Linear Algebra I

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1 Vector Spaces

2 Linear Transformations

Linear transformations, elementary properties of linear transformations, kernel and injectivity, rank plus nullity, surjectivity, isomorphisms.

2.1 Linear Transformations

Definition 2.1 (Linear transformation)**.** *Let U and V be vector spaces over the same field $\mathbb{F}$. A function $T : U \to V$ is called a* **linear transformation** *if:*

(L1) for all $u, v \in U$, we have $T(u + v) = T(u) + T(v)$;

(L2) for all $u \in U$ and $a \in \mathbb{F}$, we have $T(au) = aT(u)$.

Example 2.2. *Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x + y, x + z)$. Show that T is a linear transformation.*

Example 2.3. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + 1, x + y)$. Test whether T is a linear transformation or not.*

Example 2.4 (Linear function). *In Calculus, we call a function* $f : \mathbb{R} \to \mathbb{R}$ *linear if* f *is given by the formula* $y = ax + b$. *Is there a relation between these functions and the linear transformation that we introduced?*

Example 2.5. *Let* $T : C^1(a, b) \to C(a, b)$ *defined by* $T(f) = f'$. *Show that* T *is a linear transformation.*

Example 2.6. *Let* U *be any vector space over* $\mathbb{F}$. *Let* $T : U \to U$ *defined by* $T(u) = u$ *for all* $u \in U$. *Show that* T *is a linear transformation (called the* **identity transformation***).*

Example 2.7. *Let* U *and* V *be vector spaces over* $\mathbb{F}$. *Let* $T : U \to V$ *defined by* $T(u) = 0_V$ *for all* $u \in U$. *Show that* T *is a linear transformation (called the* **zero transformation***).*

Example 2.8. *Suppose* $U = 2^{\Omega}$ *and* $V = 2^{\Delta}$ *where* Δ *is a subset of* Ω. *Define a function* $T : U \to V$ *via* $T(A) = A \cap \Delta$ *for each vector-subset* A. *This is a linear transformation.*

Just like with the subspace test, there is an easier, one-condition check of linearity.

Theorem 2.9 (Linearity check). *A function* $T : U \to V$ *is a linear transformation if and only if for all* $u, v \in U$ *and all* $a \in \mathbb{F}$, *we have* $T(au + v) = aT(u) + T(v)$.

Theorem 2.10. *Suppose* $T : U \to V$ *is a linear transformation. Then*

(1) $T(0_U) = 0_V$;

(2) $T(-u) = -T(u)$ *for all* $u \in U$;

(3) $T(u - v) = T(u) - T(v)$ *for all* $u, v \in U$.

Theorem 2.11. *Let U and V be finite dimensional vector spaces over $\mathbb{F}$, and let $S = \{u_1, \cdots, u_n\}$ be a basis for U. Then for any set $\{v_1, \cdots, v_n\}$ of n vectors (not necessarily distinct) in V, there is a unique linear transformation $T : U \to V$ such that $T(u_i) = v_i$ for $i = 1, \cdots, n$.*

Example 2.12. *Let $S = \{(1, 0), (2, 1)\}$ be a basis for $\mathbb{R}^2$. Find a linear transformation $T : \mathbb{R}^2 \to P_2(\mathbb{R})$ such that $T(1, 0) = 1+x$ and $T(2, 1) = x-x^2$.*

Definition 2.13. *Let U and V be vector spaces over $\mathbb{F}$. For any linear transformations $S, T : U \to V$ and $r \in \mathbb{F}$, we define $S + T : U \to V$ by $(S+T)(u) = S(u)+T(u)$ for all $u \in U$ and $rT : U \to V$ by $(rT)(u) = rT(u)$ for all $u \in U$.*

Example 2.14. *Let* $S, T : \mathbb{R}^2 \to \mathbb{R}^2$ *be linear transformations defined by* $T(x, y) = (x, 2y)$ *and* $S(x, y) = (y, x)$*. Find* $2T$*,* $S + T$ *and* $3T - 4S$*.*

Theorem 2.15. *Let* U *and* V *be vector spaces over* $\mathbb{F}$*. For any linear transformations* $S, T : U \to V$ *and* $r \in \mathbb{F}$*,* $S+T$ *and* rT *are linear transformations.*

Definition 2.16. *The set of all linear transformations from* U *to* V *is denoted by* $L(U, V)$*.*

Remark 2.17. *Let* U *and* V *be vector spaces over* $\mathbb{F}$*. Then* $L(U, V)$ *is a vector space over* $\mathbb{F}$*.*

2.2 Kernel and Image of a Linear Transformation

Definition 2.18. *Suppose U and V be vector spaces over $\mathbb{F}$ and $T \in L(U, V)$.*

(1) The **kernel** *of T, denoted by* **Ker** *T, is the set*

$$\operatorname{Ker} T = \{u \in U \mid T(u) = 0\}.$$

(2) The **image (range)** *of T, denoted by* **Im** *T, is the set*

$$ImT = \{v \in V \mid v = T(u) \text{ for some } u \in U\} = \{T(u) \mid u \in U\}.$$

Example 2.19. *Let $T \in L(\mathbb{R}^3, \mathbb{R}^2)$ defined by $T(x, y, z) = (x + y + z, 0)$. Find $KerT$ and ImT.*

Example 2.20. *Let $T \in L(\mathbb{R}^2, \mathbb{R}^2)$ defined by $T(x, y) = (y, x)$. Find $KerT$ and ImT.*

Theorem 2.21. *Suppose U and V be vector spaces over $\mathbb{F}$ and $T \in L(U, V)$. Then*

(1) $KerT$ is a subspace of U.

(2) ImT is a subspace of V.

Theorem 2.22. *Let U and V be vector spaces over $\mathbb{F}$ and $T \in L(U, V)$. Then T is one-to-one if and only if $KerT = \{0\}$.*

Remark 2.23. *Let U and V be vector spaces over $\mathbb{F}$ and $T \in L(U, V)$. Then T is onto if and only if $ImT = V$.*

Definition 2.24. *Let U and V be vector spaces over $\mathbb{F}$ and $T \in L(U, V)$.*

(1) If $KerT$ is a finite dimensional subspace of U, then the dimension of $KerT$ is called the **nullity** *of T, and denoted by $n(T)$ (or $\dim KerT$).*

(2) If ImT is a finite dimensional subspace of V, then the dimension of ImT is called the **rank** *of T, and denoted by $r(T)$ (or $\dim ImT$).*

Example 2.25. *In Example* 2.19*, $n(T) = 2$ and $r(T) = 1$, while in Example* 2.20*, $n(T) = 0$ and $r(T) = 2$.*

Theorem 2.26 (Rank+nullity)**.** *Suppose $T \in L(U, V)$ and suppose that* $\dim U < \infty$. *Then* $\dim ImT + \dim KerT = \dim U$.

Example 2.27. *Find a linear transformation $T \in L(\mathbb{R}^2, \mathbb{R}^2)$ such that $\{(1, 2)\}$ is a basis for ImT.*

Example 2.28. *Find a linear transformation* $T \in L(\mathbb{R}^3, \mathbb{R}^2)$ *such that* $\{(1, 0, -2), (0, 3, 1)\}$ *is a basis for* $KerT$.

2.3 Isomorphisms and inverse mappings

Definition 2.29. *Let* U, V *and* W *be vector spaces over* $\mathbb{F}$, $T \in L(U, V)$ *and* $S \in L(V, W)$. *We define the function* $S \circ T : U \to W$ *by* $(S \circ T)(u) = S(T(u))$ *for all* $u \in U$. $S \circ T$ *is called the composition of* S *and* T.

Theorem 2.30. *If* $T \in L(U, V)$ *and* $S \in L(V, W)$, *then* $S \circ T \in L(U, W)$.

Example 2.31. *Let* $T \in L(\mathbb{R}^3, \mathbb{R}^2)$ *define by* $T(x, y, z) = (z - y, 2x)$ *and* $S \in L(\mathbb{R}^2, \mathbb{R}^3)$ *define by* $S(x, y) = (2y, x - y, x)$. *Find* $S \circ T$ *and* $T \circ S$.

Definition 2.32. *Let U and V be vector spaces over $\mathbb{F}$, $T \in L(U, V)$ and $S \in L(V, U)$.*

(1) If $S \circ T = I_U$, then S is called a **left inverse** *to T.*

(2) If $T \circ S = I_V$, then S is called a **right inverse** *to T.*

(3) S is called an **inverse** *to T if S is a left inverse to T and S is a right inverse to T.*

Example 2.33. *Let $T \in L(\mathbb{R}^2, \mathbb{R}^3)$ defined by $T(x, y) = (x, x + 2y, x - y)$. Show that $S \in L(\mathbb{R}^3, \mathbb{R}^2)$ defined by $S(x, y, z) = (x, -2x + y + z)$ is a left inverse to T.*

Example 2.34. *Let $T \in L(\mathbb{R}^2, P_1(\mathbb{R}))$ defined by $T(a, b) = (a + b) + bx$. Show that $S \in L(P_1(\mathbb{R}), \mathbb{R}^2)$ defined by $S(a + bx) = (a - b, b)$ is an inverse to T.*

Example 2.35. *Let $T \in L(\mathbb{R}^3, \mathbb{R}^2)$ defined by $T(x, y, z) = (x + y, 2x - z)$. Show that $S \in L(\mathbb{R}^2, \mathbb{R}^3)$ defined by $S(x, y) = (\frac{1}{2}y, x - \frac{1}{2}y, 0)$ is a right inverse to T.*

Theorem 2.36. *Let U and V be vector spaces over $\mathbb{F}$ and $T \in L(U,V)$. If T has left and right inverses, then they must be equal.*

Definition 2.37. *Let U and V be vector spaces over $\mathbb{F}$. $T \in L(U,V)$ is called an* **isomorphism** *(or* **invertible***, or* **non-singular***) if there exist $S \in L(V,U)$ such that $S \circ T = I_U$ and $T \circ S = I_V$, and we write $S = T^{-1}$.*

Definition 2.38. *Let U and V be vector spaces over $\mathbb{F}$. U is said to be* **isomorphic** *to V, denoted by $U \cong V$, if there exist $T \in L(U,V)$ which is an isomorphism.*

Example 2.39. *In Example* 2.34*, T is an isomorphism, and $P_1(\mathbb{R}) \cong \mathbb{R}^2$.*

Theorem 2.40. *If $T : U \to V$ is an isomorphism, then T^{-1} is a linear transformation.*

Theorem 2.41. *Let U and V be vector spaces over $\mathbb{F}$ and $T \in L(U,V)$. Then T has a right inverse if and only if T is onto.*

Theorem 2.42. *Let U and V be vector spaces over $\mathbb{F}$ and $T \in L(U, V)$. Then T has a left inverse if and only if T is one-to-one.*

Theorem 2.43. *Let U and V be vector spaces over $\mathbb{F}$ and $T \in L(U, V)$. Then T is an isomorphism if and only if T is one-to-one and onto.*

Theorem 2.44. *Let U and V be finite dimensional vector spaces over $\mathbb{F}$. Then $U \cong V$ if and only if* $\dim U = \dim V$.

Example 2.45. *Let $\mathbb{F}$ be any field and U be an n-dimensional vector space over $\mathbb{F}$. Then $U \cong \mathbb{F}^n$.*

Example 2.46. *Let* $\mathbb{F}$ *be any field. Then* $P_n(\mathbb{F}) \cong \mathbb{F}^{n+1}$*, and* $M_{m\times n}(\mathbb{F}) \cong \mathbb{F}^{mn}$*.*

Example 2.47. $\mathbb{C} \cong \mathbb{R}^2$*, where* $\mathbb{C}$ *is a space over* $\mathbb{R}$*.*

Definition 2.48. *Let* U *and* V *be finite dimensional vector spaces over* $\mathbb{F}$ *such that* $\dim U = m$ *and* $\dim V = n$*. Let* $H = \{u_1, u_2, \cdots, u_m\}$ *be a basis for* U *and* $S = \{v_1, v_2, \cdots, v_n\}$ *be a basis for* V*. Let* $T \in L(U, V)$*. Then* $T(u_i) \in V$ *for* $i = 1, \cdots, m$*.* $T(u_i)$ *can expressed uniquely as a linear combination of the vectors in* S*.*

$$T(u_1) = a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n$$

$$T(u_2) = a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n$$

$$\vdots$$

$$T(u_m) = a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n$$

Let $M_T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$*. The matrix* M_T *is the* **matrix of** T *relative to the bases* H *and* S*, and it is denoted by* (M_T, H, S) *or* M_T*.*

Example 2.49. *Let* $T \in L(\mathbb{R}^3, \mathbb{R}^2)$ *defined by* $T(x, y, z) = (x + y, x + z)$*. Find* M_T *relative to the standard bases for* $\mathbb{R}^3$ *and* $\mathbb{R}^2$*.*

Theorem 2.50. *Let $H = \{u_1, u_2, \cdots, u_m\}$ be a basis for U and $S = \{v_1, v_2, \cdots, v_n\}$ be a basis for V. Let $T \in L(U, V)$ and $u \in U$. Let $X = (x_1, x_2, \cdots, x_m)$, and $Y = (y_1, y_2, \cdots, y_n)$ be vector coordinates of u and $T(u)$ with respect to the bases H and S, respectively. Let M_T be a matrix of T relative to the bases H and S. Then $Y = XM_T$.*

Remark 2.51. *Let $G = \{u_1, u_2, \cdots, u_m\}$, $H = \{v_1, v_2, \cdots, v_n\}$ and $J = \{w_1, w_2, \cdots, w_p\}$ be bases for vector spaces U, V and W over $\mathbb{F}$, respectively. Let $T \in L(U, V)$ and $S \in L(V, W)$. Let M_T be a matrix of T relative to the bases G and H. Let M_S be a matrix of S relative to the bases H and J. Then the matrix of $S \circ T$ relative to the bases G and J is $M_T M_S$(i.e. $M_{S\circ T} = M_T M_S$).*

Remark 2.52. *Let $G = \{u_1, u_2, \cdots, u_m\}$ and $H = \{v_1, v_2, \cdots, v_n\}$ be bases for vector spaces U and V over $\mathbb{F}$, respectively. Let $T, S \in L(U, V)$ and $r \in \mathbb{F}$. Let M_T, M_S, M_{S+T}, and M_{rT} be matrices of $T, S, S + T$, and rT relative to the bases G and H, respectively. Then*

(1) $M_{S+T} = M_S + M_T$.

(2) $M_{rT} = rM_T$.

Remark 2.53. *Let U and V be finite dimensional vector spaces over $\mathbb{F}$ and $T \in L(U, V)$. T is an isomorphism if and only if the matrix for T relative to any pair of bases of U and V is invertible.*

Theorem 2.54. *Let U be a finite dimensional vector space over $\mathbb{F}$ and $T, S \in L(U, U)$. If $S \circ T = I$, then $T \circ S = I$.*

Theorem 2.55. *Let $\mathbb{F}$ be any field and $M \in M_{m\times n}(\mathbb{F})$. Then there exist a linear transformation $T \in L(\mathbb{F}^m, \mathbb{F}^n)$ such that the matrix for T relative to standard bases for $\mathbb{F}^m$ and $\mathbb{F}^n$ is M (i.e. $M_T = M$).*

Theorem 2.56. *Let M and N be $n \times n$ matrices over a field $\mathbb{F}$. If $MN = I$, then $NM = I$.*

Theorem 2.57. *Let U and V be finite dimensional vector spaces over $\mathbb{F}$ such that $\dim U = m$ and $\dim V = n$. Let $T \in L(U, V)$ such that $\dim ImT = r$. Then there exist bases for U and V such that the matrix for T relative to these bases is $M_T = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. (This form is called a* **Normal form***").*

Example 2.58. *Let* $T \in L(\mathbb{R}^3, \mathbb{R}^2)$ *defined by* $T(x, y, z) = (x + y, x + z)$. *Find bases for* $\mathbb{R}^3$ *and* $\mathbb{R}^2$ *for which the matrix for* T *relative to these bases is of normal form.*

Remark 2.59. *Let* $M, M^* \in M_{m\times n}(\mathbb{F})$ *and let* $\dim U = m$ *and* $\dim V = n$. *Then* M *and* M^* *are matrices for a linear transformation* $T \in L(U, V)$ *relative to distinct pair of bases of* U *and* V *if and only if there exist invertible matrices* $P \in M_{m\times m}(\mathbb{F})$ *and* $Q \in M_{n\times n}(\mathbb{F})$ *such that* $M^* = PMQ^{-1}$.