Linear Algebra II

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1 Vector Spaces

2 Linear Transformations

3 Eigenvalues and Eigenvectors

Definition 3.1 (**Eigenvector**). *Suppose $T : U \to U$ is a linear transformation of U. A vector $u \in U$ is an **eigenvector** of T with respect to $\lambda \in \mathbb{F}$ if $T(u) = \lambda u$.*

Example 3.2. *Let $U = \mathbb{R}^2$ and $T \in L(\mathbb{R}^2, \mathbb{R}^2)$ defined by $T(x, y) = (x + y, 2y)$. Determine some eigenvectors of T.*

Note that $u = 0$ is an eigenvector of T with respect to any $\lambda \in \mathbb{F}$. Indeed, $T(0) = 0 = \lambda 0$, so the condition holds, and 0 is an eigenvector with respect to λ. Interestingly, in all other cases the scalar λ that works for an eigenvector u is unique.

Theorem 3.3. *Suppose $u \neq 0$ is an eigenvector for a linear transformation $T : U \to U$. Then there is only one $\lambda \in \mathbb{F}$ such that $T(u) = \lambda u$.*

Definition 3.4 (**Eigenvalue**). *A scalar $\lambda \in \mathbb{F}$ is an **eigenvalue** of a linear transformation $T : U \to U$ if $T(u) = \lambda u$ for a nonzero eigenvector $u \in U$.*

Example 3.5. *Let $U = \{f : f : \mathbb{R} \to \mathbb{R}$ is a differentiable function$\}$ and $\mathbb{F} = \mathbb{R}$. Let $T \in L(U, U)$ defined by $T(f) = f'$. Determine some eigenvalues of T.*

Theorem 3.6. *Let U be an n-dimensional vector space over $\mathbb{F}$ and $T \in L(U,U)$. Let M and M^* be matrices of T associate with bases $G = \{u_1, \cdots, u_n\}, G^* = \{u_1^*, \cdots, u_n^*\}$ for U, respectively. Then for each $\lambda \in \mathbb{F}$, $|M - \lambda I_n| = |M^* - \lambda I_n|$, where I_n is an n-by-n identity matrix.*

Definition 3.7 (Characteristic polynomial). *Let U be an n-dimensional vector space over $\mathbb{F}$ and $T \in L(U,U)$. Let M be a matrix of T associate with any basis for U. The determinant $\Delta(t) := |M - tI_n| = (-1)^n t^n + b_{n-1}t^{n-1} + \cdots + b_1 t + b_0$ is called the* ***characteristic polynomial*** *of T.*

Example 3.8. *Let $U = \mathbb{R}^2$ and $T \in L(\mathbb{R}^2, \mathbb{R}^2)$ defined by $T(x,y) = (y, -x)$. Find characteristic polynomial of T.*

Theorem 3.9. *Let U be an n-dimensional vector space over $\mathbb{F}$, $T \in L(U,U)$ and $\Delta(t)$ be the characteristic polynomial of T. Then λ is an eigenvalue of T if and only if λ is a root of $\Delta(t)$, that is, $\Delta(\lambda) = 0$.*

Example 3.10. *In Example 3.8,* $\Delta(t) = t^2 + 1$. $\Delta(t) \neq 0 \; \forall t \in \mathbb{R}$. *Therefore,* T *has no eigenvalues.*

Example 3.11. *Let* $U = \mathbb{R}^2$ *and* $T \in L(\mathbb{R}^2, \mathbb{R}^2)$ *defined by* $T(x, y) = (x + y, 2y)$. *Find eigenvalues and eigenvectors of* T.

Example 3.12. *Let* $U = \mathbb{C}^2$, $\mathbb{F} = \mathbb{C}$ *and* $T \in L(\mathbb{C}^2, \mathbb{C}^2)$ *defined by* $T(x, y) = (y, -x)$. *Find eigenvalues and eigenvectors of* T.

Example 3.13. *Let* $U = \mathbb{C}^2$, $\mathbb{F} = \mathbb{R}$ *and* $T \in L(\mathbb{C}^2, \mathbb{C}^2)$ *defined by* $T(x, y) = (y, -x)$. *Find eigenvalues and eigenvectors of* T.

Remark 3.14. *Let U be an n-dimensional vector space over $\mathbb{F}$ and $T \in L(U, U)$. Then T has at most n distinct eigenvalues.*

Definition 3.15 (Eigenspace). *Suppose $\lambda \in \mathbb{F}$. The* ***eigenspace*** *of $T : U \to U$ corresponding to λ is the following set:*

$$U_\lambda = \{u \in U : T(u) = \lambda u\}.$$

Theorem 3.16. *The subset U_λ is a subspace of U for all $\lambda \in \mathbb{F}$. Furthermore, $U_\lambda = \ker(T - \lambda I)$, where I is the identity transformation.*

We note that $U_\lambda \neq 0$ if and only if λ is an eigenvalue of T. This leads to a practical method of computing eigenvalues and eigenvectors of T. Note that λ is an eigenvalue of T if and only if $U_\lambda \neq 0$, that is, if U_λ contains nonzero vectors. Hence we have the following result.

Theorem 3.17. *A scalar $\lambda \in \mathbb{F}$ is an eigenvalue of T if and only if $\ker(T - \lambda I) \neq 0$.* □

Definition 3.18 (Geometric multiplicity). *Let U be vector space over $\mathbb{F}$, $T \in L(U, U)$ and λ be an eigenvalue of T. The dimension of U_λ is called the* ***geometric multiplicity*** *of λ and denoted by $GM(\lambda)$.*

Definition 3.19 (Algebraic multiplicity). *Let U be an n-dimensional vector space over $\mathbb{F}$, $T \in L(U, U)$ and λ is an eigenvalue of T. The* ***algebraic multiplicity*** *of λ is defined to be its multiplicity as a root of the characteristic polynomial of T, and denoted by $AM(\lambda)$.*

Remark 3.20. *$AM(\lambda) = n$ if and only if $\Delta(t) = (t - \lambda)^n g(t)$, where $g(t)$ is a polynomial of t and $g(\lambda) \neq 0$.*

Example 3.21. *Let $U = \mathbb{R}^2$ and $T \in L(\mathbb{R}^2, \mathbb{R}^2)$ defined by $T(x, y) = (x, x + y)$. Find algebraic multiplicity and geometric multiplicity of each eigenvalue of T.*

Theorem 3.22. *Let U be an n-dimensional vector space over $\mathbb{F}$, $T \in L(U,U)$ and λ is an eigenvalue of T. Then $GM(\lambda) \leq AM(\lambda)$.*

Theorem 3.23. *Let U be an n-dimensional vector space over $\mathbb{F}$, $T \in L(U,U)$ and λ is an eigenvalue of T. If $AM(\lambda) = 1$, then $GM(\lambda) = 1$.*

Theorem 3.24. *Let U be a vector space over $\mathbb{F}$ and $T \in L(U,U)$. Let $u_1, \cdots, u_m$ be eigenvectors of T associate with distinct eigenvalues $\lambda_1, \cdots, \lambda_m$. Then $\{u_1, \cdots, u_m\}$ is linearly independent.*

Theorem 3.25. *Let U be a vector space over $\mathbb{F}$ and $T \in L(U,U)$. Let λ and σ be distinct eigenvalues of T. Then $U_\lambda \cap U_\sigma = \{0\}$.*

Definition 3.26. *Let U be a finite dimensional vector space over $\mathbb{F}$ and $T \in L(U,U)$. A basis S of U **diagonalizes** T if the matrix of T with respect to S is a diagonal matrix.*

Definition 3.27 (**Diagonalizable**)**.** *Let U be a vector space over $\mathbb{F}$ and $T \in L(U,U)$. If there exist a basis S of U which diagonalizes T, then T is said to be **diagonalizable**.*

Example 3.28. *Let $U = \mathbb{R}^2$ and $T \in L(\mathbb{R}^2, \mathbb{R}^2)$ defined by $T(x,y) = (y,x)$. Let $S_1 = \{(1,0),(0,1)\}$, $S_2 = \{(0,1),(1,0)\}$, $S_3 = \{(1,1),(2,3)\}$ and $S_4 = \{(1,1),(1,-1)\}$. Which of these bases diagonalizes T.*

Remark 3.29. *Let U be an n-dimensional vector space over $\mathbb{F}$ and $T \in L(U,U)$. T is diagonalizable if and only if*

1. *The characteristic polynomial of T, $\Delta(t)$, must be of the form $\Delta(t) = \alpha(t-\lambda_1)^{r_1}\cdots(t-\lambda_k)^{r_k}$, where $r_1+\cdots+r_k = n$, $\lambda_i \in \mathbb{F}$, $\lambda_i \neq \lambda_j$ whenever $i \neq j$, and $\alpha \in \mathbb{F}-\{0\}$.*

2. *For each eigenvalue λ_i of T, $AM(\lambda_i) = GM(\lambda_i)$ for $i = 1, \cdots, k$.*

Example 3.30. *In Example 3.11, T is a diagnalizable since it satisfies the conditions of Remark 3.29.*

Example 3.31. *In Example 3.21, T is not diagnalizable since $AM(1) \neq GM(1)$.*

Example 3.32. *In Example 3.8, T is not diagnalizable since $\Delta(t) = t^2+1 \neq 0 \; \forall t \in \mathbb{R}$.*

Remark 3.33. *Let U be an n-dimensional vector space over $\mathbb{F}$ and $T \in L(U,U)$. Then T is diagonalizable if and only if there exist a basis for U consisting of eigenvectors of T.*

Theorem 3.34. *Let U be an n-dimensional vector space over $\mathbb{F}$ and $T \in L(U,U)$. If T has n distinct eigenvalues, then T is diagonalizable.*

Example 3.35. *Let* $U = \mathbb{R}^2$ *and* $T \in L(\mathbb{R}^2, \mathbb{R}^2)$ *defined by* $T(x, y) = (y, x)$. *Find a basis for* $\mathbb{R}^2$ *which diagonalizes* T.