Salahaddin University-Erbil
College of Engineering
Department of Water Resources Engineering
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Mathematics III

Functions of Several Variables
(Ch. 14)

19th lecture

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Some Notes about Partial Derivatives

1. If $z = f(x)$, a function of single independent variable x, we get dz/dx.
2. If $z = f(x_1, x_2, ... x_n)$, a function of two or more independent variables $x_1, x_2, ..., x_n$, we get $\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \dots, \frac{\partial z}{\partial x_n}$
3. If $z = u + v$, where $u = f(x, y)$, $v = \phi(x, y)$ then z is a function of x and y.

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}; \qquad \frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

4. If $z = uv$, where $u = f(x, y)$, $v = \phi(x, y)$ then:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}; \qquad \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$$

5. If $z = \frac{u}{v}$, where $u = f(x, y)$, $v = \phi(x, y)$ then:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}; \qquad \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$$

Higher Order Partial Derivatives

Since the first order partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are themselves function of x and y they can be further differentiates partially w.r.t x as well as y. These are called second order partial derivatives of z. The usual notations for these second order derivatives are:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \quad \text{or} \quad f_{xx}(x, y);$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \quad \text{or} \quad f_{yy}(x, y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \quad \text{or} \quad f_{xy}(x, y);$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \quad \text{or} \quad f_{yx}(x, y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \quad \text{or} \quad f_{yx}(x, y) = f_{xy}(x, y)$$

$f_{yx} = (f_y)_x$ Differentiate first with respect to y, then with respect to x

Higher Order Partial Derivatives

EXAMPLE 1

If $z = x\cos y + y\,e^{x}$, find $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial y \partial x}$, $\dfrac{\partial^2 z}{\partial y^2}$, $\dfrac{\partial^2 z}{\partial x \partial y}$

EXAMPLE 2

If $z = x^2 y + \cos y + y\sin x$, find $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial y \partial x}$, $\dfrac{\partial^2 z}{\partial y^2}$, $\dfrac{\partial^2 z}{\partial x \partial y}$

Chain Rule for Functions of Two Variables

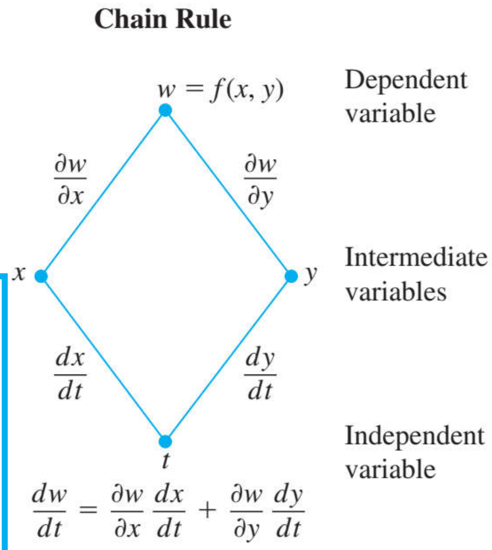
The Chain Rule for functions of a single variable says that when $w = f(x)$ is a differentiable function of x and $x = g(t)$ is a differentiable function of t, w is a differentiable function of t and dw/dt can be calculated by the formula:

$$\frac{dw}{dx} = \frac{dw}{dx}\frac{dx}{dt}$$

If $w = f(x, y)$ is differentiable and if $x = x(t), y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = f_x\big(x(t), y(t)\big) * x'(t) + f_y\big(x(t), y(t)\big) * y'(t)$$

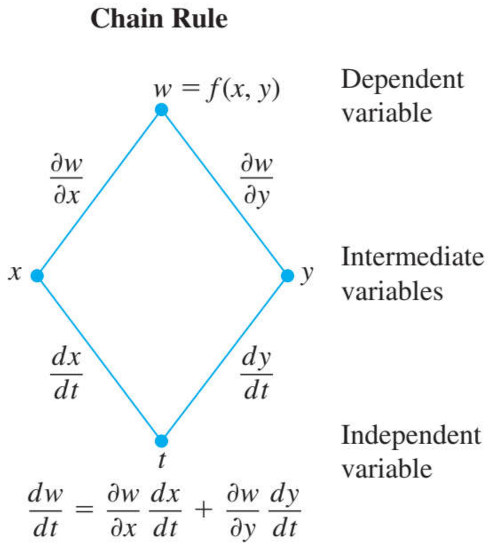
or $$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$ or $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$



Branch Diagram or Tree Diagram

Chain Rule for Functions of Two Variables

$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ or $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$ is called chain rule for functions of one independent variable t, and two intermediate variables x, y.



The ***branch diagram*** in the margin provides a convenient way to remember the Chain Rule. The "true" independent variable in the composite function is t, whereas x and y are intermediate variables (controlled by t) and w is the dependent variable.

EXAMPLE 1

Use the Chain Rule to find the derivative of $w = xy$ w.r.t. t along the path $x = \cos t$, $y = \sin t$. What is the derivative's value at $t = \pi/2$?