Linear Algebra II

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4 Euclidean Vector Spaces

Definition 4.1 (Inner Product). *Let U be any real vector space. An* ***inner product*** *on U is a function $\langle , \rangle : U \times U \to \mathbb{R}$ that assigns to each pairs of vectors u, v of U a real number $\langle u, v \rangle$ satisfying*

1. *For all $u, v \in U, \langle u, v \rangle = \langle v, u \rangle$.*
2. *For all $u, v \in U$ and $r \in \mathbb{R}, \langle ru, v \rangle = r\langle u, v \rangle$.*
3. *For all $u, v, w \in U, \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$.*
4. *For all $u \in U$,*

$$(i)\ \langle u, u \rangle \geq 0,$$
$$(ii)\ If\ \langle u, u \rangle = 0,\ then\ u = 0.$$

Example 4.2. *Let $U = \mathbb{R}^2$ and $u = (x_1, x_2), v = (y_1, y_2) \in \mathbb{R}^2$. Define $\langle , \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $\langle u, v \rangle = x_1y_1 + x_2y_2$. Show that $\langle , \rangle$ is an inner product on $\mathbb{R}^2$. (It is called* ***standard inner product on*** *$\mathbb{R}^2$)*

Example 4.3. *Let* $U = \mathbb{R}^2$ *and* $u = (x_1, x_2), v = (y_1, y_2) \in \mathbb{R}^2$*. Define* $\langle , \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ *by* $\langle u, v\rangle = x_1y_1 - x_2y_1 - x_1y_2 + 3x_2y_2$*. Show that* $\langle , \rangle$ *is an inner product on* $\mathbb{R}^2$*.*

Example 4.4. *Let* $U = \mathbb{R}^n$ *and* $u = (x_1, \cdots, x_n), v = (y_1, \cdots, y_n) \in \mathbb{R}^n$*. Define* $\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *by* $\langle u, v\rangle = x_1y_1 + \cdots + x_ny_n$*. Show that* $\langle , \rangle$ *is an inner product on* $\mathbb{R}^n$*. (It is called* ***standard inner product on*** $\mathbb{R}^n$*)*

Example 4.5. *Let* $U = P_n(\mathbb{R})$ *and* $u(x), v(x) \in P_n(\mathbb{R})$*. Define* $\langle , \rangle : P_n(\mathbb{R}) \times P_n(\mathbb{R}) \to \mathbb{R}$ *by* $\langle u(x), v(x)\rangle = \int_0^1 u(x)v(x)dx$*. Show that* $\langle , \rangle$ *is an inner product on* $P_n(\mathbb{R})$*. (It is called* ***standard inner product on*** $P_n(\mathbb{R})$*)*

Example 4.6. *Let* $U = M_{mn}(\mathbb{R})$ *and* $u = (a_{ij}), v = (b_{ij}) \in M_{mn}(\mathbb{R})$*. Define* $\langle , \rangle : M_{mn}(\mathbb{R}) \times M_{mn}(\mathbb{R}) \to \mathbb{R}$ *by* $\langle u, v\rangle = \sum_{j=1}^n \sum_{i=1}^m a_{ij}b_{ij}$*. Show that* $\langle , \rangle$ *is an inner product on* $M_{mn}(\mathbb{R})$*. (It is called* ***standard inner product on*** $M_{mn}(\mathbb{R})$*)*

Example 4.7. *Let U be an n-dimensional vector space over $\mathbb{R}$ and $S = \{u_1, \cdots, u_n\}$ be a basis for U. Let $u, v \in U$. Then there exist $x_1, \cdots, x_n, y_1, \cdots, y_n \in \mathbb{R}$ such that $u = x_1u_1 + \cdots + x_nu_n$ and $v = y_1u_1 + \cdots + y_nu_n$. Define $\langle , \rangle : U \times U \to \mathbb{R}$ by $\langle u, v\rangle = x_1y_1 + \cdots + x_ny_n$. Show that $\langle , \rangle$ is an inner product on U.*

Theorem 4.8. *Let U be any real vector space and $\langle , \rangle$ be an inner product on U. Then for all $u, v, w \in U$, $r \in \mathbb{R}$, the following is hold*

1. $\langle u, rv\rangle = r\langle u, v\rangle$.
2. $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$.
3. $\langle u, 0\rangle = \langle 0, u\rangle = 0$.
4. *If* $u \neq 0$, *then* $\langle u, u\rangle > 0$.

Definition 4.9. *A real vector space U which has defined on it an inner product $\langle , \rangle$ is called* ***Euclidean vector space*** *or* ***inner product space****. We denote it by $\langle U, \langle , \rangle\rangle$.*

Example 4.10. *Let $U = \mathbb{R}^2$ and $u = (x_1, x_2), v = (y_1, y_2) \in \mathbb{R}^2$. Define $\langle , \rangle_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $\langle u, v\rangle_1 = x_1y_1 + x_2y_2$, by Example 4.2, $\langle , \rangle_1$ is an inner product on $\mathbb{R}^2$.*
Define $\langle , \rangle_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $\langle u, v\rangle_2 = x_1y_1 - x_2y_1 - x_1y_2 + 3x_2y_2$, by Example 4.3, $\langle , \rangle_2$ is an inner product on $\mathbb{R}^2$.
Therefore $\langle \mathbb{R}^2, \langle , \rangle_1\rangle$ and $\langle \mathbb{R}^2, \langle , \rangle_2\rangle$ are different Euclidean vector spaces.

Theorem 4.11 (**Cauchy-Schwarz inequality**). *If u and v are any two vectors in an Euclidean vector space U, then $\langle u, v\rangle^2 \leq \langle u, u\rangle\langle v, v\rangle$.*

Definition 4.12. *Let $\langle U, \langle , \rangle \rangle$ be an Euclidean vector space. The* ***length*** *or* ***norm*** *of a vector $u \in U$ defined as $||u|| = \sqrt{\langle u, u \rangle}$.*

Example 4.13. *Let $\langle M_{22}(\mathbb{R}), \langle , \rangle \rangle$ be an Euclidean vector space. Find $|| \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} ||$.*

Note 4.14. *The Cauchy-Schwarz inequality can be reformulated as follows: If u and v are any two vectors in an Euclidean vector space U, then $|\langle u, v \rangle| \leq ||u|| \cdot ||v||$.*

Theorem 4.15. *Let $\langle U, \langle , \rangle \rangle$ be an Euclidean vector space. Then*

1. *For all $u \in U$, $||u|| \geq 0$.*
2. *For all $u \in U$, $||u|| = 0$ if and only if $u = 0$.*
3. *For all $u \in U$ and $r \in \mathbb{R}$, $||ru|| = |r| \cdot ||u||$.*
4. *For all $u, v \in U$, $||u + v|| \leq ||u|| + ||v||$. (Triangle inequality)*

Definition 4.16. *Let $\langle U, \langle , \rangle \rangle$ be an Euclidean vector space. The* ***distance*** *between vectors $u, v \in U$ defined as $d(u, v) = ||u - v||$.*

Example 4.17. *Let $\langle M_{22}(\mathbb{R}), \langle , \rangle \rangle$ be an Euclidean vector space. Let*

$$u = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } v = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}.$$

Find $d(u, v)$.

Theorem 4.18. *Let $\langle U, \langle , \rangle \rangle$ be an Euclidean vector space. Then*

1. *For all $u, v \in U, d(u, v) \geq 0$.*
2. *For all $u, v \in U, d(u, v) = 0$ if and only if $u = v$.*
3. *For all $u, v \in U, d(u, v) = d(v, u)$.*
4. *For all $u, v, w \in U, d(u, v) \leq d(u, w) + d(w, v)$.*

Definition 4.19. *Let u and v are any two non-zero vectors in an Euclidean vector space $\langle U, \langle , \rangle \rangle$. The* ***angle*** *$\theta$ between u and v is defined by $cos\theta = \frac{\langle u,v \rangle}{||u|| ||v||}$, where $0 \leq \theta \leq \pi$.*

Example 4.20. *In* $\langle P_1(\mathbb{R}), \langle, \rangle\rangle$, *find the angle between* $u(x) = x$ *and* $v(x) = 3x - 2$.

Definition 4.21. *Let* u *and* v *are any two vectors in an Euclidean vector space* $\langle U, \langle, \rangle\rangle$. *Then* u *and* v *are* ***perpendicular*** *or* ***orthogonal*** *if* $\langle u, v\rangle = 0$.

Theorem 4.22 (Pythagorean Theorem). *Let* u *and* v *are any two non-zero vectors in an Euclidean vector space* $\langle U, \langle, \rangle\rangle$. *If* u *and* v *are perpendicular, then* $||u + v||^2 = ||u||^2 + ||v||^2$.

Definition 4.23. *Let* S *be a subset of an Euclidean vector space* $\langle U, \langle, \rangle\rangle$. S *is said to be* ***orthogonal*** *if any two distinct vectors in* S *are orthogonal.*

Definition 4.24. *Let* S *be a subset of an Euclidean vector space* $\langle U, \langle, \rangle\rangle$. S *is said to be* ***orthonormal*** *if* S *is orthogonal and each element of* S *has length* 1.

Example 4.25. *Let* $S = \{(1,0,3), (-3,0,1), (0,1,0)\}$. *Then* S *is an orthogonal basis of* $\mathbb{R}^3$ *with usual inner product but it is not orthonormal.*

Example 4.26. *Let* $S = \{(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$. *Then* S *is an orthonormal basis of* $\mathbb{R}^2$ *with usual inner product.*

Theorem 4.27. *Let* $S = \{u_1, \cdots, u_n\}$ *be an orthogonal set of non-zero vectors in an Euclidean vector space* $\langle U, \langle , \rangle \rangle$. *Then* S *is linearly independent.*

Theorem 4.28. *Let* $S = \{u_1, \cdots, u_n\}$ *be an orthogonal basis of an Euclidean vector space* $\langle U, \langle , \rangle \rangle$. *Then for any* u *in* U, $u = \frac{\langle u, u_1 \rangle}{||u_1||^2} u_1 + \cdots + \frac{\langle u, u_n \rangle}{||u_n||^2} u_n$.

Corollary 4.29. *Let $S = \{u_1, \cdots, u_n\}$ be an orthonormal basis of an Euclidean vector space $\langle U, \langle, \rangle\rangle$. Then for any u in U, $u = \langle u, u_1\rangle u_1 + \cdots + \langle u, u_n\rangle u_n$.*

Example 4.30. *Let $u_1 = (1, 1, 1), u_2 = (0, 1, -1), u_3 = (-2, 1, 1)$. Show that $S = \{u_1, u_2, u_3\}$ is a basis for $\mathbb{R}^3$ and $u = (3, -1, 2)$ is a LC of the vectors in S.*

Remark 4.31. *Let $S = \{u_1, \cdots, u_n\}$ be an orthogonal set of non-zero vectors in an Euclidean vector space $\langle U, \langle, \rangle\rangle$. Then $H = \{\frac{u_1}{||u_1||}, \cdots, \frac{u_n}{||u_n||}\}$ is orthonormal and $[S] = [H]$.*

Remark 4.32 (Gram-Schmidt orthogonalization process). *For every subspace of a finite dimensional Euclidean space $\langle U, \langle, \rangle\rangle$, there exist an orthogonal basis.*
Let $S = \{u_1, \cdots, u_n\}$ be any basis to a subspace M. To transform S to an orthogonal basis $T = \{v_1, \cdots, v_n\}$ we process as follows
Let $v_1 = u_1$,
$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{||v_1||^2} v_1$,
$\vdots$
$v_n = u_n - \frac{\langle u_n, v_{n-1}\rangle}{||v_{n-1}||^2} v_{n-1} - \frac{\langle u_n, v_{n-2}\rangle}{||v_{n-2}||^2} v_{n-2} - \cdots - \frac{\langle u_n, v_1\rangle}{||v_1||^2} v_1$.

Example 4.33. *Find orthogonal basis for $P_1(\mathbb{R})$ with usual inner product.*

Example 4.34. *Let M be a subspace of $\mathbb{R}^4$ that has a basis $S = \{(1, -2, 0, 1), (-1, 0, 0, -1), (0, 0, 0, 1)\}$. Find orthonormal basis for M with usual inner product.*

Definition 4.35. *Let M be a subspace of a vector space U over F. N is said to be* ***complement space*** *to M if N is a subspace and $U = M \oplus N$.*

Theorem 4.36. *Every subspace M of a finite dimensional vector space U has a complement space.*

Example 4.37. *Let $M = \{(x, y) : x = 3y\}$ be a subspace of $\mathbb{R}^2$. Let $N_1 = [\{(1, 0)\}], N_2 = [\{(0, 1)\}]$. Then it is clear that $\mathbb{R}^2 = M \oplus N_1$ and $\mathbb{R}^2 = M \oplus N_2$.*

Definition 4.38. *Let M be a subspace of an Euclidean vector space $\langle U, \langle , \rangle \rangle$. Let $M^{\perp}$ be the set of all vectors in U which are orthogonal to every vector in M. $M^{\perp} = \{u \in U : \langle u, v \rangle = 0 \ \ \forall v \in M\}$.*

Theorem 4.39. *Let M be a subspace of an Euclidean vector space $\langle U, \langle , \rangle \rangle$. Then*

1. *$M^{\perp}$ is a subspace.*
2. *$M \cap M^{\perp} = \{0\}$.*
3. *$U = M \oplus M^{\perp}$.*
4. *$(M^{\perp})^{\perp} = M$.*

Definition 4.40. *Let M be a subspace of an Euclidean vector space $\langle U, \langle , \rangle \rangle$. The space $M^{\perp}$ is called an **orthogonal complement** of M.*

To find $M^{\perp}$, we process as follows

1. Find a basis $\{u_1, \cdots, u_n\}$ for M.
2. Extend $\{u_1, \cdots, u_n\}$ to $\{u_1, \cdots, u_n, v_1, \cdots, v_m\}$ be a basis for U.
3. By using GSOP, find $\{w_1, \cdots, w_n, z_1, \cdots, z_m\}$ which is orthogonal basis for U.
   Therefore, $M^{\perp} = [\{z_1, \cdots, z_m\}]$.

Example 4.41. *Find an orthogonal complement to the subspace $M = \{(x, y) : x - 2y = 0\}$ of $\mathbb{R}^2$ with usual inner product.*

Definition 4.42. *Let M be a subspace of an Euclidean vector space $\langle U, \langle , \rangle \rangle$ and $u \in U$. Then $u = v + w$ for a unique $v \in M$ and $w \in M^{\perp}$. v is called the **projection** of u on a subspace M.*

Example 4.43. *Find the projection of $u = (3, 4)$ on the subspace $M = \{(x, y) : x - 2y = 0\}$ of $\mathbb{R}^2$ with usual inner product.*

To find a projection of a vector u on a subspace M, we process as follows

1. Find an orthogonal basis $\{u_1, \cdots, u_n\}$ for M.
2. Put $v = \frac{\langle u, u_1 \rangle}{||u_1||^2} u_1 + \cdots + \frac{\langle u, u_n \rangle}{||u_n||^2} u_n$. Therefore, v is a projection of u on M.

Definition 4.44. *Let $\langle U, \langle , \rangle \rangle$ be an Euclidean vector space and $T : U \to U$ be a linear transformation. T is said to be **orthogonal transformation** if $\forall u, v \in U, \langle T(u), T(v) \rangle = \langle u, v \rangle$.*

Example 4.45. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = \frac{1}{\sqrt{2}}(x - y, x + y)$, where $\mathbb{R}^2$ is an Euclidean vector space with usual inner product. Show that T is an orthogonal transformation.*

Example 4.46. *Let* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $T(x, y) = (x, x + y)$, *where* $\mathbb{R}^2$ *is an Euclidean vector space with usual inner product. Show that* T *is not orthogonal transformation.*

Theorem 4.47. *Let* $\langle U, \langle , \rangle\rangle$ *be an Euclidean vector space and* $T : U \to U$ *be a LT. The following are equivalent*

1. T *is an orthogonal transformation.*
2. T *preserves the length,* $\forall u \in U$ $||T(u)|| = ||u||$.
3. *For all unit vector* u , $T(u)$ *is a unit vector.*

Theorem 4.48. *Let* $\langle U, \langle , \rangle\rangle$ *be a finite dimensional Euclidean vector space and* $T : U \to U$ *be an orthogonal transformation. Then* T *is an isomorphism.*

Theorem 4.49. *The composition of two orthogonal transformations is an orthogonal transformation.*

Theorem 4.50. *Let $S = \{u_1, \cdots, u_n\}$ be an orthonormal basis for an Euclidean vector space $\langle U, \langle , \rangle\rangle$ and $T : U \to U$ be an orthogonal transformation. Let M be a matrix for T with respect to S. Then the following are hold*

1. *Every row of M is of length 1, which is regarded as a vector in $\mathbb{R}^n$ with standard inner product.*
2. *Rows of M are orthogonal.*
3. *$M^{-1} = M^t$.*
4. *(1) and (2) are hold for columns.*

Definition 4.51. *Let $M \in M_{nn}(\mathbb{R})$. M is said to be* ***orthogonal matrix*** *if $MM^t = I$.*

Theorem 4.52. *Let $M \in M_{nn}(\mathbb{R})$ and M is an orthogonal matrix. Let $S = \{u_1, \cdots, u_n\}$ be an orthonormal basis for an Euclidean vector space $\langle U, \langle , \rangle\rangle$. Then there exist an orthogonal transformation $T : U \to U$ such that the matrix for T with respect to S is M.*