Q1. Find a basis for $kerT$ and a basis for $ImT$ for each of the following.

a. $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (2x, y + z)$.

b. $T: P_1(\mathbb{C}) \to \mathbb{C}$ defined by $T(a_0 + a_1 x) = 2a_0 - ia_1$ where $P_1(\mathbb{C})$ and $\mathbb{C}$ are spaces over $\mathbb{C}$.

c. $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x, 2x, 3x)$.

d. $T: M_{2\times2}(\mathbb{R}) \to \mathbb{R}^2$ defined by $T\begin{bmatrix} x & y \\ z & w \end{bmatrix} = (x - w, y + 2z)$.

Q2. If $S, T: \mathbb{R}^3 \to \mathbb{R}^2$ are linear transformations defined by $T(x, y, z) = (x, y)$ and $S(x, y, z) = (x, 2x)$, then find rank and nullity for $T, S, 2T + S$ and $T - 5S$.

Q3. If $U$ is a finite dimensional vector space over a field $\mathbb{F}$ and $T: U \to U$ be a one-to-one linear transformation, then show that $T$ is onto.

Q4. If $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(p(x)) = p(x)'$, then find rank and nullity of $T$.

Q5. Find rank and nullity of $T: \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $T(1, -2) = (-\sqrt{3}, -\sqrt{2})$ and $T(1, 1) = (\sqrt{2}, \sqrt{3})$.

Q6. Let $T: M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ defined by $T(A) = (A - A^t)/2$, where $A^t$ is the transpose of $A$. Then

a. Show that $T$ is a linear transformation.

b. Find $dimKerT$.

Q7. Find a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^4$ such that $ImT$ is spanned by $v_1 = (-1, 1, 0, 0)$ and $v_2 = (1, 0, 2, -3)$.

Q8. Find a linear transformation $T: M_{2\times2}(\mathbb{R}) \to \mathbb{R}^3$ such that $\left\{\begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}\right\}$ is a basis for $KerT$.

Q9. Find all possible compositions for the following.

a. $T: \mathbb{R}^3 \to M_{2\times2}(\mathbb{R}), S: \mathbb{R} \to \mathbb{R}^3$ where $T(x, y, z) = \begin{bmatrix} -x & y - z \\ z & 2x \end{bmatrix}$ and $S(x) = (2x, -x, 0)$.

b. $T: \mathbb{R}^2 \to \mathbb{R}^2$ and $S: \mathbb{R}^2 \to \mathbb{R}^2$ where $T(x, y) = (x + y, 2x - 3y)$ and $S(x, y) = (0, -x + 3y)$.

c. $T: \mathbb{C}^2 \to P_1(\mathbb{C})$ and $S: \mathbb{C}^2 \to \mathbb{C}^2$ where $\mathrm{T}(z_1, z_2) = 2z_1 - iz_1 x$ and $\mathrm{S}(z_1, z_2) = (z_1 + z_2, iz_1)$.

d. $T: \mathbb{R}^3 \to \mathbb{R}^2$ and $S: \mathbb{R}^2 \to \mathbb{R}^3$ where $T(x, y, z) = (0, x + y - z)$ and $S(x, y) = (x - 2y, 3x, 2y)$.

Q10. If $U$ is a finite dimensional vector space over a field $\mathbb{F}$ and $T: U \to U$ is a linear transformation, then show that $KerT \subset KerT^2$ and $ImT^2 \subset ImT$.

Q11. If $T: V \to W$ and $S: U \to V$ are linear transformations such that $KerT = \{0\}$ and $KerS = \{0\}$, then show that $Ker(ToS) = \{0\}$, where $T^2 = ToT$.

Q12. If $T: U \rightarrow U$ is a linear transformation such that $T^2 = ToT = 0$, then show that $T - I$ is an isomorphism.

Q13. If $T: U \rightarrow U$ is a linear transformation such that $T^2 - T + I = 0$, then show that $T^{-1}$ exists and is equal to $I - T$.

Q14. If $T: U \rightarrow U$ is a linear transformation such that $T^2 + 2T + I = 0$, then find $T^{-1}$.