# VARIABLE-ORDER FRACTIONAL DIFFERENTIAL EQUATIONS: MATHEMATICAL PHYSICS AND APPLICATION: ARTICLE REVIEW

**Subject Teacher: Dr. Sattar O.Hasan**

**The University of Salahaddin-Erbil, College of Education, Department of Physics, Erbil, Kurdistan Region-Iraq**

**Abstract:** The code for searching the fractional calculation and resolving fractional differential equations is the numerical calculation of the fractional integrals and derivatives. Due to the non-location and modernity of the fractional differential operators especially for variable-order fractional differential conditions (VO-FDEs), the precise arrangements to fractional conditions must be found in genuine applications. Therefore, it is essential for fractional differential equations to improve numerical methods. Furthermore, the (VO-FDE) models are commonly recognized as an exploration of real-world phenomena as a diverse mathematical tool, and as an alternate solution. We also have a brief overview of various physical models and deployment procedures. he most objective of this article is to help perusers in selecting the proper portrayal, show, and numerical arrangement to unravel complex issues.

**Keywords:** variable-order; fractional calculus; fractional differential equations; numerical methods; applications

## 1. INTRODUCTION

The analysis of real or complex order differential and integral operators is known as fractional calculus. Many real-world phenomena could be the most successful explained using fractional operators than using integer order calculus. The inherent multiscale existence of these operators is an interesting feature. As a consequence, memory effects (i.e., a system's response is a function of its previous history) are enabled by time-fractional operators, while non-local and scale effects are enabled by space-fractional operators. The several properties of fractional operators have spurred renewed interest in fractional calculus in recent years, resulting in a slew of applications, especially in the simulation of physical problems. As a result, Variable-Order fractional calculus became a viable option for providing a statistical foundation for accurately characterizing complex physical systems and processes. Due to their appropriateness in modeling a wide assortment of marvels over numerous areas of science and designing, counting atypical dissemination, $\mathrm{VO-FDEs}$ have drawn an expanding sum of intrigued (Chechkin, Gorenflo et al. 2005) viscoelastic mechanics (Coimbra 2003), control frameworks, petroleum building, and numerous other branches of material science and building, to title a couple of [(Awotunde, Ghanam et al. 2015), (Cai, Chen et al. 2018), (Metzler and Klafter 2000)].

Samko and Ross suggested the $\mathrm{Variable-Order}$ integral and differential concepts, as well as some simple properties, in 1993. After summarizing the findings of the $\mathrm{VO}$ fractional operators' study, Lorenzo and Hartley looked at the meanings of Variable-Order fractional operators in various ways. Following that, some additional $\mathrm{VO-FDEs}$ concept extensions and implementation potentials were explored further (Coimbra 2003). In the last ten years, it has become a science hotspot and has sparked widespread interest. Section 5 will have a more comprehensive summary. Extensive research has been done on physical simulation using $\mathrm{VO-FDEs}$ models. (Kobelev, Kobelev et al. 2003), Variable memory issues in scientific and dynamical frameworks, for illustration, where the fractal measurement reverse over time and facilitates, were shown. Coimbra and colleagues used $\mathrm{VO}$ fractional operators to study the viscoelasticity oscillator. As an expansion of the nonlinear standard differential equation, the $\mathrm{Variable-Order}$ fractional derivative is used (Sweilam and Al-Mekhlafi 2016) proposed a novel multi-strain tuberculosis model. We have looked at how $\mathrm{VO-FDE}$ models could be used to characterize transient diffusion. Following that, precise or numerical solutions for $\mathrm{VO-FDEs}$ must be sought. Obtaining the analytical solution of $\mathrm{VO-FDEs}$, on the other hand, is normally difficult. In common, numerical methods are utilized as productive created strategies for the numerical estimation of VO-FDEs (Lin, Liu et al. 2009) For the case, for the VO nonlinear fragmentary dissemination condition, analysts looked into the consistency and joining of an unused express finite-difference guess.

VO-FDEs arrangements were established by Razminia et al. Zayernouri and Karniadakis created an exponentially exact fragmentary ghastly collocation strategy for tackling linear/nonlinear Variable-Order fragmentary differential conditions. Chen et al. formulated an unused understood expository strategy for understanding the two-dimensional variable-order fragmentary permeation condition easily. To clarify atypical dissemination and wave proliferation, For the time Variable-Order fragmentary subordinate, Zhao et al. proposed a second-order guess formula. Following that, (Cao and Qiu 2016) To derive a high-order numerical method for Variable-Order FDEs , they used a second-order numerical approximation. Certain characteristics and inversion formulas of the VO operator $d(x)f(x)/dx(x)$ have been mentioned using the Riemann-Liouville definition and the Fourier transform. (Samko and Ross 1993).

In addition, in Section 2 Discussion of the VO fractional integral and derivative structures from a physical standpoint. In Section 3, We look at the Numerical methods for VO-FDEs. In Section 4, we explain the Applications of VO-fractional differential equations models.

## 2. DISCUSSION OF THE VO FRACTIONAL INTEGRAL AND DERIVATIVE STRUCTURES FROM A PHYSICAL STANDPOINT

### 2.1 In device memory characterization, VO fractional operators are used:

For operators from exterior the zone, the memory property of gadgets is characterized utilizing the Variable-Order fragmentary indispensably and subordinate. From a physical standpoint, it provides a modern way to explore complex processes, genetic effects, and self-similarity [(Sun, Chen et al. 2011), (Ramirez and Coimbra 2007)]. As a result, the use of VO fractional derivative and integral in theoretical analysis and physical simulation has gotten a lot of recognition. The VO fractional integral/derivative discusses two kinds of concepts for memory structures. To put it another way, one must explain how the device memory varies over time and in relation to spatial coordinates (Kobelev, Kobelev et al. 2003). Another is linked to order background memory; specifically, VO is inspired by previous distinction order values and has a unique memory function (Sheng, Sun et al. 2011). In (Sun, Chen et al. 2011), the difference between these two concepts in terms of describing the machine memory property was studied using the VO fractional relaxation-type differential equation. Zhang and Liu have looked into the effects of time-dependent memory and medium heterogeneity's vector spatial correlation on tracer dynamics. VO fragmentary proportional-integral-derivative was recommended by Dabiri et al. to realize the ideal protect parameters and decide bounded closed-loop reaction for straight dynamical frameworks beneath different starting conditions.

### 2.2. Fractional dynamic system of random order:

For clarifying framework unwinding, constriction, and dissemination, the arbitrary arrange fragmentary subsidiary show has as of late picked up ubiquity. Noise is often introduced into physical structures in real-world applications, such as varieties within the outside weight field in odd dissemination frameworks or a sporadic temperature field in vitality dissemination frameworks (Sun, Chen et al. 2011). This noise invariably causes the whole unit to sway. In this case, the random-order fragmentary subordinate demonstrates suits this kind of variance prepare best. As a result, random-order fragmentary differential conditions models have been created to way better get the effect of framework commotion on physical systems' energetic behavior. A consistent and arbitrary term is utilized within the fragmentary subsidiary arrange for the reenactments, where the steady term speaks to the system's normal working rate and the irregular term speaks to the variances caused by arbitrary clamors. A random-order fragmentary fundamentally is characterized as takes after:

$$I_{0+}^{\alpha_0+\varepsilon_t} f(t) = \frac{1}{\Gamma(\alpha_0 + \varepsilon_t)} \int_0^t (t-\tau)^{(\alpha_{0+\varepsilon_{t-1}})} f(\tau) d\tau, \qquad \alpha_0 + \varepsilon_t > 0. \qquad (2.2)$$

The expression for the random-order fractional derivative is at that point created as

$$D_{0+}^{\alpha_0+\varepsilon_t} f(t) = \frac{d^n}{dt^n}\left(I_{0+}^{n-\alpha_0-\varepsilon_t} f(t)\right), \qquad n-1 < \alpha_0 + \varepsilon_t < n. \qquad (2.3)$$

In real-world designing questions, the random-order fragmentary subordinate show can be utilized to quantitatively degree and depict a system's variance. Evaluating natural outflows, designing hazard expectations, framework soundness examination, and other applications are among the conceivable outcomes. The random-order fragmentary subsidiary can moreover be turned into a random-order fragmentary differential condition show to illustrate bizarre dissemination on discrete limited spaces.(Wu, Baleanu et al. 2017).

## 3. NUMERICAL METHODS FOR VO − FDEs

The induction of an explanatory approach for VO − FDEs is additionally in its early stages due to the standards of VO fragmentary administrators. An assortment of numerical estimation strategies and computational approaches have been proposed to analyze the Variable − Order FDEs models. For this reason, numerous computational approaches have been considered within the writing, counting finite difference methods (FDMs), ghastly strategies, network strategies, and spline addition strategies. [(Atangana and Cloot 2013), (Chen, Liu et al. 2014) ]. To solution a growing number of abstract physical equations, computationally efficient numerical methods have been used. (Sierociuk, Malesza et al. 2015).

**3.1. Numerical methods for time FDEs:**

As it's known, limited contrast plans for Variable-Order time and/or space Fractional Differential Equations have been broadly examined. There are a few discretization frameworks of distinctive elucidations. It is conceivable to specific the discretization of the Caputo-type $\mathrm{Variable} - \mathrm{Order}$ time-fractional subordinate as takes after:

$$\frac{\partial^{\alpha_i^{k+1}} u(x_i, t_{k+1})}{\partial t^{\alpha_i^{k+1}}} = \frac{\tau^{-\alpha_i^{k+1}}}{\Gamma(2-\alpha_i^{k+1})} \Big\{ u(x_i, t_{k+1}) - u(x_i, t_k) + \sum_{j=1}^{k} \big[u\big(x_i, t_{k+1-j}\big) - u\big(x_i, t_{k-j}\big)\big] \times [(j+1)^{1-\alpha_i^{k+1}} - j^{1-\alpha_i^{k+1}}] \Big\} + O(\tau). \quad (3.1)$$

Regarding the VO Riemann-Liouville derivative, (Cao and Qiu 2016) The taking after moved Grunwald interpretation was proposed.

$$A_{\tau,p}^{\alpha(t)} y(t) = \frac{1}{\tau^{\alpha(t)}} \sum_{k=0}^{\infty} g_k^{\alpha(t)} y(t-(k-p)\tau). \quad (3.2)$$

Furthermore, (Lorenzo and Hartley 2002) suggested the following derivation of the VO integral's Laplace transform:

$$\mathfrak{L}\left\{D_t^{-\alpha(t)} f(t)\right\} = \int_0^{\infty} e^{-st} \left( \int_0^t \frac{(t-\tau)^{\alpha(t-\tau)-1}}{\Gamma\big(\alpha(t-\tau)\big)} f(\tau) d\tau \right) dt, \alpha(t) > 0, t > 0. \quad (3.3)$$

We already looked at three limited contrast plans for $\mathrm{VO}$ time FDEs: the formal conspires, the implied conspire, and the Crank-Nicholson conspire. The precision, soundness, and meeting of these three frameworks were surveyed and summarized(Sun, Chen et al. 2012). For the $\mathrm{VO}$ time fractional derivative, the Crank-Nicholson scheme is as follows:

$$\frac{\partial^{\alpha_i^{k+1}} u(x_i,t_{k+1})}{\partial t^{\alpha_i^{k+1}}} = \frac{\tau^{-\alpha_i^{k+1}}}{\Gamma(2-\alpha_i^{k+1})}\left(u(x_i,t_{k+1}) - u(x_i,t_k) + \sum_{j=1}^{k}\left[u\left(x_i,t_{k+1-j}\right) - u\left(x_i,t_{k-j}\right)\right]\right)\left[(j+1)^{1-\alpha_i^{k+1}} - (j)^{1-\alpha_i^{k+1}}\right]. \quad (3.4)$$

Jacobi spectral collocation method was proposed by (Bhrawy and Zaky 2015) as a viable alternative to the following method of solving the two-dimension VO fractional nonlinear cable equation

$$\frac{\partial u(x,y,t)}{\partial t} = D_t^{\Upsilon 1(x,y,t)}\Delta u(x,y,t) - \mu D_t^{1-\Upsilon 2(x,y,t)}\Delta u(x,y,t) + f(x,y,t). \quad (3.5)$$

(Jiang and Liu 2017) Centered on duplicating part hypothesis and collocation strategy, an unused computational procedure for the time VO fragmentary mobile-immobile advection-dispersion show was proposed.

$$\beta_1 \frac{\partial C(x,t)}{\partial t} + \beta_2 D_t^{\Upsilon(x,t)} C(x,t) - \upsilon \frac{\partial C(x,t)}{\partial t} + D\frac{\partial^2 C(x,t)}{\partial t^2} + f(x,t), \quad (3.6)$$

Look at the VO nonlinear fractional wave condition in terms of scientific procedures. For VO fragmentary dissemination conditions, Zayernouri and Karniadakis proposed distinctive FDMs. Sierociuk et al. displayed a $\text{Variable} - \text{Order}$ subordinate calculation plot based on a lattice. The neighborhood outspread premise work strategy was utilized to illuminate the $\text{Variable} - \text{Order}$ time-fractional diffusion condition. Utilizing the duplicating part rule, Li and Wu unraveled VO fragmentary boundary esteem issues for fractional differential conditions. His point on the right B - splint with the sense of caputo and the calculation of two Fort Frankel, Moghdam and Macado suggested three levels without vision for FDM to perform the untimely equations of change-row time lines suggesting an inseparable arrangement to understand the time of the VO status distribution that shows both successful and stable. The explanatory investigation is utilized to examine the soundness and joining of numerical strategies. Atangana explored the transmit condition with VO fragmentary subsidiaries, and the steadiness, as well as the joining examination, were effectively checked(Atangana 2015). Jiang and Li affirmed the exponential joining and proposed a space-time ghostly collocation strategy for the $2 - \text{Dimension}$ VO fragmentary permeation condition (Jiang and Li 2018). Zhang and colleagues proposed a VO time-fractional mobile-immobile advection-dispersion show. An understood computational approach for the 2-Dimension VO fragmentary permeation condition in non-homogeneous permeable media was explored. The proposed method's soundness and merging were examined(Zhang, Liu et al. 2013).

## 4. APPLICATIONS OF VO − FDEs MODELS

### 4.1. Variable − Order fractional diffusion equation models:

Anomaly diffusion is a general phenomenon in which the particle distribution's growth rate or shape does not correspond to a Gaussian distribution, to the best of our knowledge. See this page for more information [(Anh, Angulo et al. 2005), (Sokolov and Klafter 2005), (Wang, Hua et al. 2018)] Warm conduction, solute exchange, groundwater spillage, a gas stream within the exceedingly heterogeneous broken or disarranged permeable fabric, unwinding in engineered or biopolymers, and seismic wave propagation are as it were a couple of the points secured. These executions drive the improvement of modern scientific and physical models. Indeed so, how to bargain with dissemination forms where the dissemination design reverse with time, spatial variety, or framework parameters remains an open address in bizarre dissemination modeling. The Variable − Order fractional derivative models have as of late been utilized to supply a thorough and orderly approach for understanding memory impacts, innate properties, and delay behavior in physical applications, to overcome the confinements of indispensably arrange fragmentary models. (Leith 2003). As a result, Variable − Order subordinate models have ended up a prevalent inquire about a device in complex bizarre dissemination modeling. (Obembe, Hossain et al. 2017) VO time-fractional diffusion models were utilized to consider liquid streams in fractal geometry or heterogeneous media. The taking after could be a generalized time-dependent non-local flux law that's utilized at distinctive scales:

$$u = -\frac{\beta K_a}{\mu^\circ} D_t^{\alpha(t)}(\nabla p). \qquad (4.1)$$

where $K_a$ is the pseudo-permeability, $\beta$ is the transmissibility transformation calculate, $\mu^\circ$ is the oil viscosity. The time-dependent diffusion system, non-uniform medium particle migration, and periodic turbulence diffusion method can all be explained using Variable-Order derivative models. (Sokolov and Klafter 2006). (Gerasimov, Kondratieva et al. 2010) For deciding fluid entrance in permeable media, a VO-FDEs is proposed.

$$\frac{\partial^{\alpha(U)}U}{\partial t^{\alpha(U)}} = \frac{\partial}{\partial x}\left(K(U)\frac{\partial U}{\partial x}\right), \tag{4.2}$$

where U denotes the amount of liquid in the container, and then

$$\frac{\partial^{\alpha(U(x,t))}U(x,t)}{\partial t^{\alpha(U(x,t))}} \lim_{\tau\to 0}\frac{\sum_{k=0}^{\infty} A_k U(x,t-k\tau)}{\tau^{\alpha(U(x,t))}}. \tag{4.3}$$

Furthermore, when describing the Variable-Order fractional derivative, another reason for computer memory may be suggested. The governing equation for time-dependent fractional anomalous diffusion is (Sun, Chen et al. 2009).

$$CD_{0+}^{\alpha(t)}c(x,t) = K\frac{\partial^2 c(x,t)}{\partial x^2}, \;\; 0 < \alpha(t) < 1, \tag{4.4}$$

*K* stands for the dispersion coefficient. Centered on the basic theory of description, other variables may be substituted for the order $\alpha(t)$. The VO time-space the subordinate odd dissemination show is characterized by:

$$D_t^{\alpha(x,t)}c(x,t) = K\frac{\partial^2 c(x,t)}{\partial x^2}, \qquad 0 < \alpha(x,t) < 1. \tag{4.5}$$

(Straka 2018) derived a vector anomalous exponent VO fractional Fokker-Planck equation.

$$D_t^{\beta(x)}p(x,t) = K\frac{\partial^2 P(x,t)}{\partial x^2}. \tag{4.6}$$

As a result, the VO-FDE models are capable of accurately describing anomalous diffusion in complex anisotropic media. The location-dependent diffusion process is defined using the space-dependent VO-FDE model. Furthermore,(Chen, Zhang et al. 2013) A concentration-dependent $\mathrm{VO}-$ FDEs demonstrate was proposed to portray the coupled chloride diffusion-binding forms in strengthened concrete.

$$\frac{\partial^{p(C_f)}C_f(x,t)}{\partial t^{p(C_f)}} = K\frac{\partial^2 C_f(x,t)}{\partial x^2}, 0 < x < +\infty, t > 0, \tag{4.7}$$

where VO of time-fractional derivative is 0< $p(C_f)$ < 1 The solute concentration, which chooses the dissemination or relocation instrument, influences the dissemination movement of certain physical, chemical, and organic areas in a few circumstances (Choong, Wong et al. 2006). The concentration-dependent VO − FDEs model has a wide variety of applications and has been used in chemical and biological diffusion processes as a result. This implementation sense motivates us to dig deeper into the VO fractional diffusion model's concentration-dependent characteristics.

$$cD_{0+}^{\alpha[C(x,t)]}c(x,t) = K\frac{\partial^2 c(x,t)}{\partial x^2}, < \alpha[c(x,t)] < 1. \qquad (4.8)$$

In addition, in numerous viable dissemination forms, physical measuring such as porosity, Reynolds number, fractal measurement, and Hurst numbers may change over time or have particular values in a few spatial positions.

$$cD_{0+}^{\alpha[f(x,t)]}c(x,t) = K\frac{\partial^2 c(x,t)}{\partial x^2}, 0 < \alpha[f(x,t)] < 1. \qquad (4.9)$$

An independent variable-function is represented by $f(x,t)$ in the above equation. Table 1 summarizes the VO − FDEs templates so that readers can understand them intuitively.

Table 1: The applications of different VO fractional diffusion models.

| **Model name** | **Governing equation** | **Physical meaning** |
|---|---|---|
| ***Content-dependent Anomalous diffusion model*** | $\frac{\partial^{\alpha(U)}U}{\partial t^{\alpha(U)}} = \frac{\partial}{\partial x}(K(U)\frac{\partial U}{\partial x})$ | Exploring liquid<br>Infiltration in porous media |
| ***VO time-space dependent anomalous diffusion model*** | $D_t^{\alpha(x,t)}c(x,t) = K\frac{\partial^2 c(x,t)}{\partial x^2}$<br>$0 < \alpha(x,t) < 1$ | Depicting concentration breakthrough curve exhibits diverse anomalous behaviors |
| ***VO fractional Fokker-Planck equation*** | $D_t^{\beta(x)}P(x,t) = K\frac{\partial^2 P(x,t)}{\partial x^2}$ | Describing spatial Heterogeneity in complex anisotropic medium |
| ***Concentration dependent VO fractional diffusion equation model*** | $\frac{\partial^{p(C_f)}C_f(x,t)}{\partial t^{p(C_f)}} = K\frac{\partial^2 C_f(x,t)}{\partial x^2}$<br>$0 < x < \infty, t > 0$ | Describing the coupled chloride<br>Diffusion -binding<br>Processes in reinforced concrete |

To streamline the more complex momentary diffusing in heterogeneous media, we utilized the cleared out and right spatial Variable-Order-FDO.

$$\frac{\partial^{\alpha(x,t)}}{\partial t^{\alpha(x,t)}} u(x,t) = -\upsilon \frac{\partial}{\partial x} u(x,t) + D + \frac{\partial^{\beta(x,t)} u(x,t)}{\partial_{+} x^{\beta(x,t)}} + D - \frac{\partial^{\beta(x,t)} u(x,t)}{\partial_{x^{\beta(x,t)}}}, \quad (4.10)$$

The positive and negative diffusion coefficients are D+ and D-, respectively. To entirety up, the $\mathrm{VO} - \mathrm{FDEs}$ demonstrate could be a well-developed and promising approach for clarifying bizarre dissemination forms in heterogeneous permeable media that are time-dependent, space-dependent or concentration-dependent (Sun, Chen et al. 2012).

**4.2. Constitutive models for Variable-Order fractional viscoelasticity.**

Designing utilizes polymers, plastics, non-Newtonian liquids, plastic, elastic, soil, oil, concrete, and other viscoelastic materials. These items are utilized in chemical, petroleum, science, medication, natural building, and other areas. Stress is a mechanical strain in viscoelastic materials. The two types of conditions that make up the rheological syndrome have been identified as: relaxation, in which tension decreases under constant strain; and creep, in which deformation constantly rises under constant stress. Then, for viscoelastic deformation, a generalized time-dependent model was suggested (Ingman and Suzdalnitsky 2004)

$$\sigma(t) = \mu D_t^{\alpha} \varepsilon(t), \qquad (4.11)$$

where $\sigma$ is the stress, $\varepsilon$ is the strain, $\mu$ is the dimensional coefficient and $D_t^{\alpha}$ is the fractional differential operator. FDE models, according to Bagley, can well explain the linear viscoelastic stress relaxation of polymers at a settled temperature. (Smit and De Vries) have documented a functional fractional derivative rheological model.

$$\sigma = E \frac{d^{\alpha} \varepsilon(t)}{dt^{\alpha}}, \qquad (4.12)$$

where E denotes the content property and $0 < \alpha < 1$. When $\alpha = 0$, the above function becomes Hooke’s law; when $\alpha = 1$ , it is the Newton viscous law. However, it is apparent that the CO fractional derivative models have not taken the heterogeneity in material features into account. From the standpoint of execution, (Sweilam and Assiri 2013) The Variable-Order FDE is a useful instrument for studying certain processes, such as nonlinear viscoelasticity oscillator function. Within the viscoelasticity areas, the VO constitutive models have picked up a divide of thought to bargain with variable mechanical behaviors based on time, space variance, or contraption parameters. The constitutive appearance can be communicated as takes after:

$$\sigma = E\frac{d^{\alpha(t)}\varepsilon(t)}{dt^{\alpha(t)}}. \qquad (4.13)$$

The constitutive show can dependably capture mechanical reactions and depict mechanical property moves. The tension in a viscoelastic material is a result of not just the actual actual strain and strain magnitude, but also the previous strain past. Pedro et al. used a Variable-Order derivative to account for the high non-linearity flow. Centered on the Variable-Order time fractional derivative, Bouras established a novel non-linear thermo-viscoelastics rheological model for high temperature creep in concrete.

$$D_t^{\alpha(T(t))}\varepsilon(t) = \frac{\sigma(t)}{\eta\alpha\big(T(t)\big)}. \qquad (4.14)$$

Li et al. recommended that a Variable-Order fragmentary differential conditions show of shape-memory behavior is best to CO-FDE models when foreseeing the memory manner of shape-memory polymers. The viscoelastic movement of a simplex molecule swaying in a non-Newtonian liquid will capture the plainly visible viscoelasticity of colloidal molecule suspensions. The Variable-Order fragmentary show is regularly utilized to ponder the stress-strain relationship of materials within the field of rheology (which includes suspension and polymer behavior) (Soon, Coimbra et al. 2005). Wu et al. have recommended a crawl demonstration based on the Variable-Order fragmentary subordinate to depict the time-dependent mechanical properties of shake amid crawl. The Abel dashpot's stress-strain organization is distributed as:

$$\varepsilon(t) = \frac{\sigma}{\eta o}\frac{t^{\beta}}{\Gamma(1+\beta)} \quad 0 \le \beta \le 1. \qquad (4.15)$$

As a result, the arrangement of the Variable-Order fragmentary subsidiary can be considered a work of time.

$$\sigma(t) = \eta_{\alpha(t)} c_{t_{k-1}} D_t^{\alpha(t)} \varepsilon(t), \quad 0 \le \alpha(t) \le 1, t_{k-1} \le t \le t_k. \qquad (4.16)$$

Based on the current exploratory discoveries, the improved crawl show based on Variable-Order fractional derivative concurs well with the exploratory information at distinctive crawl speeds. It shows how changing the derivative order in the phase can be used to explain how mechanical properties of materials evolve.

Finally, based on the run the show of utilizing the fractional-order to characterize the mechanical property, a Variable-Order fragmentary viscoelastic show up is chosen from the going at some point as of late the fragmentary viscoelastic show up. Within the Variable-Order demonstration, the variable fragmentary arrangement is thought to speak to the change of mechanical properties, such as strain softening behaviors.

### 4.3. Variable − Order fractional control model.

The fragmentary derivative illustrate has been recognized as a basic reenactment technique in a collection of common and planning ranges (Soon, Coimbra et al. 2005). As a result, the VO fractional calculus has gotten a lot of interest in the control area. Diaz and Coimbra proposed a control approach and utilized the Variable-Order arrangement as the control component to stabilize a chaotic dynamical framework. Variable-Order fractional operators can be utilized to portray Variable-Order fractional clamor and assess the VO fragmentary subordinate of an obscure flag in a loud environment utilizing wavelet investigation. For understanding VO fractional ideal control issues, (Heydari and Avazzadeh 2018) illustrated a modern computational approach centered on Legendre wavelets.

## 5. CONCLUSIONS

VO fractional calculus extends the definition of CO fractional calculus by varying the order of division or integration with time (t), space (x), and other variables. The help of Variable-Order fractional calculus empowers it to clarify a wide assortment of memory frameworks, acquired properties, and complex cycles in common. In expansion, a later investigation has been utilized to investigate the physical signification of Variable-Order fractional calculus. However, efficient numerical/approximate solutions are crucial in operation, and VO-FDE model analytical solutions are extremely difficult to obtain. We trust that this inquiry will help within the ponder of VO fractional calculus, which is aiming to clarify a few bizarre highlights of frameworks and unravel a few challenging issues in real-world applications.

## 6. REFERENCE


Anh, V., et al. (2005). "Diffusion on multifractals." Nonlinear Analysis: Theory, Methods & Applications **63**(5-7): e2043-e2056.

Atangana, A. (2015). "On the stability and convergence of the time-fractional variable order telegraph equation." Journal of Computational Physics **293**: 104-114.

Atangana, A. and A. H. Cloot (2013). "Stability and convergence of the space fractional variable-order Schrödinger equation." Advances in Difference Equations **2013**(1): 1-10.

Awotunde, A. A., et al. (2015). "Artificial boundary condition for a modified fractional diffusion problem." Boundary Value Problems **2015**(1): 1-17.

Bhrawy, A. and M. Zaky (2015). "Numerical simulation for two-dimensional variable-order fractional nonlinear cable equation." Nonlinear dynamics **80**(1): 101-116.

Cai, W., et al. (2018). "A survey on fractional derivative modeling of power-law frequency-dependent viscous dissipative and scattering attenuation in acoustic wave propagation." Applied Mechanics Reviews **70**(3).

Cao, J. and Y. Qiu (2016). "A high order numerical scheme for variable order fractional ordinary differential equation." Applied Mathematics Letters **61**: 88-94.

Chechkin, A. V., et al. (2005). "Fractional diffusion in inhomogeneous media." Journal of Physics A: Mathematical and General **38**(42): L679.

Chen, W., et al. (2013). "A variable-order time-fractional derivative model for chloride ions sub-diffusion in concrete structures." Fractional Calculus and Applied Analysis **16**(1): 76-92.

Chen, Y., et al. (2014). "Numerical solution for the variable order linear cable equation with Bernstein polynomials." Applied Mathematics and Computation **238**: 329-341.

Choong, T. S., et al. (2006). "Film-pore-concentration-dependent surface diffusion model for the adsorption of dye onto palm kernel shell activated carbon." Journal of colloid and interface science **301**(2): 436-440.

Coimbra, C. F. (2003). "Mechanics with variable‐order differential operators." Annalen der Physik **12**(11‐12): 692-703.

Gerasimov, D., et al. (2010). "An anomalous non-self-similar infiltration and fractional diffusion equation." Physica D: Nonlinear Phenomena **239**(16): 1593-1597.

Heydari, M. H. and Z. Avazzadeh (2018). "A new wavelet method for variable‐order fractional optimal control problems." Asian Journal of Control **20**(5): 1804-1817.

Ingman, D. and J. Suzdalnitsky (2004). "Control of damping oscillations by fractional differential operator with time-dependent order." Computer Methods in Applied Mechanics and Engineering **193**(52): 5585-5595.

Jiang, W. and H. Li (2018). "A space–time spectral collocation method for the two-dimensional variable-order fractional percolation equations." Computers & Mathematics with Applications **75**(10): 3508-3520.

Jiang, W. and N. Liu (2017). "A numerical method for solving the time variable fractional order mobile–immobile advection–dispersion model." Applied Numerical Mathematics **119**: 18-32.

Kobelev, Y. L., et al. (2003). Anomalous diffusion with time-and coordinate-dependent memory. Doklady Physics, Springer.

Leith, J. (2003). "Fractal scaling of fractional diffusion processes." Signal Processing **83**(11): 2397-2409.

Lin, R., et al. (2009). "Stability and convergence of a new explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation." Applied Mathematics and Computation **212**(2): 435-445.

Lorenzo, C. F. and T. T. Hartley (2002). "Variable order and distributed order fractional operators." Nonlinear dynamics **29**(1): 57-98.

Metzler, R. and J. Klafter (2000). "The random walk's guide to anomalous diffusion: a fractional dynamics approach." Physics reports **339**(1): 1-77.

Obembe, A. D., et al. (2017). "Variable-order derivative time fractional diffusion model for heterogeneous porous media." Journal of Petroleum Science and Engineering **152**: 391-405.

Ramirez, L. E. and C. F. Coimbra (2007). "A variable order constitutive relation for viscoelasticity." Annalen der Physik **16**(7 ‐ 8): 543-552.

Samko, S. G. and B. Ross (1993). "Integration and differentiation to a variable fractional order." Integral transforms and special functions **1**(4): 277-300.

Sheng, H., et al. (2011). "A physical experimental study of variable-order fractional integrator and differentiator." The european physical journal special topics **193**(1): 93-104.

Sierociuk, D., et al. (2015). "On the recursive fractional variable-order derivative: equivalent switching strategy, duality, and analog modeling." Circuits, Systems, and Signal Processing **34**(4): 1077-1113.

Smit, W. and H. De Vries (1970). "Rheological models containing fractional derivatives." Rheologica Acta **9**(4): 525-534.

Sokolov, I. M. and J. Klafter (2005). "From diffusion to anomalous diffusion: a century after Einstein's Brownian motion." Chaos: An Interdisciplinary Journal of Nonlinear Science **15**(2): 026103.

Sokolov, I. M. and J. Klafter (2006). "Field-induced dispersion in subdiffusion." Physical review letters **97**(14): 140602.

Soon, C. M., et al. (2005). "The variable viscoelasticity oscillator." Annalen der Physik **14**(6): 378-389.

Straka, P. (2018). "Variable order fractional Fokker–Planck equations derived from continuous time random walks." Physica A: Statistical Mechanics and its Applications **503**: 451-463.

Sun, H., et al. (2009). "Variable-order fractional differential operators in anomalous diffusion modeling." Physica A: Statistical Mechanics and its Applications **388**(21): 4586-4592.

Sun, H., et al. (2012). "Finite difference schemes for variable-order time fractional diffusion equation." International Journal of Bifurcation and Chaos **22**(04): 1250085.

Sun, H., et al. (2011). "A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems." The european physical journal special topics **193**(1): 185-192.

Sun, H., et al. (2011). "Random-order fractional differential equation models." Signal Processing **91**(3): 525-530.

Sweilam, N. H. and S. M. Al-Mekhlafi (2016). "Numerical study for multi-strain tuberculosis (TB) model of variable-order fractional derivatives." Journal of advanced research **7**(2): 271-283.

Sweilam, N. H. and T. A. R. Assiri (2013). "Numerical simulations for the space-time variable order nonlinear fractional wave equation." Journal of Applied Mathematics **2013**.

Wang, F., et al. (2018). "Boundary function method for inverse geometry problem in two-dimensional anisotropic heat conduction equation." Applied Mathematics Letters **84**: 130-136.

Wu, G. C., et al. (2017). "Lattice fractional diffusion equation of random order." Mathematical Methods in the Applied Sciences **40**(17): 6054-6060.

Zhang, H., et al. (2013). "A novel numerical method for the time variable fractional order mobile–immobile advection–dispersion model." Computers & Mathematics with Applications **66**(5): 693-701.