Linear Algebra I

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1 Vector Spaces

1.1 Fields

Definition 1.1. *A* **field** *is a set* $\mathbb{F}$ *on which we have two binary operations:* $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ *and* $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$*, satisfying the following axioms:*

(F1) (Associativity of addition) For all $a, b, c \in \mathbb{F}$, $a + (b + c) = (a + b) + c$.

(F2) (Commutativity of addition) For all $a, b \in \mathbb{F}$, $a + b = b + a$.

(F3) (Existence of zero) There exists $0 \in \mathbb{F}$ *such that, for all* $a \in \mathbb{F}$, $a + 0 = a = 0 + a$.

(F4) (Existence of additive inverses) For each $a \in \mathbb{F}$*, there exists* $-a \in \mathbb{F}$ *such that* $a + (-a) = 0 = -a + a$.

(F5) (Associativity of multiplication) For all $a, b, c \in \mathbb{F}$, $a(bc) = (ab)c$.

(F6) (Commutativity of multiplication) For all $a, b \in \mathbb{F}$, $ab = ba$.

(F7) (Existence of identity) There exists $1 \in \mathbb{F}$ *such that , for all* $a \in \mathbb{F}$, $a1 = a = 1a$.

(F8) (Existence of multiplicative inverses) For each $a \in F$, $a \neq 0$*, there exists* $a^{-1} \in \mathbb{F}$ *such that* $aa^{-1} = 1 = a^{-1}a$.

(F9) (Distributivity) For all $a, b, c \in \mathbb{F}$, $a(b + c) = ab + ac$ *and* $(a + b)c = ac + bc$.

The elements of $\mathbb{F}$ may be called ***numbers*** or ***scalars***, the operations $+$ and $\cdot$ will be called ***addition*** and ***multiplication***, respectively.

Example 1.2.

$\mathbb{Q}$ (the rationals), $\mathbb{R}$ (the reals), and $\mathbb{C}$ (the complex numbers) are fields, while the integers $\mathbb{Z}$ do *not* form a field. Can you see why?

1.2 Finite fields

In this course we will also need finite fields. By this we mean a field $\mathbb{F}$ which is finite as a set. So there are only finitely many numbers in this number system!

Theorem 1.3. *If $\mathbb{F}$ is a finite field then $|\mathbb{F}| = p^f$ for some prime integer p and a positive integer f. Furthermore, for each order p^f, there is exactly one field of that order.*

This theorem tells us how many finite fields exist and what are the possible orders. However, we also need to know what the fields are. We will write $\mathbb{F}_q$ for the only field of order (or size) $q = p^f$.

1.2.1 Prime order

If the order of the field is p (that is, $f = 1$) then the field $\mathbb{F}_p$ is simply $\mathbb{Z}_p$, the integers modulo p. This means that $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$ and the operations of addition and multiplication on $\mathbb{F}_p$ are the usual addition and multiplication supplemented by subtracting a suitable multiple of p in order to bring the result within the interval $[0, p-1]$.

Example 1.4. *Suppose we need to work with $\mathbb{F}_5$. Then $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$, and here are examples of addition:*

$$2 + 2 = 4,$$

$$4 + 3 = 7 - 5 = 2.$$

In the first example, the sum, 4 is already in the range, so we do not need to subtract 5. In the second example, we subtract 5 from the sum, 7, in order to bring the result into the range.

Here also are the examples of multiplication:

$$2 \cdot 2 = 4,$$

$$4 \cdot 4 = 16 - 5 - 5 - 5 = 1.$$

In the first example, we are in the range, so we do not subtract any 5s. In the second example, we need to subtract 5 three times in order for the result to be in the target interval.

1.2.2 Non-prime order

In the non-prime case the above method does *not* work! If you need, for example, the field $\mathbb{F}_4$ of order 4 then you *cannot* simply take $\mathbb{Z}_4$, the integers modulo 4. This is not a field! Instead you can take $\mathbb{F}_4 = \{0, 1, a, b\}$, where addition and multiplication is given by the following tables.

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

·	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

1.3 Vector Spaces

The following is the main definition of the course.

Definition 1.5. *Suppose $\mathbb{F}$ is a field. A* **vector space** *over $\mathbb{F}$ is a set V together with two operations: $+ : V \times V \to V$ and $\cdot : \mathbb{F} \times V \to V$, satisfying the following axioms:*

(VS1) (Associativity of addition) For all $u, v, w \in V$, $u+(v+w) = (u+v)+w$.

(VS2) (Commutativity of addition) For all $u, v \in V$, $u + v = v + u$.

(VS3) (Existence of zero vector) There exists $0 \in V$ such that $v+0 = 0+v+0$ for all.

(VS4) (Existence of additive inverses) For each $u \in V$, there exists $-u \in V$ such that $u + (-u) = 0 = -u + u$.

(VS5) (Associativity of multiplication) For all $a, b \in \mathbb{F}$ and $u \in V$, $a(bu) = (ab)u$.

(VS6) (Distributivity) For all $a, b \in \mathbb{F}$ and $u, v \in V$, $a(u + v) = au + av$ and $(a + b)v = av + bv$.

(VS7) (Identity multiplication) For all $u \in V$, $1v = v$.

We refer to the elements of V as ***vectors***, as opposed to *scalars*, which are the numbers, that is, the elements of $\mathbb{F}$.

Note that the product of a scalar and a vector is always to be written in this order: first the scalar and then the vector; never the other way around.

The operations $+$ and $\cdot$ will be called ***vector addition*** and ***scalar multiplication***, respectively.

Example 1.6. *Let* $\mathbb{R}$ *be the field of real numbers,* $V = \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$*.*
$(x_1, y_1) = (x_2, y_2)$ *if and only if* $x_1 = x_2$ *and* $y_1 = y_2$ *We define the vector addition by* $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ *[coordinate-wise addition] and scalar multiplication by* $a(x_1, y_1) = (ax_1, ay_1)$ *[coordinate-wise scalar multiplication]. Show that* $\mathbb{R}^2$ *is a vector space over* $\mathbb{R}$*.*

Example 1.7. *Let $\mathbb{R}$ be the field of real numbers, $V = \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$.*
$(x_1, y_1) = (x_2, y_2)$ if and only if $x_1 = x_2$ and $y_1 = y_2$ We define the vector addition by $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ [coordinate-wise addition] and scalar multiplication by $a(x_1, y_1) = (ax_1, a^2y_1)$. Test whether $\mathbb{R}^2$ is a vector space over $\mathbb{R}$.

Example 1.8. *Let $\mathbb{F}$ be any field and $n \in \mathbb{Z}^+$, $\mathbb{F}^n = \{(x_1, \cdots, x_n) : x_i \in \mathbb{F} \; for \; i = 1, \cdots, n\}$.*
$(x_1, \cdots, x_n) = (y_1, \cdots, y_n)$ if and only if $x_i = y_i$ for $i = 1, \cdots, n$.
We define the vector addition by $(x_1, \cdots, x_n) + (y_1, \cdots, y_n) = (x_1 + y_1, \cdots, x_n + y_n)$ [coordinate-wise addition]
and scalar multiplication by $a(x_1, \cdots, x_n) = (ax_1, \cdots, ax_n)$ [coordinate-wise scalar multiplication]. Show that $\mathbb{F}^n$ is a vector space over $\mathbb{F}$.

Example 1.9. *Let $\mathbb{F}$ be any field and $m, n \in \mathbb{Z}^+$, $M_{mn}(\mathbb{F}) = \{A : A \; is \; m - by - n \; matrix \; in \; \mathbb{F}\}$.*
Let $+$ be the sum of two matrices and $\cdot$ be the multiplication of a matrix by a constant. Show that $M_{mn}(\mathbb{F})$ is a vector space over $\mathbb{F}$.

Example 1.10. *Let $\mathbb{F}$ be any field and $n \in \mathbb{Z}^+$, $P_n(\mathbb{F}) = \{a_0 + a_1x + \cdots + a_nx^n : a_i \in \mathbb{F} \; for \; i = 0, 1, \cdots, n\}$.*
$a_0 + a_1x + \cdots + a_nx^n = b_0 + b_1x + \cdots + b_nx^n$ if and only if $a_i = b_i$ for $i = 0, 1, \cdots, n$.
We define the vector addition by $(a_0 + a_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$ [coordinate wise addition]
and scalar multiplication by $a(a_0 + a_1x + \cdots + a_nx^n) = aa_0 + aa_1x + \cdots + aa_nx^n$ [coordinate wise scalar multiplication]. Show that $P_n(\mathbb{F})$ is a vector space over $\mathbb{F}$.

Example 1.11. *Let* $X \neq \emptyset$ *and* $V = \{f : f : X \to \mathbb{R}\ is\ a\ function\}$*. The vector addition is defined as the sum of two functions* $(f+g)(x) = f(x)+g(x)$ *for all* $x \in X$*. The scalar multiplication is defined by* $(af)(x) = af(x)$ *for all* $x \in X$*. Show that* V *is a vector space over* $\mathbb{R}$*.*

Example 1.12. *Let* I *be any interval subset of* $\mathbb{R}$ *and* $C(I) = \{f : f : I \to \mathbb{R}\ is\ a\ continuous\ function\}$*. Show that* $C(I)$ *is a vector space over* $\mathbb{R}$ *.*

Example 1.13. *Let* I *be any interval subset of* $\mathbb{R}$*,* $n \in \mathbb{Z}^+$ *and* $C^n(I) = \{f : f : I \to \mathbb{R}\ is\ a\ differentiable\ function\ whose\ nth{-}derivative\ is\ continuous\}$*. Show that* $C^n(I)$ *is a vector space over* $\mathbb{R}$ *.*

Example 1.14 (Subsets of a set)**.** *The set* 2^Ω *of all subsets of a set* Ω *is a vector space over* $\mathbb{F}_2$*. The addition is provided by the operation of* symmetric difference *(often denoted by* $\triangle$*). Recall, that for two subsets* A *and* B *of* Ω*, the symmetric difference* $A \triangle B$ *of* A *and* B *consists of all elements that are in* A*, but not in* B*, and all elements that are in* B*, but not in* A*. So*

$$A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

How do we define multiplication with scalars?

1.4 Basic properties

Here are some basic properties of vector spaces.

Theorem 1.15 (Elementary properties)**.** *Suppose* V *is a vector space over a field* $\mathbb{F}$*. Then the following hold:*

(1) (Cancellation in sums) For $u, v, w \in V$*, if* $u + v = u + w$ *then* $v = w$*.*

(2) For all $u \in V$*,* $0u = 0$*.*

(3) For all $u \in V$*,* $(-1)u = -u$*.*

(4) Also, for all $a \in \mathbb{F}$*,* $a0 = 0$*.*

Theorem 1.16. *Suppose V is a vector space over a field $\mathbb{F}$. Then*

(1) The zero vector is unique.

(2) For each $u \in V$, $-u$ is unique.

Theorem 1.17. *Suppose V is a vector space over $\mathbb{F}$. Then:*

(1) For $a \in \mathbb{F}$ and $u \in V$, if $au = 0$ then either $a = 0$ or $u = 0$.

(2) (Cancellation in products)

  (a) For $0 \neq a \in \mathbb{F}$ and $u, v \in V$, if $au = av$ then $u = v$.

  (b) Also, for $a, b \in \mathbb{F}$ and $0 \neq u \in V$, if $au = bu$ then $a = b$.

1.5 Subspaces

Definition 1.18. *Suppose V is a vector space over $\mathbb{F}$. A subset $U \subseteq V$ is a* **subspace** *if it is a vector space over $\mathbb{F}$ in its own right with respect to addition and multiplication inherited from V.*

Definition 1.19. *Suppose V is a vector space over $\mathbb{F}$. We say that a subset $U \subseteq V$ is* **closed for addition** *if, for all $u, v \in U$, $u + v \in U$.*

Definition 1.20. *Suppose V is a vector space over $\mathbb{F}$. A subset $U \subseteq V$ is* **closed for multiplication with scalars** *if, for all $u \in U$ and all $a \in \mathbb{F}$, $au \in U$.*

Theorem 1.21 (Subspace criterion I)**.** *Suppose V is vector space over $\mathbb{F}$. A* nonempty *subset $U \subseteq V$ is a subspace if and only if U is closed for addition and multiplication with scalars.* $\square$

Note that the empty subset $U = \emptyset$ is not a subspace, because (VS3) fails!! There is no zero vector in this U. This is why we have to exclude the empty set in the above theorem.

The two conditions in the theorem can be replaced by a single condition.

Theorem 1.22 (Subspace criterion, II)**.** *A non-empty subset U of a vector space V over $\mathbb{F}$ is a subspace of V if and only if, for all $u, v \in U$ and all $a \in \mathbb{F}$, we have that $au + v \in U$.*

Example 1.23. *In every vector space V, the subset $U = V$ is obviously a subspace. We say that a subspace $U \subseteq V$ is* **proper** *if $U \neq V$, that is, U is strictly smaller than V.*

Hence $U = V$ is the only subspace of V that is not proper.

Similarly, in every vector space V, the subset $\{0\}$ is a subspace. This subspace is called the **trivial** *subspace. We will often write 0 for the trivial subspace.*

Example 1.24. *In $\mathbb{R}^2$, the subset $U = \{(x, y) \mid y = 2x\}$ is a subspace.*

Example 1.25. *In $\mathbb{R}^2$, the subset $U = \{(x, y) \mid y = mx;\ m \in \mathbb{R}\}$ is a subspace.*

Example 1.26. *In $\mathbb{R}^3$, the subset $U = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ is a subspace.*

Example 1.27. *In $\mathbb{R}^3$, the subset $U = \{(x, y, 1) \mid x, y \in \mathbb{R}\}$ is not a subspace.*

Example 1.28 ($\mathbb{R}^2$ and $\mathbb{R}^3$)**.** *In the plane $\mathbb{R}^2$, the subspaces are:*

(1) the trivial subspace containing only the origin 0;

(2) the straight lines through the origin; and

(3) the improper subspace—the entire plane.

Similarly, in the 3D space $\mathbb{R}^3$, we have the following subspaces:

(1) the trivial subspace;

(2) the straight lines through the origin;

(3) the planes through the origin; and

(4) the improper subspace—the entire space $\mathbb{R}^3$.

Example 1.29. *In $\mathbb{R}^4$, let $U = \{(x, y, z, w) \mid xw = 0\}$. Is U a subspace of $\mathbb{R}^4$? Explain.*

Example 1.30. *In $M_{2\times 2}(\mathbb{R})$, the subset $U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a + b = c \right\}$ is a subspace.*

1.6 Intersection of subspaces

Theorem 1.31 (Intersection of subspaces)**.** *If $\{U_i \mid i \in I\}$ is a collection of subspaces of a vector space V, then the intersection $W := \cap_{i\in I} U_i$ is again a subspace of V.*

Example 1.32. *In* $P_2(\mathbb{R})$*, let* $U_1 = \{a + bx + cx^2 \mid a + 2b - c = 0\}$ *and* $U_2 = \{a + bx + cx^2 \mid a + b + 2c = 0\}$*. Find* $U_1 \cap U_2$*.*

Note that the union of two subspaces need not be a subspace in general. For example, in $\mathbb{R}^2$, if $U_1 = \{(x, y) \mid y = x\}$ and $U_2 = \{(x, y) \mid y = -x\}$, then $U_1 \cup U_2$ is not a subspace.

1.7 Generated subspace

Definition 1.33 (Generated subspace)**.** *For a vector space* V *and a subset* $X \subseteq V$*, the subspace* generated *by* X *(denoted* $\langle X \rangle$*) is the unique minimal subspace of* V *containing* X*.*

If $X = A \cup B \cup C$, say, we may write $\langle A, B, C \rangle$ instead of $\langle A \cup B \cup C \rangle$. Also, we will write u in place $\{u\}$.

Example 1.34. *If the set* X *consists of a single vector,* u*, then every subspace containing* u *also contains all multiples of* u*. That is,* $\langle u \rangle = \{au \mid a \in \mathbb{F}\}$*.*

For example, if $V = P_3(\mathbb{R})$ and $u = x+1$ then $\langle u \rangle$ contains polynomials 0, $x+1$, $\frac{1}{2}x+\frac{1}{2}$, $-3x-3$, and in fact, $\langle u \rangle = \{a(x+1) \mid a \in \mathbb{R}\} = \{ax+a \mid a \in \mathbb{R}\}$.

Example 1.35. *Similarly, if* $X = \{u, v\}$ *then* $\langle u, v\rangle = \{au + bv \mid a, b \in \mathbb{F}\}$*.*

For example, if $V = \mathbb{R}^2$ and $u = (2, 0)$ and $v = (0, 4)$, then $\langle u, v\rangle = \{a(2, 0) + b(0, 4) \mid a, b \in \mathbb{R}\} = \{(2a, 4b) \mid a, b \in \mathbb{R}\}$. For each $(x, y) \in \mathbb{R}^2$, $(x, y) = \frac{x}{2}(2, 0) + \frac{y}{4}(0, 4)$ which implies $(x, y) \in \langle u, v\rangle$. Thus, $\langle u, v\rangle = \mathbb{R}^2$.

Example 1.36. *If* $\mathbb{F} = \mathbb{F}_2$ *and* $V = 2^\Omega$*, where* $\Omega = \{red, blue, yellow, green\}$*, and* $u = \{red, yellow\}$ *and* $v = \{blue, yellow, green\}$*, then find* $\langle u, v\rangle$*.*

1.8 Sum of subspaces, direct sum

Definition 1.37 (Sum of subspaces)**.** *Suppose* U *and* W *are two subspaces of a vector space* V*. Then we define* $U + W = \{u + w \mid u \in U, w \in W\}$*.*

Theorem 1.38. *If* U *and* W *are subspaces of a vector space* V*, then* $U + W$ *is also a subspace of* V*.* □

Example 1.39. *In* $\mathbb{R}^2$*, let* $U = \{(x, 0) \mid x \in \mathbb{R}\}$ *and* $W = \{(0, y) \mid y \in \mathbb{R}\}$*. Find* $U + W$*.*

Example 1.40. *In* $\mathbb{R}^3$*, let* $U = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ *and* $W = \{(0, y, z) \mid y, z \in \mathbb{R}\}$*. Find* $U + W$*.*

Definition 1.41 (Direct sum)**.** *Suppose* U *and* W *are subspaces of a vector space* V*. We say that* $D = U + W$ *is the* **direct sum** *of* U *and* W *(denoted* $D = U \oplus W$*) if for every* $d \in D$ *there exists only one choice of* $u \in U$ *and* $w \in W$ *such that* $d = u + w$*.*

Example 1.42. *In Example* 1.39*,* $\mathbb{R}^2$ *is the direct sum of* U *and* W*. That is,* $\mathbb{R}^2 = U \oplus W$*. However, in Example* 1.40 $\mathbb{R}^3$ *is not the direct sum of* U *and* W

Theorem 1.43 (Direct sum criterion)**.** *For subspaces U and W of a vector space V, we have that $D = U + W$ is the direct sum of U and W if and only if $U \cap W = 0$.*

Example 1.44. *In $\mathbb{R}^2$, let $U = \{(x, y) \mid y = x\}$ and $W = \{(x, y) \mid y = -x\}$. Show that $\mathbb{R}^2 = U \oplus W$.*

1.9 Linear combinations and independence

Definition 1.45. *Let $S = \{u_1, u_2, \cdots, u_k\}$ be a set of vectors in a vector space V over a field $\mathbb{F}$. A vector u in V is called a* **linear combination** *of the vectors in S if*

$$u = a_1u_1 + a_2u_2 + \cdots + a_ku_k$$

for some $a_1, a_2, \cdots, a_k \in \mathbb{F}$.

Example 1.46. *In $\mathbb{R}^2$, test whether the vector $u = (1, 2)$ is a linear combination of the vectors in $S = \{(1, 1), (1, -1)\}$ or not.*

Example 1.47. *In $\mathbb{R}^3$, show that the vector $u = (1, 2, 3)$ is a linear combination of the vectors in $S = \{(1, 1, 1), (-1, 0, 1)\}$.*

Definition 1.48. *Let $S = \{u_1, u_2, \cdots, u_k\}$ be a set of vectors in a vector space V over a field $\mathbb{F}$. The set S* **spans** *V, or V is* **spanned** *by S (denoted $V = \langle S \rangle$), if every vector in V is a linear combination of the vectors in S.*

Example 1.49. *In $\mathbb{R}^2$, let $S = \{(1, 2), (0, -3)\}$. Show that $\mathbb{R}^2 = \langle S \rangle$.*

Example 1.50. *In $\mathbb{R}^3$, let $S = \{(1,1,0),(1,0,1),(0,1,1)\}$. Show that $\mathbb{R}^3 = \langle S \rangle$.*

Definition 1.51 (Finite-dimensional spaces)**.** *A vector space V over a field $\mathbb{F}$ is called* **finite-dimensional** *if there exist a finite subset S of V such that $V = \langle S \rangle$. Otherwise, we call the vector space* **infinite dimensional**.

Definition 1.52. *Let $S = \{u_1, u_2, \cdots, u_k\}$ be a set of distinct vectors in a vector space V over a field $\mathbb{F}$. Then S is said to be* **linearly dependent** *if there exist scalars $a_1, a_2, \cdots, a_k \in \mathbb{F}$ not all zero, such that*

$$a_1 u_1 + a_2 u_2 + \cdots + a_k u_k = 0.$$

Otherwise, S is called **linearly independent**. *That is, S is linearly independent if $a_1 u_1 + a_2 u_2 + \cdots + a_k u_k = 0$ holds only when*

$$a_1 = a_2 = \cdots = a_k = 0.$$

Example 1.53. *In $\mathbb{R}^3$, let $S = \{(1,2,0),(-1,0,1),(0,1,1)\}$. Show that S is linearly independent.*

Example 1.54. *In $P_2(\mathbb{R})$, let $S = \{2 + x + x^2, x + 2x^2, 2 + 2x + 3x^2\}$. Test whether S is linearly dependent or linearly independent.*

Theorem 1.55. *Let $S = \{u_1, u_2, \cdots, u_n\}$ be a set of nonzero vectors in a vector space V over a field $\mathbb{F}$. Then S is linearly dependent if and only if one of the vectors u_j is a linear combination of the preceding vectors in S.*

Example 1.56. *In $\mathbb{R}^3$, let $S = \{(1, 2, -1), (1, -2, 1), (-3, 2, -1), (2, 0, 0)\}$. Test whether S is linearly dependent or not.*

1.10 Bases

Definition 1.57 (Basis). *A set of vectors* $S = \{u_1, u_2, \cdots, u_k\}$ *in a vector space* V *over a field* $\mathbb{F}$ *is called a* **basis** *for* V *if* S *spans* V *(that is* $V = \langle S \rangle$*) and* S *is linearly independent.*

Theorem 1.58 (Existence of bases). *Every vector space has a basis.*

Example 1.59. *In* $\mathbb{R}^2$*, let* $S = \{(1,0),(0,1)\}$*. Then* S *is a basis for* $\mathbb{R}^2$*, called the* **standard (natural) basis** *for* $\mathbb{R}^2$*.*

Example 1.60. *In* $\mathbb{R}^3$*, let* $S = \{(1,0,0),(0,1,0),(0,0,1)\}$*. Then* S *is a basis for* $\mathbb{R}^3$*, called the* **standard (natural) basis** *for* $\mathbb{R}^3$*.*

Example 1.61. *In* $P_2(\mathbb{R})$*, let* $S = \{1, x, x^2\}$*. Then* S *is a basis for* $P_2(\mathbb{R})$*, called the* **standard (natural) basis** *for* $P_2(\mathbb{R})$*.*

Example 1.62. *The set* $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ *is a basis for* $M_{2\times 2}(\mathbb{R})$*, called the* **standard (natural) basis** *for* $M_{2\times 2}(\mathbb{R})$*.*

Example 1.63. *Show that the set* $U = \{1 + x^2, -1 + x, 2 + 2x\}$ *is a basis for* $P_2(\mathbb{R})$*.*

Theorem 1.64. *If* $S = \{u_1, u_2, \cdots, u_n\}$ *is a basis for a vector space* V*, then every vector in* V *can be written in one and only one way as a linear combination of the vectors in* S*.*

Theorem 1.65. *If* $S = \{u_1, u_2, \cdots, u_n\}$ *is a set of nonzero vectors spanning a vector space* V*, then* S *contains a basis* T *for* V*.*

Theorem 1.66. *If* $S = \{u_1, u_2, \cdots, u_n\}$ *is a basis for a vector space* V *and* $T = \{v_1, v_2, \cdots, v_r\}$ *is a linearly independent set of vectors in* V*, then* $n \geq r$*.*

Corollary 1.67. *If* $S = \{u_1, u_2, \cdots, u_n\}$ *and* $T = \{v_1, v_2, \cdots, v_m\}$ *are bases for a vector space* V*, then* $n = m$*.*

Definition 1.68. *The* **dimension** *of a nonzero vector space* V *over a field* $\mathbb{F}$ *is the number of vectors in a basis for* V*. We often write* **dim** V *(or* $\mathbf{dim}_{\mathbb{F}} V$*) for the dimension of* V*. We also define the dimension of* $\{0\}$ *to be zero.*

Example 1.69.

1. $\dim \mathbb{R}^2 = 2$.
2. $\dim \mathbb{R}^n = n$.
3. If $\mathbb{F}$ is any field, then $\dim_{\mathbb{F}} \mathbb{F}^n = n$.
4. $\dim_{\mathbb{R}} \mathbb{C}^2 = 4$.
5. $\dim P_n(\mathbb{R}) = n + 1$.

Theorem 1.70. *If S is a linearly independent set of vectors in a finite-dimensional vector space V, then there is a basis T for V which contains S.*

Example 1.71. *Find a basis for $\mathbb{R}^3$ containing the vectors $(1, 0, 1)$ and $(0, -1, 1)$.*

Theorem 1.72. *Let V be an $n-$dimensional vector space.*

1. *If $S = \{u_1, u_2, \cdots, u_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.*
2. *If $S = \{u_1, u_2, \cdots, u_n\}$ spans V, then S is a basis for V.*

Example 1.73. *Show that the set $S = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ is a basis for $\mathbb{R}^3$.*

Theorem 1.74. *Suppose U is a subspace of a finite-dimensional vector space V. Then*

1. $\dim U \leq \dim V$.
2. *If* $\dim U = \dim V$ *then* $U = V$.

1.11 Dimension of the sum of subspaces

Theorem 1.75 (Dimension of sum)**.** *Suppose V is a vector space and suppose U and W are finite-dimensional subspaces of V. Then*

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

Example 1.76. *In $\mathbb{R}^3$, let $U = \{(x, y, z) \mid z = x + 2y\}$ and $W = \{(x, y, z) \mid x = -y\}$. Find $\dim(U + W)$ and $U + W$.*

Theorem 1.77. *Suppose U and W are finite-dimensional subspaces of a vector space V. Let $S = U + W$. Then $S = U \oplus W$ if and only if $\dim S = \dim U + \dim W$.* □

1.12 Coordinates

Definition 1.78. *Let* $S = \{u_1, u_2, \cdots, u_n\}$ *be a basis for a vector space* V *over a field* $\mathbb{F}$. *Then every vector* u *in* V *can be written as* $u = a_1u_1 + a_2u_2 + \cdots + a_nu_n$, *where* $a_1, a_2, \cdots, a_n \in \mathbb{F}$. *The vector* $X = (a_1, a_2, \cdots, a_n)$ *is called a* **coordinate vector** *of* u *relative to the basis* S.

Example 1.79. *In* $P_2(\mathbb{R})$, *find the coordinate vector of* $u = 3 - x^2$ *relative to the basis* $S = \{3, -1 + x, x^2\}$.

Definition 1.80. *Let* $S = \{u_1, u_2, \cdots, u_n\}$ *and* $S^* = \{u_1^*, u_2^*, \cdots, u_n^*\}$ *be bases for a vector space* V *over a field* $\mathbb{F}$. *Then every vector* $u_i \in S$ *can be written as a linear combination of the vectors in* S^*. *That is,*
$u_1 = a_{11}u_1^* + a_{12}u_2^* + \cdots + a_{1n}u_n^*$,
$u_2 = a_{21}u_1^* + a_{22}u_2^* + \cdots + a_{2n}u_n^*$,
$\vdots$
$u_n = a_{n1}u_1^* + a_{n2}u_2^* + \cdots + a_{nn}u_n^*$. *Let* $P = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$. *Then* P *is called a* **transition matrix** *from the basis* S *to the basis* S^*.

Example 1.81. *Find the transition matrix from the basis* $S = \{(2, 3), (0, 3)\}$ *for* $\mathbb{R}^2$ *to the basis* $S^* = \{(-1, 0), (3, 3)\}$.

Example 1.82. *Let* $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$ *be a transition matrix from the basis* $S = \{(2, 1), (0, 3)\}$ *for* $\mathbb{R}^2$ *to the basis* $S^* = \{u_1^*, u_2^*\}$. *Find* u_1^* *and* u_2^*.

Theorem 1.83. *Let S and S^* be two bases for a fdvs V over a field $\mathbb{F}$. Let P be a transition matrix from S to S^*. If X is a coordinate vector of $u \in V$ with respect to S, then $X^* = XP$ is a coordinate vector of u with respect to S^*.*

Theorem 1.84. *Let S and S^* be bases for a vector space V. If there exist a matrix $P = (p_{ij})$ such that for any $u \in V$, the vector coordinate of u is X and X^* relative to S and S^*, respectively, and $X^* = XP$, then P is a transition matrix from S to S^*.*

Theorem 1.85. *Let S, S^* and S^{**} be bases for a vector space V. Let P be a transition matrix from S to S^* and Q be a transition matrix from S^* to S^{**}. Then PQ is a transition matrix from S to S^{**}.*

Theorem 1.86. *Let S and S^* be bases for a vector space V. If P is a transition matrix from S to S^*, then P^{-1} is a transition matrix from S^* to S.*