Abstract Algebra I
Chapter 1
Group Theory

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Two Important Groups

1. The Group of Integers Modulo n

Definition 1.2.1: Let n be a fixed positive integer. Two integers a and b are said to be **congruent modulo** n, written

$$a \equiv b(mod\ n),$$

if and only if the difference $a - b$ is divisible by n. That is, $a \equiv b(mod\ n)$ if and only if $a - b = kn$ for some integer k.

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Example 1.2.2:

If $n = 7$, we have

1. $3 \equiv 24 \pmod 7$,
2. $-5 \equiv 2 \pmod 7$,
3. $-8 \equiv -50 \pmod 7$, etc.

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If $a - b$ is not divisible by n, we say that a is incongruent to b modulo n and, in this case, write $a \not\equiv b (\text{mod } n)$.

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Example 1.2.3:

1. $7 \not\equiv 2 \pmod 4$,
2. $17 \not\equiv 3 \pmod 5$.

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Theorem 1.2.4: Let n be a fixed positive integer and a, b be arbitrary integers. Then $a \equiv b(mod\ n)$ if and only if a and b have the same remainder when divided by n.

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Example: $15 \equiv 7 \pmod 4$ because $15 = 4 \cdot 3 + 3$ and $7 = 4 \cdot 1 + 3$

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(5) if $a \equiv b (mod\ n)$, then $ac \equiv bc (mod\ n)$,

(6) if $a \equiv b (mod\ n)$, then $a^k \equiv b^k (mod\ n)$ for every positive integer k.

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Note 1.2.6: The converse of part (5) fails to be true. For instance, $2 \cdot 4 \equiv 2 \cdot 1 \pmod{6}$, yet $4 \not\equiv 1 \pmod{6}$.

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Theorem 1.2.7: If $ca \equiv cb \pmod{n}$ and c is relatively prime to n, then $a \equiv b \pmod{n}$.

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Definition 1.2.8: For an arbitrary integer a, let $[a]$ denote the set of all integers congruent to a modulo n:

$$[a] = \{x \in \mathbb{Z} \mid x \equiv a(mod\ n)\}$$

$$= \{x \in \mathbb{Z} \mid x = a + kn \text{ for some integer } k\}.$$

We call $[a]$ the **congruence class**, modulo n, determined by a and refer to a as a **representative** of this class.

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Example 1.2.9: If we deal with congruence modulo 3, then

$$[0] = \{x \in \mathbb{Z} \mid x = 3k \text{ for some } k \in \mathbb{Z}\}$$
$$= \{\cdots, -9, -6, -3, 0, 3, 6, 9, \cdots\}.$$

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Also

$$[1] = \{x \in \mathbb{Z} \mid x = 1 + 3k \text{ for some } k \in \mathbb{Z}\}$$
$$= \{\cdots, -8, -5, -2, 1, 4, 7, 10, \cdots\}.$$

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Similarly,

$$[2] = \{x \in \mathbb{Z} \mid x = 2 + 3k \text{ for some } k \in \mathbb{Z}\}$$
$$= \{\cdots, -7, -4, -1, 2, 5, 8, 11, \cdots\}.$$

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Observe that every integer lies in one of these three classes. Integers in the same congruence class are congruent modulo 3, while integers in different classes are incongruent modulo 3. In the above illustration, for instance,

$$[-7] = [2] = [11] = [35] = \cdots$$

For convenience, one often selects the smallest nonnegative integer from each class to represent it.
For the general case of congruence modulo n, we write

$$\mathbb{Z}_n = \{[0], [1], [2], \cdots, [n-1]\}.$$

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Theorem 1.2.10: Let n be a positive integer and $\mathbb{Z}_n$ be as defined before. Then

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(4) $\cup\{[a] \mid a \in \mathbb{Z}\} = \mathbb{Z}$.

Two Important Groups

Definition 1.2.11: A binary operation $+_n$ may be defined on $\mathbb{Z}_n$ as follows: for each $[a], [b] \in \mathbb{Z}_n$, let $[a] +_n [b] = [a + b]$.

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Prove that $+_n$ is well defined.

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Example 1.2.12: Suppose we consider congruence modulo 7. Then

$$[3] +_7 [6] = [3 + 6] = [9] = [2],$$

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and

$$[10] +_7 [-15] = [10 - 15] = [-5] = [2].$$

The Group of Integers Modulo n

Theorem 1.2.13: For each positive integer n, the mathematical system $(\mathbb{Z}_n, +_n)$ forms a commutative group, known as the *group of integers modulo n*.

Example 1.2.14: The operation table for $(\mathbb{Z}_4, +_4)$ is

$+_4$	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

The Group of Integers Modulo n

For simplicity, we often write $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$. With this notation, the above operation table assumes the form

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

2. The Symmetric Group on n Symbols

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We can represent f in a two-line form

$$f = \begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix}.$$

The Symmetric Group on n Symbols

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$$i = f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \; f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \; f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \; f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \; f_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

The Symmetric Group on n Symbols

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Thus, $S_3 = \{f_1, f_2, f_3, f_4, f_5, f_6\}$.

The Symmetric Group on n Symbols

A typical multiplication, say $f_4 \circ f_6$, proceeds as follows:

$$f_4 \circ f_6 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = f_2.$$

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On the other hand, we have

$$f_6 \circ f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = f_3,$$

so that multiplication of permutations is not commutative.

The Symmetric Group on n Symbols

Theorem 1.2.18: The pair $(S_n, \circ)$ forms a group, known as the **symmetric group on n symbols**, which is noncommutative for $n \geq 3$.

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The identity for $(S_n, \circ)$ is the permutation $i = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$ and the inverse of any permutation $f \in S_n$ is described by

$$f^{-1} = \begin{pmatrix} f(1) & f(2) & \cdots & f(n) \\ 1 & 2 & \cdots & n \end{pmatrix}.$$

Definition 1.2.19: Let $n_1, n_2, \cdots, n_k$ be k distinct integers between 1 and n. If a permutation $f \in S_n$ is such that

$$\begin{aligned} f(n_i) &= n_{i+1} \text{ for } 1 \leq i < k, \\ f(n_k) &= n_1, \text{ and} \\ f(n) &= n \text{ for } n \notin \{n_1, n_2, \cdots, n_k\}, \end{aligned}$$

then f is said to be a $k-$**cycle**, or a **cycle of length** k.

The Symmetric Group on n Symbols

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Remark 1.2.20: Any two-line form can be written in the form $(n_1 \;\; n_2 \;\; \cdots \;\; n_k)$.

The Symmetric Group on n Symbols

Example 1.2.21: In the symmetric group $(S_5, \circ)$, for example, we have

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 2 & 4 & 3 \end{pmatrix} = (2\ \ 5\ \ 3).$$

The Symmetric Group on n Symbols

Example 1.2.21: In the symmetric group $(S_5, \circ)$, for example, we have

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 2 & 4 & 3 \end{pmatrix} = (2\ \ 5\ \ 3).$$

Example 1.2.22: Write all elements of $(S_3, \circ)$ in cycle form.

The Symmetric Group on n Symbols

Theorem 1.2.23: Every permutation $f \in S_n$ can be written as a commutative product of cycles, no two of which have an element in common.

The Symmetric Group on n Symbols

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Example 1.2.24: In the symmetric group $(S_5, \circ)$,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix} = (1\ \ 5\ \ 3) \circ (2\ \ 4).$$

Definition 1.2.25: Any cycle of length two is called a **transposition**.

The Symmetric Group on n Symbols

Definition 1.2.25: Any cycle of length two is called a **transposition**.

Example 1.2.26: List all transpositions in the group $(S_3, \circ)$.

Corollary 1.2.27: Every permutation may be expressed as the product of transpositions.

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A $k-$cycle $(1 \;\; 2 \;\; \cdots \;\; k)$ can be written as a product of $k-1$ transpositions in the following manner:

$$(1 \;\; 2 \;\; \cdots \;\; k) = (1 \;\; k) \circ (1 \;\; k-1) \circ \cdots \circ (1 \;\; 2).$$

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Example 1.2.28:
In $(S_3, \circ)$, $(1 \;\; 2 \;\; 3) = (1 \;\; 3) \circ (1 \;\; 2)$.

The Symmetric Group on n Symbols

Definition 1.2.29: A permutation is **even** if it can be written as a product of even number of transpositions, and is **odd** if it can be written as a product of odd number of transpositions.

The Symmetric Group on n Symbols

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Example 1.2.30: List all even and odd permutations in the group $(S_3, \circ)$.

The Symmetric Group on n Symbols

Definition 1.2.31: The subset of S_n consisting of all even permutations of n letters with the operation $\circ$ is called the **alternating group** and denoted by $(A_n, \circ)$.

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Example 1.2.32: $A_3 = \{i, (1\ \ 2\ \ 3), (1\ \ 3\ \ 2)\}$.

The Symmetric Group on n Symbols

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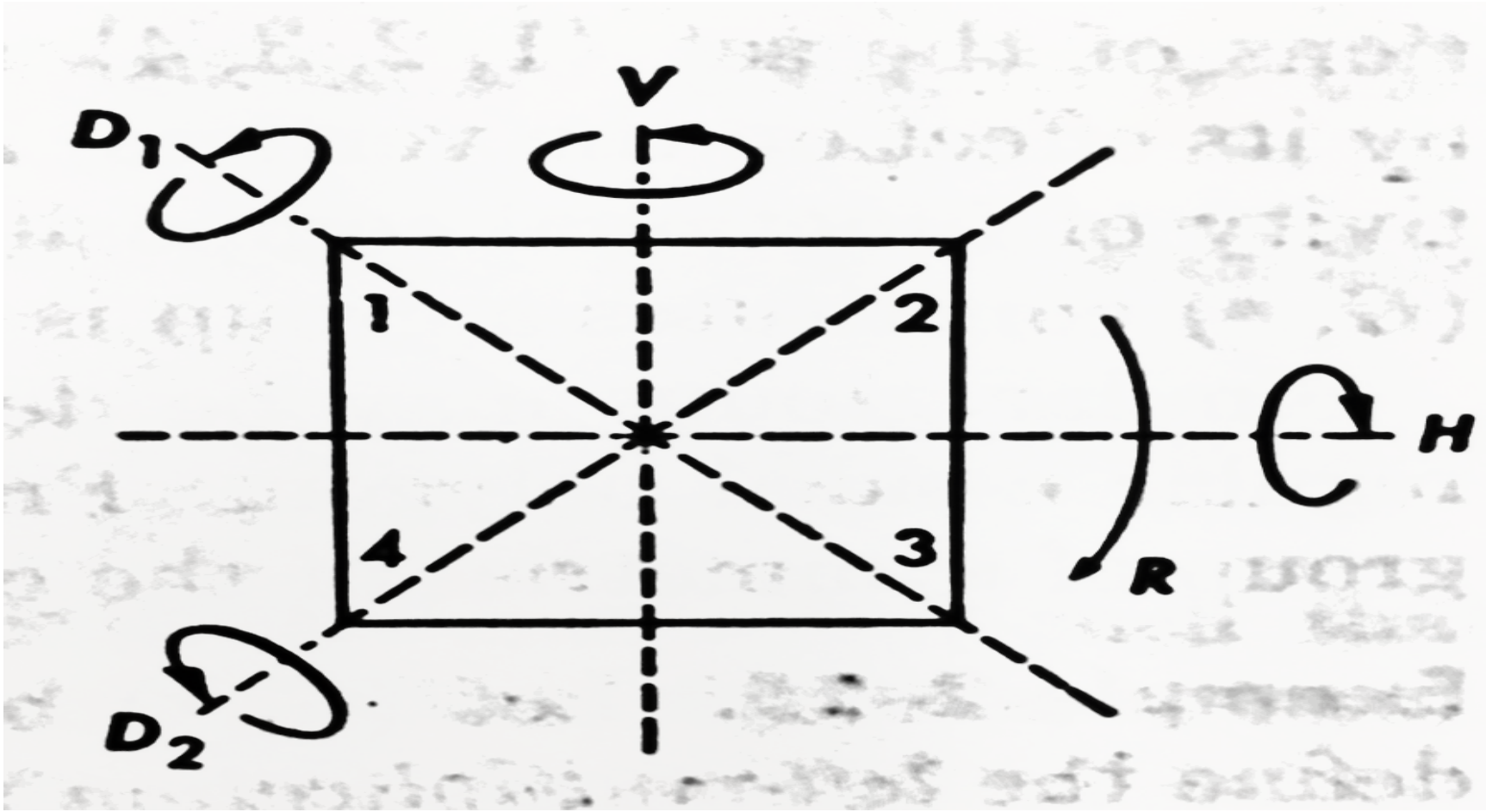
Example 1.2.32: $A_3 = \{i, (1\ \ 2\ \ 3), (1\ \ 3\ \ 2)\}$.

Example 1.2.33: The elements of A_4 are:

Dihedral Group of Order 8(D_8)



Dihedral Group of Order 8(D_8)

$*$	R_{90}	R_{180}	R_{270}	R_{360}	H	V	D_1	D_2
R_{90}	R_{180}	R_{270}	R_{360}	R_{90}	D_1	D_2	V	H
R_{180}	R_{270}	R_{360}	R_{90}	R_{180}	V	H	D_2	D_1
R_{270}	R_{360}	R_{90}	R_{180}	R_{270}	D_2	D_1	H	V
R_{360}	R_{90}	R_{180}	R_{270}	R_{360}	H	V	D_1	D_2
H	D_2	V	D_1	H	R_{360}	R_{180}	R_{270}	R_{90}
V	D_1	H	D_2	V	R_{180}	R_{360}	R_{90}	R_{270}
D_1	H	D_2	V	D_1	R_{90}	R_{270}	R_{360}	R_{180}
D_2	V	D_1	H	D_2	R_{270}	R_{90}	R_{180}	R_{360}