R^2.

Solution: Let $A=(x_1,x_2)$, $B=(y_1,y_2)$, $C=(z_1,z_2)\in R^2$ and $r\in R$

1- $\langle A,B\rangle= x_1y_1-x_2y_1-x_1y_2+3x_2y_2= y_1x_1-y_1x_2-y_2x_1+3y_2x_2=\langle B,A\rangle$

2- $\langle rA,B\rangle=(rx_1)y_1-(rx_2)y_1-(rx_1)y_2+(3rx_2)y_2=r(x_1y_1-x_2y_1-x_1y_2+3x_2y_2)= r\langle A,B\rangle$

3- $\langle A+B,C\rangle=(x_1+y_1)z_1-(x_2+y_2)z_1-(x_1+y_1)z_2+3(x_2+y_2)z_2=x_1z_1-x_2z_1-x_1z_2+3x_2z_2+y_1z_1-y_2z_1-y_1z_2+3y_2z_2=\langle A,C\rangle+\langle B,C\rangle$

4- $\langle A,A\rangle=x_1^2-x_2x_1-x_1x_2+3x_2^2= x_1^2-2x_1x_2+3x_2^2= x_1^2-2x_1x_2+x_2^2+2x_2^2=(x_1-x_2)^2+2x_2^2\geq 0$ and if $\langle A,A\rangle=0$, then $(x_1-x_2)^2+2x_2^2=0$, then $x_1-x_2=0$ and $x_2=0$, then $x_1=x_2=0$, then A=O.

Example4.3: Let $V=R^n$ and $A=(x_1,\ldots,x_n)$, $B=(y_1,\ldots,y_n)\in R^n$, define $\langle,\rangle: R^n\times R^n\to R$ by $\langle A,B\rangle=x_1y_1+\ldots+x_ny_n$. Show that $\langle,\rangle$ is an inner product on R^n.

(It is called standard inner product on R^n)

Solution: Let $A=(x_1,\ldots,x_n)$, $B=(y_1,\ldots,y_n)$, $C=(z_1,\ldots,z_n)\in R^n$ and $r\in R$

1- $\langle A,B\rangle= x_1y_1+\ldots+x_ny_n= y_1x_1+\ldots+y_nx_n=\langle B,A\rangle$

2- $\langle rA,B\rangle=(rx_1)y_1+\ldots+(rx_n)y_n=r(x_1y_1+\ldots+x_ny_n)= r\langle A,B\rangle$

3- $\langle A+B,C\rangle= (x_1+y_1)z_1+\ldots+(x_n+y_n)z_n= x_1z_1+\ldots+x_nz_n+ y_1z_1+\ldots+y_nz_n=\langle A,C\rangle+\langle B,C\rangle$

4- $<A,A>=x_1^2+\ldots+x_n^2\geq 0$ and If $<A,A>=0$, then $x_1^2+\ldots+x_n^2=0$, then $x_1=\ldots=x_n=0$, then $A=O$.

Example4.4: Let $V=P_n(R)$ and $A(x),B(x)\in P_n(R)$, define $<,>: P_n(R)\times P_n(R)\to R$ by $<A(x),B(x)>=\int_0^1 A(x)B(x)dx$. Show that $<,>$ is an inner product on $P_n(R)$.

(It is called standard inner product on $P_n(R)$)

Solution: Let $A(x), B(x), C(x)\in P_n(R)$ and $r\in R$

1- $<A(x),B(x)>=\int_0^1 A(x)B(x)dx=\int_0^1 B(x)A(x)dx=<B(x),A(x)>$

2- $<rA(x),B(x)>=\int_0^1 (rA(x))B(x)dx=r\int_0^1 A(x)B(x)dx=r<A(x),B(x)>$

3- $<A(x)+B(x),C(x)>=\int_0^1 (A(x)+B(x))C(x)dx=\int_0^1 A(x)C(x)dx+\int_0^1 B(x)C(x)dx=<A(x),C(x)>+<B(x),C(x)>$

4- $<A(x),A(x)>=\int_0^1 A(x)A(x)dx=\int_0^1 A^2(x)dx\geq 0$ and If $<A(x,)A(x)>=0$, then $\int_0^1 A^2(x)dx=0$, then $A(x)=0\ \forall x\in[0,1]$, then the equation $A(x)=0$ has infinitely many solution, but this equation has at most n roots$\Rightarrow A(x)=0\ \forall x\in R\Rightarrow A(x)$ is zero polynomial.

Example4.5: Let $V=M_{mn}(R)$ and $A=(a_{ij}), B=(b_{ij})\in M_{mn}(R)$, define $<,>: M_{mn}(R)\times M_{mn}(R)\to R$ by $<A,B>=\sum_{j=1}^{n}\sum_{i=1}^{m}a_{ij}b_{ij}$. Show that $<,>$ is an inner product on $M_{mn}(R)$.

(It is called standard inner product on $M_{mn}(R)$)

Solution: (obvious).

Example4.6: Let V be n-dimensional vector space over R and $S=\{A_1,\ldots,A_n\}$ be a basis for V. Let $A,B\in V$, then there exist $x_1,\ldots,x_n,y_1,\ldots,y_n\in R$ such that $A=x_1A_1+\ldots+x_nA_n$ and $B=y_1A_1+\ldots+y_nA_n$ define$<,>: V\times V\to R$ by $<A,B>=x_1y_1+\ldots+x_ny_n$. Show that $<,>$ is an inner product on V.

Solution: (obvious).

Theorem4.1: Let V be any real vector space and $<,>$ be an inner product on V, then for all $A,B,C\in V$, $r\in R$ the following is hold

1- $<A,rB>=r<A,B>$

2- $<A,B+C>=<A,B>+<A,C>$

3- $<A,O>=<O,A>=o$

4- If $A\neq O$, then $<A,A>>0$.

Proof of (1): $<A,rB>=<rB,A>=r<B,A>=r<A,B>$.

Proof of (2): $<A,B+C>=<B+C,A>=<B,A>+<C,A>=<A,B>+<A,C>$.

Proof of (3): $<O,A>=<oO,A>=o<O,A>=o$ and from condition(1) it follows that $<A,O>=<O,A>=o$.

Proof of (4): If $A\neq O$. By Axiom4(1), $<A,A>\geq 0$. Suppose that $<A,A>=O$, by Axiom4(2), $A=O$, contradiction, then $<A,A>>O$.

Definition4.2: Let V be any real vector space which has defined on it an inner product $<,>$ is called Euclidean vector space or inner product space. We denoted by $<V,<,>>$.

Example: Let $V=R^2$ and $A=(x_1,x_2)$, $B=(y_1,y_2)\in R^2$, define $<,>_1$: $R^2\times R^2\to R$ by $<A,B>_1=x_1y_1+x_2y_2$, by Example4.1, $<,>_1$ is an inner product on R^2. Define $<,>_2$: $R^2\times R^2\to R$ by $<A,B>=x_1y_1-x_2y_1-x_1y_2+3x_2y_2$ by Example4.2, $<,>_2$ is an inner product on R^2. Therefore $<R^2,<,>_1>$ and $<R^2,<,>_2>$ are different Euclidean vector space.

Theorem4.2:"Cauchy-Schwarz inequality" If A and B are any two vectors in an Euclidean vector space V, then $<A,B>^2\leq<A,A><B,B>$.

Proof: Consider the vector A+xB where x is a scalar By Axiom4(1), $<A+xB,A+xB>\geq 0$. Since $<A+xB,A+xB>=<A,A>+2<A,B>x+<B,B>x^2$. Let $c=<A,A>$, $b=<A,B>$ and $a=<B,B>$ if we fix A and B, then $c+2bx+ax^2$ is a quadratic polynomials in x and $c+2bx+ax^2\geq 0\ \forall x\in R$. This means that this polynomial either has no real roots or if it has two equal real roots. These possibilities can only occur if the discriminant is ≤ 0. Hence $(2b)^2-4ac\leq 0\Rightarrow b^2\leq ac\Rightarrow$ $<A,B>^2\leq<A,A><B,B>$.

Definition4.3: Let $<V,<,>>$ be an Euclidean vector space, the length or norm of a vector $A\in V$ defined as $\|A\|=\sqrt{<A,A>}$.

Example: Let $<M_{22}(R),<,>>$ be an Euclidean vector space. Find $\left\|\begin{bmatrix}1 & 2\\ -1 & 3\end{bmatrix}\right\|$.

Solution: $\left\|\begin{bmatrix}1 & 2\\ -1 & 3\end{bmatrix}\right\|=\sqrt{1+4+1+9}=\sqrt{15}$

The Cauchy-Schwarz inequality can be reformulate as follows if A and B are any two vector in an Euclidean vector space $<V,<,>>$, then $|<A,B>|\leq\|A\|\,\|B\|$.

Proof: Since $<A,B>^2\leq<A,A><B,B>$, but $\|A\|^2=<A,A>$ and $\|B\|^2=<B,B>\Rightarrow$ $<A,B>^2\leq\|A\|^2\|B\|^2=(\|A\|\,\|B\|)^2$, by taking square root $|<A,B>|\leq\|A\|\,\|B\|$.

Theorem4.3: Let $<V,<,>>$ be an Euclidean vector space, then

1- For all $A\in V$, $\|A\|\geq 0$.

2- For all $A\in V$, $\|A\|=0$ iff $A=O$.

3- For all $A\in V$ and $r\in R$ $\|rA\|=|r|\|A\|$.

4- For all $A,B\in V$, $\|A+B\|\leq\|A\|+\|B\|$. (Triangle inequality)

Proof of (1): Since $\|A\|=\sqrt{<A,A>}\geq 0\Rightarrow\|A\|\geq 0$.

Proof of (2): If $\|A\|=0\Rightarrow <A,A>=0$, by Axiom4(2) A=O. If $A=O\Rightarrow\|A\|^2=<O,O>=0\Rightarrow\|A\|=0$. Therefore, $\|A\|=0$ iff A=O.

Proof of (3): $\|rA\|^2=<rA,rA>=rr<A,A>=r^2\|A\|^2$, by taking square root $\|rA\|=|r|\|A\|$

Proof of (4): $\|A+B\|^2=<A+B,A+B>=<A,A>+2<A,B>+<B,B>=\|A\|^2+2<A,B>+\|B\|^2\leq\|A\|^2+2|<A,B>|+\|B\|^2\leq\|A\|^2+2\|A\|\|B\|+\|B\|^2=(\|A\|+\|B\|)^2$, by taking square root, $\|A+B\|\leq\|A\|+\|B\|$.

Definition4.4: Let <V,<,>> be an Euclidean vector space, the distance between vectors $A,B\in V$ defined as $d(A,B)=\|A-B\|$.

Example: Let $<M_{22}(R),<,>>$ be an Euclidean vector space. Let $A=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}2 & 1\\ -1 & 3\end{bmatrix}$ find d(A,B)

Solution: $d(A,B)=\left\|\begin{bmatrix}-1 & 1\\ 3 & -2\end{bmatrix}\right\|=\sqrt{15}$.

Theorem4.4: Let <V,<,>> be an Euclidean vector space, then

1- For all $A,B\in V$, $d(A,B)\geq 0$.

2- For all $A,B\in V$, $d(A,B)=0$ iff A=B.

3- For all $A,B\in V$ $d(A,B)=d(B,A)$.

4- For all $A,B,C\in V$, $d(A,B)\leq d(A,C)+d(C,B)$.

Proof of (1): Since $d(A,B)=\|A-B\|\geq 0$ and from Theorem4.3(1).

Proof of (2): $A=B\Leftrightarrow A-B=0\Leftrightarrow\|A-B\|=0\Leftrightarrow d(A,B)=0$ and from Theorem4.3(2).

Proof of (3): $d(A,B)=\|A-B\|=\|(-1)(B-A)\|=|-1|\|B-A\|=d(B,A)$.

Proof of (4): $d(A,B)=\|A-B\|=\|A-C+C-B\|\leq\|A-C\|+\|C-B\|=d(A,C)+d(C,B)$.

Definition4.5: Let A and B are any two non-zero vector in an Euclidean vector space <V,<,>>. The angle θ between A and B defined $\cos\theta=\frac{<A,B>}{\|A\|\|B\|}$ where $0\leq\theta\leq\pi$.

Example: In $<P_1(R),<,>>$. Find angle between A(x)=x and B(x)=3x-2.

Solution: $\langle A(x),B(x)\rangle=\int_0^1 x(3x-2)dx=0 \Rightarrow \cos\theta=0 \Rightarrow \theta=\frac{\pi}{2}$.

Definition4.6: Let A and B are any two vector in an Euclidean vector space $\langle V,\langle,\rangle\rangle$. Then A and B are perpendicular or orthogonal if $\langle A,B\rangle=0$.

Theorem4.5: "Pythagorean Theorem" Let A and B are any two vector in an Euclidean vector space $\langle V,\langle,\rangle\rangle$. If A and B are perpendicular then $\|A+B\|^2=\|A\|^2+\|B\|^2$

Proof: Since $\|A+B\|^2=\langle A+B,A+B\rangle=\langle A,A\rangle+2\langle A,B\rangle+\langle B,B\rangle=\|A\|^2+0+\|B\|^2=\|A\|^2+\|B\|^2$

Definition4.7: Let S be a subset of an Euclidean vector space $\langle V,\langle,\rangle\rangle$. S is said to be orthogonal if any two distinct vector in S are orthogonal.

Definition4.8: Let S be a subset of an Euclidean vector space $\langle V,\langle,\rangle\rangle$. S is said to be orthonormal if S is orthogonal and each element of S has length 1.

Example: Let $A=\{(1,0,3),(-3,0,1),(0,1,0)\}$. Then A is an orthogonal basis of R^3 with usual inner product but it is not orthonormal.

Example: Let $A=\{(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}),(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})\}$. Then A is an orthonormal basis of R^2 with usual inner product.

Theorem4.6: Let $A=\{A_1,\ldots,A_n\}$ be an orthogonal set of non-zero vector in an Euclidean vector space $\langle V,\langle,\rangle\rangle$. Then A is linearly independent.

Proof: Let $c_1A_1+\ldots+c_nA_n=O$ …(1). Taking the inner product of both sides of (1) with A_i we get $\langle A_i,c_1A_1+\ldots+c_nA_n\rangle=\langle A_i,O\rangle=0$, then $c_1\langle A_i,A_1\rangle+\ldots+c_{i-1}\langle A_i,A_{i-1}\rangle+c_i\langle A_i,A_i\rangle+c_{i+1}\langle A_i, A_{i+1}\rangle+\ldots+c_n\langle A_i,A_n\rangle=0 \Rightarrow 0+\ldots+0+c_i\|A_i\|^2+0+\ldots+0=0 \Rightarrow c_i\|A_i\|^2=0$, but $\|A_i\|^2\neq 0 \Rightarrow c_i=0$ for $i=1,\ldots,n \Rightarrow$ Therefore, A is linearly independent.

Theorem4.7: Let $S=\{A_1,\ldots,A_n\}$ be an orthogonal basis of an Euclidean vector space $\langle V,\langle,\rangle\rangle$. Then for any A in V, $A=\frac{\langle A,A_1\rangle}{\|A_1\|^2}A_1+\ldots+\frac{\langle A,A_n\rangle}{\|A_n\|^2}A_n$.

Proof: Let $A\in V$, since S is a basis of $V \Rightarrow A=x_1A_1+\ldots+x_nA_n$ for some $x_i\in R$. $\langle A,A_i\rangle=\langle x_1A_1+\ldots+x_nA_n,A_i\rangle=x_1\langle A_1,A_i\rangle+\ldots+x_{i-1}\langle A_{i-1},A_i\rangle+x_i\langle A_i,A_i\rangle+x_{i+1}\langle A_{i+1},A_i\rangle+\ldots+ x_n\langle A_n,A_i\rangle=x_i\langle A_i,A_i\rangle \Rightarrow x_i=\frac{\langle A,A_i\rangle}{\|A_i\|^2} \Rightarrow A=\frac{\langle A,A_1\rangle}{\|A_1\|^2}A_1+\ldots+\frac{\langle A,A_n\rangle}{\|A_n\|^2}A_n$.

Corollary4.1: Let $S=\{A_1,\ldots,A_n\}$ be an orthonormal basis of an Euclidean vector space $\langle V,\langle,\rangle\rangle$. Then for any A in V, $A=\langle A,A_1\rangle A_1+\ldots+\langle A,A_n\rangle A_n$.

Proof: By Theorem4.7, $A=\frac{<A,A_1>}{\|A_1\|^2}A_1+\ldots+\frac{<A,A_n>}{\|A_n\|^2}A_n$, since $\|A_i\|^2=1$, then $A=<A,A_1>A_1+\ldots+<A,A_n>A_n$.

Example: Let $A_1=(1,1,1)$, $A_2=(0,1,-1)$, $A_3=(-2,1,1)$. Show that $S=\{A_1,A_2,A_3\}$ is a basis for R^3 and $A=(3,-1,2)$ is a LC of vectors in S.

Solution: Since S is orthogonal set by Theorem4.7, S is LI, by Theorem1.12, S is a basis of R^3, by Theorem4.7, $A=\frac{<A,A_1>}{\|A_1\|^2}A_1+\frac{<A,A_2>}{\|A_2\|^2}A_2+\frac{<A,A_3>}{\|A_3\|^2}A_3$.

By simple calculation $<A,A_1>=4$, $<A,A_2>=-3$, $<A,A_3>=-5$, $\|A_1\|^2=3$, $\|A_2\|^2=2$ and $\|A_3\|^2=6\Rightarrow A=\frac{4}{3}A_1+(-\frac{3}{2})A_2+(-\frac{5}{6})A_3$.

Remark4.1: Let $S=\{A_1,\ldots,A_n\}$ be an orthogonal set of non-zero vectors in an Euclidean vector space <V,<,>>. Then $H=\{\frac{A_1}{\|A_1\|},\ldots,\frac{A_n}{\|A_n\|}\}$ is orthonormal and [S]=[H].

Remark4.2: "Gram-Schmidt orthogonalization process" For every subspace of a finite dimensional Euclidean space <V,<,>>, there exist an orthogonal basis.

Let $S=\{A_1,\ldots,A_n\}$ be any basis to a subspace M. To transform S to an orthogonal basis $T=\{B_1,\ldots,B_n\}$ we process as follows

Let $B_1=A_1$,

$B_2=A_2-\frac{<A_2,B_1>}{\|B_1\|^2}B_1$, …,

$B_n=A_n-\frac{<A_n,B_{n-1}>}{\|B_{n-1}\|^2}B_{n-1}-\frac{<A_n,B_{n-2}>}{\|B_{n-2}\|^2}B_{n-2}-\ldots-\frac{<A_n,B_1>}{\|B_1\|^2}B_1$

Example: Find orthogonal basis for $P_1(R)$ with usual inner product.

Solution: Let {1,x} be a standard basis for $P_1(R)$, by GSOP, $B_1=A_1=1$,

$B_2=A_2-\frac{<A_2,B_1>}{\|B_1\|^2}B_1=x-\frac{<x,1>}{\|1\|^2}1$ but $<x,1>=\int_0^1 x\,dx=\frac{1}{2}$ and $\|1\|^2=\int_0^1 dx=1\Rightarrow B_2=x-\frac{1}{2}\Rightarrow\{1, x-\frac{1}{2}\}$ is orthogonal basis for $P_1(R)$.

Example: Let M be a subspace of R^4 has a basis S={(1,-2,0,1),(-1,0,0,-1),(0,0,0,1)}. Find orthonormal basis for M with usual inner product.

Solution: By GSOP, $B_1=A_1=(1,-2,0,1)$,

$B_2=A_2-\frac{<A_2,B_1>}{\|B_1\|^2}B_1=(-1,0,0,-1)-\frac{<(1,-2,0,1),(-1,0,0,-1)>}{\|(1,-2,0,1)\|^2}(1,-2,0,1)=(-\frac{2}{3},-\frac{2}{3},0,-\frac{2}{3})$.

$B_3=A_3-\frac{<A_3,B_2>}{\|B_2\|^2}B_2-\frac{<A_3,B_1>}{\|B_1\|^2}B_1=(0,0,0,1)-(-\frac{2}{3}\frac{3}{4})(-\frac{2}{3},-\frac{2}{3},0,-\frac{2}{3})-\frac{1}{6}(1,-2,0,1)=(-\frac{1}{2},0,0,\frac{1}{2})$

$T=\{(1,-2,0,1),(-\frac{2}{3},-\frac{2}{3},0,-\frac{2}{3}),(-\frac{1}{2},0,0,\frac{1}{2})\}$ is orthogonal basis for M.

Normalizing the vector $B_1,B_2,B_3 \Rightarrow H=\{(\frac{1}{\sqrt{6}},\frac{-2}{\sqrt{6}},0,\frac{1}{\sqrt{6}}),(\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,\frac{-1}{\sqrt{3}}),(-\frac{1}{\sqrt{2}},0,0,\frac{1}{\sqrt{2}})\}$ is orthonormal basis for M.

Definition4.9: Let M be a subspace of a vector space V over F. N is said to be complement space to M if N is a subspace and $V=M\oplus N$.

Theorem4.8: For every subspace M of a finite dimensional vector space V, has a complement space.

Proof: Let $S=\{A_1,\dots,A_m\}$ be a basis for $M \Rightarrow S=\{A_1,\dots,A_m\}$ is a LI set, By Remark1.4, there exist $\{B_1,\dots,B_n\}$ such that $\{A_1,\dots,A_m,B_1,\dots,B_n\}$ is a basis for V. Let $N=[\{B_1,\dots,B_n\}] \Rightarrow V=M\oplus N$.

Example: Let $M=\{(x,y): x=3y\}$ be a subspace of R^2 let $N_1=[\{(1,0)\}]$, $N_2=[\{(0,1)\}]$ $\Rightarrow$ It is clear that $R^2=M\oplus N_1$ and $R^2=M\oplus N_2$.

Definition4.10: Let M be a subspace of an Euclidean vector space $<V,<,>>$. Let $M^\perp$ be the set of all vector in V which orthogonal to every vector in M. $M^\perp=\{A\in V: <A,B>=0\ \forall B\in M\}$.

Theorem4.9: Let M be a subspace of an Euclidean vector space $<V,<,>>$. Then

1- $M^\perp$ is a subspace

2- $M\cap M^\perp=\{O\}$

3- $V=M\oplus M^\perp$

4- $(M^\perp)^\perp=M$.

Proof of (1): Let $A,B\in M^\perp$ and $r\in R$. Let $C\in M$ $<A+B,C>=<A,C>+<B,C>=0+0=0 \Rightarrow A+B\in M^\perp$. Since $<rA,C>=r<A,C>=r0=0 \Rightarrow rA\in M^\perp$. Therefore, $M^\perp$ is a subspace of V.

Proof of (2): $A\in M\cap M^\perp \Rightarrow A\in M$ and $A\in M^\perp \Rightarrow <A,A>=0 \Rightarrow A=O \Rightarrow M\cap M^\perp=\{O\}$.

Proof of (3): By GSOP, let $S=\{A_1,\dots,A_n\}$ be orthonormal basis for M. For all $A\in V$, let $B=<A,A_1>A_1+\dots+<A,A_n>A_n \Rightarrow B\in M$. Let $C=A-B \Rightarrow A=B+C$ we must prove that $C\in M^\perp$

$<A_k,C>=<A_k,A-B>=<A_k,A>-<A_k,B>=<A_k,A>-<A_k,<A,A_1>A_1+\ldots+<A,A_n>A_n>=<A_k,A>-$
$<A,A_1><A_k,A_1>+\ldots+<A,A_n><A_k,A_n>=<A_k,A>-<A,A_k><A_k,A_k>=<A_k,A>-<A,A_k>\|A_k\|^2=$
$<A_k,A>-<A_k,A>=0$.

For all $D\in M\Rightarrow D=r_1A_1+\ldots+r_nA_n$. $<D,C>=<r_1A_1+\ldots+r_nA_n,C>=r_1<A_1,C>+\ldots+r_n<A_n,C>=r_10+\ldots+r_n0=0\Rightarrow C\in M^{\perp}$.

Proof of (4): Let $A\in M$, for all $B\in M^{\perp}$, $<A,B>=0\Rightarrow A\in(M^{\perp})^{\perp}\Rightarrow M\subset(M^{\perp})^{\perp}$.

Let $A\in(M^{\perp})^{\perp}\Rightarrow<A,B>=0$ for all $B\in M^{\perp}$. Since $A\in(M^{\perp})^{\perp}\Rightarrow A\in V$, by (3), A=C+D for some $C\in M$ $D\in M^{\perp}\Rightarrow<C,D>=0$ and $<A,D>=0\Rightarrow<A-C,D>=0\Rightarrow<D,D>=0\Rightarrow D=O\Rightarrow$ $A=C+D=C+O=C\in M\Rightarrow A\in M\Rightarrow(M^{\perp})^{\perp}\subset M$. Therefore $(M^{\perp})^{\perp}=M$.

Definition4.11: Let M be a subspace of an Euclidean vector space <V,<,>>. The space $M^{\perp}$ is called orthogonal complement of M.

To find $M^{\perp}$, we process as follows

1- Find a basis $\{A_1,\ldots,A_n\}$ for M.
2- Extend $\{A_1,\ldots,A_n\}$ to $\{A_1,\ldots,A_n, B_1,\ldots,B_m\}$ be a basis for V.
3- By using GSOP find $\{C_1,\ldots,C_n, D_1,\ldots,D_m\}$ which is orthogonal basis for V.

Therefore, $M^{\perp}=[\{D_1,\ldots,D_m\}]$

Example: Find orthogonal complement to M={(x,y): x-2y=0} of R^2 with usual inner product.

Solution: {(2,1)} is a basis for M⇒{(2,1),(0,1)} is a basis for R^2. By using GSOP, $B_1=A_1=(2,1)$,

$B_2=A_2-\frac{<A_2,B_1>}{\|B_1\|^2}B_1=(0,1)-\frac{<(0,1),(2,1)>}{\|(2,1)\|^2}(2,1)=(-\frac{2}{5},\frac{4}{5})$. $T=\{(2,1),(-\frac{2}{5},\frac{4}{5})\}$ is orthogonal basis for $R^2\Rightarrow M^{\perp}=[\{(-\frac{2}{5},\frac{4}{5})\}]=\{a(-\frac{2}{5},\frac{4}{5}): a\in R\}=\{(x,y): 2x+y=0\}$

Definition4.12: Let M be a subspace of an Euclidean vector space <V,<,>> and $A\in V$, then A=B+C for a unique $B\in M$ and $C\in M^{\perp}$, B is called a projection of A on a subspace M.

Example: Find projection of A=(3,4) on a subspace M={(x,y): x-2y=0} of R^2 with usual inner product.

Solution: (3,4)=(4,2)+(-1,2) and $(4,2)\in M$ and $(-1,2)\in M^{\perp}\Rightarrow(4,2)$ is a projection of (3,4) on a subspace M.

To find a projection of a vector A on a subspace M, we process as follows

1- Find orthogonal basis $\{A_1,\ldots,A_n\}$ for M.

2- Put $B=\frac{<A,A_1>}{\|A_1\|^2}A_1+\ldots+\frac{<A,A_n>}{\|A_n\|^2}A_n$. Therefore, B is a projection of A on M.

Definition4.13: Let $<V,<,>>$ be an Euclidean vector space and $T:V\rightarrow V$ be a LT. T is said to be orthogonal transformation if $\forall A,B\in V$, $<T(A),T(B)>=<A,B>$.

Example: Let $T:R^2\rightarrow R^2$ defined by $T(x,y)=\frac{1}{\sqrt{2}}(x-y,x+y)$ where R^2 is an Euclidean vector space with usual inner product. T is orthogonal transformation.

Example: Let $T:R^2\rightarrow R^2$ defined by $T(x,y)=(x,x+y)$ where R^2 is an Euclidean vector space with usual inner product. T is not orthogonal transformation.

Solution: Let $A=(1,0)$ and $B=(0,1)\Rightarrow$ $<A,B>=0$ and $<T(A),T(B)>=<(1,1),(1,0)>=1\Rightarrow <T(A),T(B)>\neq<A,B>$.

Theorem4.10: Let $<V,<,>>$ be an Euclidean vector space and $T:V\rightarrow V$ be a LT. The following are equivalent

1- T is orthogonal transformation.

2- T is preserve length, $\forall A\in V$ $\|T(A)\|=\|A\|$

3- For all unit vector A , T(A) is a unit vector.

Proof: (1)⇒(2) $\forall A\in V$, $<T(A),T(A)>=<A,A>\Rightarrow\|T(A)\|^2=\|A\|^2\Rightarrow\|T(A)\|=\|A\|$.

(2)⇒(1) $\forall A,B\in V$, $<T(A+B),T(A+B)>-<T(A-B),T(A-B)>=<T(A),T(A)>+<T(B),T(B)>+2<T(A),T(B)>-(<T(A),T(A)>+<T(B),T(B)>-2<T(A),T(B)>)=4<T(A),T(B)>\Rightarrow\|T(A+B)\|^2-\|T(A-B)\|^2=4<T(A),T(B)>\Rightarrow 4<A,B>=4<T(A),T(B)>\Rightarrow <T(A),T(B)>=<A,B>$.

(2)⇒(3) Let A be a unit vector$\Rightarrow$ $\|A\|=1\Rightarrow\|T(A)\|=\|A\|=1\Rightarrow$ T(A) is a unit vector.

(3)⇒(2) Let $A\in V$ and $A\neq O\Rightarrow\frac{A}{\|A\|}$ is a unit vector$\Rightarrow T(\frac{A}{\|A\|})$ is unit vector$\Rightarrow$

$\left\|T(\frac{A}{\|A\|})\right\|=1\Rightarrow\frac{\|T(A)\|}{\|A\|}=1\Rightarrow\|T(A)\|=\|A\|$.

Theorem4.11: Let $<V,<,>>$ be a finite dimensional Euclidean vector space and $T:V\rightarrow V$ be an orthogonal transformation. Then T is an isomorphism.

Proof: Let $A\in KerT\Rightarrow T(A)=O\Rightarrow<T(A),T(A)>=0\Rightarrow<A,A>=0\Rightarrow A=O\Rightarrow KerT=\{O\}$. Since V is finite dimensional vector space, by Remark2.2, $dimV=dimKerT+dimImT\Rightarrow dimV=dimImT$ by Theorem1.13, $V=ImT\Rightarrow$ T is onto$\Rightarrow$T is an isomorphism.

Theorem4.12: Composition of two orthogonal transformations is orthogonal transformation.

Proof: Let $T:V\to V$ and $S:V\to V$ be orthogonal transformation. $\forall\ A\in V$, $\|(SoT)(A)\|=\|S(T(A))\|=\|T(A)\|=\|A\|$, by Theorem4.10, SoT is orthogonal transformation.

Theorem4.13: Let $S=\{A_1,\ldots,A_n\}$ be orthonormal basis for an Euclidean vector space $<V,<,>>$ and $T:V\to V$ be an orthogonal transformation. Let M be a matrix for T with respect to S, then the following are hold

1- Every row of M is of length 1, which is regarded as a vector in R^n with standard inner product.

2- Rows of M are orthogonal.

3- $M^{-1}=M^T$.

4- (1) and (2) are hold for column.

Proof: Let $M=(a_{ij})$, from the definition of a matrix of a LT, $T(A_i)=a_{i1}A_1+\ldots+a_{in}A_n$ for $i=1,\ldots,n$ $\Rightarrow$ for $i,j=1,\ldots,n$ $<A_i,A_j>=<T(A_i),T(A_j)>=<a_{i1}A_1+\ldots+a_{in}A_n,a_{j1}A_1+\ldots+a_{jn}A_n>$ $=\sum_{k=1}^{n}\sum_{p=1}^{n}a_{ik}a_{jp}<A_k,A_p>=\sum_{k=1}^{n}a_{ik}a_{jk}<A_k,A_k>=\sum_{k=1}^{n}a_{ik}a_{jk}$. Let $X_i=(a_{i1},\ldots,a_{in})$, X_i is represent i-th row of $M\Rightarrow$ $<X_i,X_j>=\sum_{k=1}^{n}a_{ik}a_{jk}=<A_i,A_j>$. **(1)** $\|X_i\|^2=<X_i,X_i>=<A_i,A_i>=\|A_i\|^2\Rightarrow\|X_i\|=\|A_i\|=1$. **(2)** for each $i\neq j$ $<X_i,X_j>=<A_i,A_j>=0$. **(3)** To calculate $MM^T=(<X_i,X_j>)$ where X_i is the i-th row of M and X_j is the j-th column of M^T $\Rightarrow$ X_j is the j-th row of M, from (1) and (2), it follows that $<X_i,X_j>=1$ if $i=j$ and $<X_i,X_j>=0$ if $i\neq j$ $\Rightarrow MM^T=I\Rightarrow M^{-1}=M^T$. **(4)** Directly from (1),(2),(3), column of M are row of M^T and $M^TM=I$.

Definition4.14: Let $M\in M_{nn}(R)$. M is said to be orthogonal matrix if $MM^T=I$.

Theorem4.14: Let $M\in M_{nn}(R)$ and M is an orthogonal matrix, let $S=\{A_1,\ldots,A_n\}$ be an orthonormal basis for an Euclidean vector space $<V,<,>>$, then there exist an orthogonal transformation $T:V\to V$ such that the matrix for T with respect to S is M.

Proof: Let $A\in V$, $A=x_1A_1+\ldots+x_nA_n$ define $T:V\to V$ by $T(A)=y_1A_1+\ldots+y_nA_n$ where $(y_1,\ldots,y_n)=(x_1,\ldots x_n)M$. It is clear that T is LT. $\|T(A)\|^2=<T(A),T(A)>=y_1^2+\ldots+y_n^2$ $\Rightarrow\|T(A)\|^2=(y_1,\ldots,y_n)(y_1,\ldots,y_n)^T=(x_1,\ldots x_n)M((x_1,\ldots x_n)M)^T=(x_1,\ldots x_n)MM^T(x_1,\ldots x_n)^T=(x_1,\ldots x_n)(x_1,\ldots x_n)^T=x_1^2+\ldots x_n^2=\|A\|^2\Rightarrow\|T(A)\|=\|A\|\Rightarrow$ T is orthogonal transformation. The vector coordinate of A_i With respect to S is $X=(0,\ldots,0,1,0,\ldots 0)$, then $(y_1,\ldots,y_n)=(0,\ldots,0,1,0,\ldots 0)M=(a_{i1},\ldots a_{in})$, then $T(A_i)=a_{i1}A_1+\ldots+a_{in}A_n$. Therefore, $M_T=M$.