Practice sheet #2

Q1. Find eigenvalues and eigenvectors for each of the following.

a. $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (3x + 3y, x + 5y)$.
b. $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x + y + z, 2y + z, 2y + 3z)$.
c. $T: P_1(\mathbb{R}) \to P_1(\mathbb{R})$ defined by $T(a + bx) = -b + ax$.
d. $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(a_0 + a_1x + a_2x^2) = (5a_0 + 6a_1 + 2a_2) - (a_1 + 8a_2)x + (a_0 - 2a_2)x^2$.
e. $T: M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2c & a + c \\ b - 2c & d \end{bmatrix}$.
f. $T: \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(z_1, z_2) = (z_1 - z_2, 2z_1)$, where $\mathbb{C}^2$ is a space over $\mathbb{C}$.

Q2. Show that the characteristic polynomial for any linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is

$\Delta(t) = t^2 - tr(M)t + \det(M)$, where $tr(M)$ is the trace of a matrix M, which is the sum of diagonal elements of M, and M is a matrix of T relative to any basis for $\mathbb{R}^2$.

Q3. Show that the linear transformation $T: U \to U$ is not isomorphism if and only if $\lambda = 0$ is an eigenvalue of T.

Q4. If $T: U \to U$ is an isomorphism and $u \in U$ is an eigenvector of T with respect to an eigenvalue λ, then show that u is an eigenvector of T^{-1} with respect to an eigenvalue ${}^{1}/_{\lambda}$.

Q5. If $u \in U$ is an eigenvector of a linear transformation $T: U \to U$ with respect to an eigenvalue λ, then show that u is an eigenvector of T^n with respect to an eigenvalue λ^n for any positive integer n.

Q6. If $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (0, x, y)$, then find characteristic polynomial for each of T, T^2 and T^3.

Q7. Find a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $u = (1, 2)$ is an eigenvector with respect to an eigenvalue $\lambda = 5$.

Q8. Find a linear transformation $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ such that $u = 1 + x$ is an eigenvector with respect to an eigenvalue $\lambda = 2$ and $v = -2x^2$ is an eigenvector with respect to an eigenvalue $\lambda = 4$.