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Riemann-Stieltjes Integration

Final Year Research Project

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A Project submitted in partial fulfillment of the requirements for the Degree of B.Sc. in Mathematics

Abstract

In this report, the goals of this project are to define the Riemann-Stieltjes Integral, and explain the properties of this integral. The process of Riemann Integration which is taught in Real Analysis classes is a specific case of the Riemann-Stieltjes Integration.Thus many of the terms and properties used to describe Riemann Integration are discussed in this project and they are extended to the Riemann-Stieltjes integral. This project therefore provides a careful introduction to the theory of Riemann-Stieltjes Integration, and explains some examples of this integral.

Keywords: Upper Riemann-Stieltjes sum**,** Lower Riemann-Stieltjes sum**,** Upper Riemann-Stieltjes Integral**,** Lower Riemann-Stieltjes Integral**.**

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1-Introduction

Until 1820 s the integral $\int_a^b f$ denoted the area enclosed by the curve f from a tob and it is defined as $(b) - F(a)$, where F is the function defined on R such that $\acute{F}$(x) = $f(x)$. Many the classical Riemann integral (Bernhard Riemann 1826-1866) of ordinary calculus has many applications including the computation of areas between two curves, volumes of solids of revolution, arc length, work performed by a variable force, hydrostatic pressures, moments and centers of mass, consumers surplus, present value of future income, expected value, and variance.

In this project we investigate a generalization of the Riemann integral called the Riemann-Stieltjes integral (Thomas Joannes Stieltjes 1856 – 1894) and its application. That is a special case of the Riemann-Stieltjes integral is the Riemann integral of ordinary calculus. This project emphasizes on the usefulness of Riemann-Stieltjes Integral in many areas of Mathematics, Statistics, Mechanics and real-life problems.

For a start, we must recognize that the process of Riemann integration which is taught in calculus classes is a specific of Riemann –Stieltjes integration.Thus many of the terms and properties used to describe Riemann integration will be discussed in this project since they carry over to Riemann-Stieltjes integration.

In the study of elementary calculus, a good intuitive feel for the idea of Riemann integration of the form $\int_a^b f$(x)dx is the enclosed area " S " of the region illustrated in figure 1.1 that lies between the lines $x = a$ and $x = b$ the x-axis, and the curve $y = f(x)$.

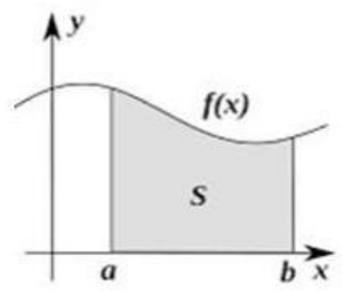

Figure 1.1: Graph of a Riemann Integral

Integration has been an important part of mathematics since the 17th century, when Gottfried Wilhelm Leibniz and Isaac Newton constructed the Fundamental Theorem of Calculus, which says that the area is essentially the same as taking an anti-derivative. In the 19th Century, Cauchy investigated integrals of continuous functions. The integral that he used was later redefined by Riemann, which he used to investigate integrals of discontinuous functions. His definition was far easier to understand and to teach than all of the previous ones, which resulted in the Riemann Integral becoming the standard integral to be taught to students in mathematics class which helps, give an insight to the Riemann-Stieltjes Integral (a generalization of Riemann Integral). An important generalization of the Riemann-Stieltjes Integral is the Lesbegue -Stieltjes Integral.

2- Definition of Terms:

In this section we recall the following definitions of some mathematical terms which will be used in the course of this research work.

Definitions 2.1 (Partition of an Interval)

A partition P of interval $[a, b]$ is a finite set of points $P = \{ x_0 , x_1 , \dots, x_n \}$ such that $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$.

Definition 2.2 (Norm of a partition)

The length of the biggest sub-interval in to which partition divides $[a, b]$, is called a norm it means that define the norm $\|P\|$ of P by $\|P\| = \max \Delta x_i$, where $1 \leq i \leq n$.

Definition 2.3 (refinement)

Another partition Q of the given interval $[a, b]$ is defined as a refinement of the partition, if $P \subset Q$ (that is Q contains all the points of P and possibly some other points). Given two partitions P_1 and P_2 of $[a, b]$, we say that Q is their common refinement if $Q = P_1 \cup P_2$.

Definition 2.4 (monotonically increasing function)

Let f be a real – valued function defined on an interval, a function f is called monotonically increasing on I if $f(x) \leq f(y)$ for all $x , y \in I$ with $x < y$.

Definition 2.5 (identity function)

The function f is called the identity function if each element of set A has an image on itself it means that $f(a) = a$ for all $a \in A$ .it is denoted by I .

Definition 2.6 (Least Upper Bound or Supremum)

Let A be a nonempty subset of R is a set of real numbers. An element $\alpha \in R$ is called the least upper bound of A if

1. α is an upper bound of A
2. $\alpha \leq \beta$ for every upper bound β of, and denoted by $\sup(A) = \alpha$.

Definition 2.7 (Greatest Lower Bound or Infimum)

Let A be a nonempty subset of R that is bounded below. An element $\gamma \in R$ is called the greatest lower bound or infimum of A if

1. γ is an lower bound of A
2. $\gamma \geq \beta$ for every lower bound of A , and denoted by $\inf(A) = \gamma$.

Definition 2.8 (bounded)

A real – valued function f defined on a set A is called bounded if there exists a real number M such that $|f(x)| \leq M$, for all $\in A$.

Definition 2.9 (continuous)

A function f is continuous at a point c in its domain D if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ for all $x \in D$ with $|x - c| < \delta$.

The function f is continuous in its domain D if it is continuous at every point of its domain.

Definition 2.10 (uniformly continuous)

Let A be a subset of R and $f: A \rightarrow R$ we say that f is uniformly continuous on A if for all $\varepsilon > 0$, there exists $\delta > 0$, such that if $x, y \in A$ with $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

3- Riemann-Stieltjes Integration

Presented here is the definition and notation of the Riemann-Stieltjes Integral.

Definition 3.1: Let α be a monotone increasing function on $[a, b]$, and let f be a bounded real-valued function on $[a, b]$.

For each partition $P = \{ x_0 , x_1 , \dots , x_n \}$ of $[a, b]$

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}), i = 1 , \dots , n .$$

Since α is monotone increasing, $\Delta\alpha_i \geq 0$ for all i. Let

$$m_i = \inf \{f(x): x \in [x_{i-1} , x_i]\} ,$$

$$M_i = \sup \{f(x): x \in [x_{i-1} , x_i]\} .$$

The upper Riemann-Stieltjes sum of f with respect to α and the partition P, is defined by

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \, \Delta\alpha_i .$$

The lower Riemann-Stieltjes sum of f with respect to α and the partition P, is defined by

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \, \Delta\alpha_i .$$

By the given conditions and definitions $m_i \leq M_i$ and $\Delta\alpha_i \geq 0$, we know

$$L(P, f, \alpha) \leq U(P, f, \alpha) .$$

Let P be any partition of $[a, b]$. Then the following sum can be found

$$\sum_{i=1}^{n} \Delta\alpha_i = (\alpha(x_1) - \alpha(x_0)) + (\alpha(x_2) - \alpha(x_1)) + \cdots + (\alpha(x_n) - \alpha(x_{n-1}))$$
$$= \alpha(x_n) - \alpha(x_0)$$
$$= \alpha(b) - \alpha(a).$$

Since $M_i \leq M$ and $m_i \geq m$ for all i and $\Delta\alpha_i \geq 0$,

$$\sum_{i=1}^{n} M_i \, \Delta\alpha_i \leq \sum_{i=1}^{n} M \, \Delta\alpha_i = M \sum_{i=1}^{n} \Delta\alpha_i = M \, [\alpha(b) - \alpha(a)].$$

$$\sum_{i=1}^{n} m_i \, \Delta\alpha_i \geq \sum_{i=1}^{n} m \, \Delta\alpha_i = m \sum_{i=1}^{n} \Delta\alpha_i = m \, [\alpha(b) - \alpha(a)].$$

Therefor

$$U(P, f, \alpha) \leq M \, [\alpha(b) - \alpha(a)]$$

And

$$L(P, f, \alpha) \geq m \, [\alpha(b) - \alpha(a)].$$

Thus, if $m \leq f(x) \leq M$ for all $x \in [a, b]$, then $m \leq m_i \leq M_i \leq M$ for each $i = 1,2, \dots , n$, and

$$m \, [\alpha(b) - \alpha(a)] \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq M \, [\alpha(b) - \alpha(a)]$$

for all partitions P of $[a, b]$.

Definition 3.2: Let f be a bounded real-valued function on $[a, b]$, and α is a monotone increasing function on $[a, b]$.

The upper Riemann-Stieltjes Integral of f with respect to α over $[a, b]$, denoted $\overline{\int_a^b} f\, d\alpha$ and the lower Riemann-Stieltjes Integral of f with respect to α over $[a, b]$, denoted $\underline{\int_a^b} f\, d\alpha$ are defined

$$\overline{\int_a^b} f\, d\alpha = \inf\{U(P, f, \alpha) : P \text{ is a partition of } [a, b]\},$$

$$\underline{\int_a^b} f\, d\alpha = \sup\{ L(P, f, \alpha) : P \text{ is a partition of } [a, b]\},$$

A function f is said to be Riemann-Stieltjes Integrable with respect to α on $[a, b]$, and write $f \in R(\alpha)[a, b]$, provided that

$$\int_a^b f\, d\alpha = \overline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\, d\alpha$$

Theorem 3.1: If P and Q are partitions of $[a, b]$and Q is finer than P, then

$$L(P, f, \alpha) \leq L(Q, f, \alpha) \leq U(Q, f, \alpha) \leq U(P, f, \alpha).$$

That is, $L(P, f, \alpha)$ increases with P and $U(P, f, \alpha)$ decreases with P.

Proof. Since Q is obtained from P by adding finitely many points, by induction, we only need to prove the case when Q is obtained from P by adding one extra point. So let

$$P = \{ x_0, x_1, \dots, x_{k-1}, x_k, \dots, x_n \}, \quad Q = \{ x_0, x_1, \dots, x_{k-1}, y, x_k, \dots, x_n \}$$

Where $x_{k-1} < y < x_k$. Then

$$L(P, f, \alpha) = \sum_{i=1}^n m_i\, (f)\Delta\alpha_i, \quad \text{where} \quad m_i(f) = \inf_{x\in[x_{i-1}, x_i]} f(x),$$

$$L(Q, f, \alpha) = \sum_{i=1}^{k-1} m_i\, (f)\Delta\alpha_i + \Big(\inf_{x\in[x_{k-1}, y]} f(x)\Big)\, (\alpha(y) - \alpha(x_{k-1})) + \Big(\inf_{x\in[y, x_k]} f(x)\Big)\, (\alpha(x_k) - \alpha(y)) + \sum_{i=k+1}^n m_i\, (f)\Delta\alpha_i .$$

Note that

$$\inf_{x\in[x_{k-1}, y]} f(x) \geq \inf_{x\in[x_{k-1}, x_k]} f(x), \qquad \inf_{x\in[y, x_k]} f(x) \geq \inf_{x\in[x_{k-1}, x_k]} f(x).$$

Hence, since α is increasing, we have that

$$\Big(\inf_{x\in[x_{k-1}, y]} f(x)\Big)\, (\alpha(y) - \alpha(x_{k-1})) + \Big(\inf_{x\in[y, x_k]} f(x)\Big)\, (\alpha(x_k) - \alpha(y)) \geq \Big(\inf_{x\in[x_{k-1}, x_k]} f(x)\Big)\, (\alpha(y) - \alpha(x_{k-1}) + \alpha(x_k) - \alpha(y)) = m_k\, (f)\, \Delta\alpha_k .$$

Consequently,

$$L(Q, f, \alpha) \geq \sum_{i=1}^{k-1} m_i\, (f)\Delta\alpha_i + m_k\, (f)\, \Delta\alpha_k + \sum_{i=k+1}^n m_i\, (f)\Delta\alpha_i = L(P, f, \alpha)$$

To proof of $U(Q, f, \alpha) \leq U(P, f, \alpha)$

$$U(P, f, \alpha) = \sum_{i=1}^n M_i\, (f)\Delta\alpha_i, \quad \text{where} \quad M_i(f) = \sup_{x\in[x_{i-1}, x_i]} f(x),$$

$$U(Q, f, \alpha) = \sum_{i=1}^{k-1} M_i\, (f)\Delta\alpha_i + \Big(\sup_{x\in[x_{k-1}, y]} f(x)\Big)\, (\alpha(y) - \alpha(x_{k-1})) + \Big(\sup_{x\in[y, x_k]} f(x)\Big)\, (\alpha(x_k) - \alpha(y)) + \sum_{i=k+1}^n M_i\, (f)\Delta\alpha_i .$$

Note that

$\sup_{x\in[x_{k-1},y]} f(x) \leq \sup_{x\in[x_{k-1},x_k]} f(x)$, $\sup_{x\in[y,x_k]} f(x) \leq \sup_{x\in[x_{k-1},x_k]} f(x)$.

Hence, since α is increasing, we have that

$(\sup_{x\in[x_{k-1},y]} f(x))\,(\alpha(y) - \alpha(x_{k-1})) + (\sup_{x\in[y,x_k]} f(x))\,(\alpha(x_k) - \alpha(y)) \leq (\sup_{x\in[x_{k-1},x_k]} f(x))\,(\alpha(y) - \alpha(x_{k-1}) + \alpha(x_k) - \alpha(y)) = M_k(f)\,\Delta\alpha_k$.

Consequently,

$$U(Q,f,\alpha) \leq \sum_{i=1}^{k-1} M_i(f)\Delta\alpha_i + M_k(f)\,\Delta\alpha_k + \sum_{i=k+1}^{n} M_i(f)\Delta\alpha_i = U(P,f,\alpha) .$$

Example 3.1: Evaluate $\int_0^1 x\, dx^2$ by using only the definition of the integral.

Solution: Since f is continuous function and $\alpha(x)$ is monotonically increasing in interval [0,1]. so $\int_0^1 f d\alpha$ exist.

Dividing the interval [0,1] into n equal parts, we get a partition

$$P = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{r-1}{n}, \frac{r}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n} = 1\}$$

Let $I_i = [\frac{i-1}{n}, \frac{i}{n}]$ is i^{th} subinterval of $[a,b]$. where $i = 1, 2, 3, \ldots, n$

Then $\Delta\alpha i = \alpha\left(\frac{i}{n}\right) - \alpha\left(\frac{i-1}{n}\right)$

$= \left(\frac{i}{n}\right)^2 - \left(\frac{i-1}{n}\right)^2$

$\Delta\alpha i = \frac{1}{n^2}(2i - 1)$

Let $m_i = inf\{ f(x): x \in I_i\}$. And $M_i = \sup\{ f(x): x \in I_i\}$.

Then $m_i = f\left(\frac{i-1}{n}\right) = \frac{i-1}{n}$, and $M_i = f\left(\frac{i}{n}\right) = \frac{i}{n}$

So, upper Riemann Stieltjes sum $U(P,f,\alpha) = \sum_{i=1}^{n} M_i\,\Delta\alpha_i$

$= \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n^2}(2i - 1)$

$$U(P,f,\alpha) = \frac{1}{n^3}\sum_{i=1}^{n} (2i^2 - i)$$

$$= \frac{1}{n^3}\left[\mathbf{2} \cdot \frac{\boldsymbol{n(n+1)(2n+1)}}{\mathbf{6}} - \frac{\boldsymbol{n(n+1)}}{\mathbf{2}}\right]$$

$$= \frac{\mathbf{1}}{\mathbf{3}}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - \left(\frac{\mathbf{1}}{\boldsymbol{n}} + \frac{\mathbf{1}}{\boldsymbol{n^2}}\right)$$

So, $\overline{\int_0^1} x\, dx^2 = \inf\{ (P,f,\alpha) : P$ is a partition of $[0,1]\}$,

$= \frac{2}{3}$ …………………. 1

Now, $L(P, f, \alpha) = \sum_{i=1}^{n} m_i \, \Delta\alpha_i$

$= \sum_{i=1}^{n} \left(\frac{i-1}{n}\right) \cdot \frac{1}{n^2} (2i - 1)$

$= \frac{1}{n^3} \sum_{i=1}^{n} (2i^2 - 3i + 1)$

$L(P, f, \alpha) = \frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{3} - \frac{3}{2} \cdot \left(\frac{1}{n} + \frac{1}{n^2}\right) + \frac{n}{n^3}$

So, $\underline{\int_0^1} x \, dx^2 = \sup\{L(P, f, \alpha) : P$ is a partition of $[0,1]\}$,

$= \frac{2}{3}$ ………………… 2

From (1) and (2), we get

$\overline{\int_0^1} x \, dx^2 = \underline{\int_0^1} x \, dx^2 = \frac{2}{3}$.

Thus the given function $f(x) = x$ is Riemann - Stiltjes Integral with respect to $\alpha(x) = x^2$.

So, $\int_0^1 x \, dx^2 = \frac{2}{3}$.

Example 3.2: Evaluate $\int_0^3 x \, d[x]$, where $[x]$ is the greatest integer not exceeding.

Solution: Here $\alpha(x) = [x]$ is monotonic increasing function on [0,3] given by

$\alpha(x) = 0$, for $0 \leq x < 1$

$\alpha(x) = 1$, for $1 \leq x < 2$

$\alpha(x) = 2$, for $2 \leq x < 3$

And $\alpha(x) = 3$, for x = 3

Consider any partition of P of [0,3] as follows.

$P = \{0 = x_0, x_l, \ldots, x_{l-1}, x_l = 1, x_{l+1}, \ldots, x_{m-1}, x_m = 2, x_{m+1}, \ldots, x_{n-1}, x_n = 3\}$

Then by definition, we have

$U(P, f, \alpha) = \sum_{i=1}^{l-1} M_i \, \Delta\alpha_i + M_l \, \Delta\alpha_l + \sum_{i=l+1}^{m-1} M_i \, \Delta\alpha_i + M_m \, \Delta\alpha_m + \sum_{i=m+1}^{n-1} M_i \, \Delta\alpha_i + M_n \, \Delta\alpha_n$

$= \sum_{i=1}^{l-1} M_i (0-0) + 1(1\text{-}0) + \sum_{i=l+1}^{m-1} M_i (1-1) + 2(2\text{-}1) + \sum_{i=m+1}^{n-1} M_i (2-2) + 3(3\text{-}2) = 6$

So, $\overline{\int_0^3} x \, d[x] = \inf\{(P, f, \alpha) : P$ is a partition of $[0,3]\} = 6$

Similarly, $\underline{\int_0^3} x \, d[x] = \sup\{L(P, f, \alpha) : P$ is a partition of $[0,3]\} = 6$

Since, $\overline{\int_0^3} x \, [x] = \underline{\int_0^3} x \, d[x]$, hence $\int_0^3 x \, d[x]$ is Riemann -Stiltjes Integral of x over [0,3]

with respect to $\alpha(x) = [x]$ and $\int_0^3 x \, d[x] = 6$.

4- Properties and Theorems of Riemann – Stieltjes Integration

Theorem 4.1: (Criterion for Integrability)

A bounded function f is in $R(\alpha)[a,b]$ if and only if for each $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

Proof: (Sufficiency for Integrability)

Let $\varepsilon > 0$. Assume that there exists a partition P of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$. Then $U(P,f,\alpha) < L(P,f,\alpha) + \varepsilon$, and thus

$$\overline{\int_a^b} f\,d\alpha \le U(P,f,\alpha) < L(P,f,\alpha) + \varepsilon \le \underline{\int_a^b} f\,d\alpha + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, this prove that $\overline{\int_a^b} f\,d\alpha \le \underline{\int_a^b} f\,d\alpha$ and hence $\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f\,d\alpha$; so $f \in R(\alpha)[a,b]$.

(Necessity for Integrability). Assume $\in R(\alpha)[a,b]$; namely, $\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f\,d\alpha$.

Let $\varepsilon > 0$. Then there exist partitions P_1 and P_2 of $[a,b]$ such that

$$U(P_1,f,\alpha) < \overline{\int_a^b} f\,d\alpha + {}^{\varepsilon}/_{2} \ , \quad L(P_2,f,\alpha) > \underline{\int_a^b} f\,d\alpha - {}^{\varepsilon}/_{2} \ .$$

Let $P = P_1 \cup P_2$ be the common refinement of P_1 and P_2 . then P is a partition of $[a,b]$, and using $\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f\,d\alpha$, we have

$U(P,f,\alpha) - L(P,f,\alpha) \le U(P_1,f,\alpha) - L(P_2,f,\alpha) < (\overline{\int_a^b} f\,d\alpha + {}^{\varepsilon}/_{2}) - (\underline{\int_a^b} f\,d\alpha - {}^{\varepsilon}/_{2}) = \varepsilon.$

Theorem 4.2: Let f be a real valued function on $[a,b]$ and α be a monotone increasing function on $[a,b]$. If f is continuous on $[a,b]$, then f is integrable with respect to α on $[a,b]$.

Proof: Supposed that f is continuous function on $[a,b]$

f is uniformly continuous on $[a,b]$

Then by definition $\forall \varepsilon > 0$, $\exists\ \delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{\alpha(b)-\alpha(a)}$ ………. (1)

$\forall\ x, y \in$ [a, b] and where $|x - y| < \delta$

Now, $P = \{a = x_0, x_1, \ldots\ x_n = b\}$ is a partition of [a, b] with $\|P\| < \delta$

Now, since f is continuous function on $[a,b]$ then f is bounded

So, $\exists\ c_i, d_i \in [x_{i-1}, x_i]$ such that $f(c_i) = m_i$ and $f(d_i) = M_i$

Where $m_i = \inf\{ f(x): x \in [x_{i-1}, x_i]\}$ and $M_i = \sup\{ f(x): x \in [x_{i-1}, x_i]\}$

$|M_i - m_i| = |f(d_i) - f(c_i)| < \frac{\varepsilon}{\alpha(b)-\alpha(a)}$ [from (1)] whenever $|di - c_i| \le |x_i - x_{i-1}| < \delta$

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i < \frac{\varepsilon}{\alpha(b)-\alpha(a)}\sum_{i=1}^{n}\Delta\alpha_i$$

$$= \frac{\varepsilon}{\alpha(b)-\alpha(a)}\,\alpha(b) - \alpha(a)$$

$$< \varepsilon$$

Then $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

So, by the Criterion for Integrability, $f \in R(\alpha)[a, b]$.

Theorem 4.3: let f be a real valued function on $[a, b]$ and α be a monotone increasing function on $[a, b]$. If f is monotone on $[a, b]$ and α is continuous on $[a, b]$, then f is integrable with respect to α on $[a, b]$

Proof: α is continuous and monotonically increasing

So, for any positive integer n ∃ a partition P such that $P = \{ a = x_0, x_1, \dots, x_n = b\}$ such that $\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n}$, $i = 1,2,\dots,n$

Since α is continuous, such a choice is possible by the intermediate value theorem. Assume f is monotone increasing function on $[a, b]$. Then

So, $m_i = \inf\{f(x): x \in [x_{i-1}, x_i]\} = f(x_{i-1})$

$M_i = \sup\{f(x): x \in [x_{i-1}, x_i]\} = f(x_i)$

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n}(M_i - m_i)\,\Delta\alpha_i$$

$$= \frac{\alpha(b)-\alpha(a)}{n}\sum_{i=1}^{n} f(x_i) - f(x_{i-1})$$

$$= \frac{\alpha(b)-\alpha(a)}{n}[f(x_1) - f(x_0) + f(x_2) - f(x_1) + \cdots + f(x_n) - f(x_{n-1})]$$

$$= \frac{\alpha(b)-\alpha(a)}{n}[f(x_n) - f(x_0)]$$

$$= \frac{\alpha(b)-\alpha(a)}{n}[f(b) - f(a)]$$

Given $\varepsilon > 0$, choose $n \in N$ such that

$$\frac{\alpha(b)-\alpha(a)}{n}[f(b) - f(a)] < \varepsilon$$

For this n and corresponding partition,

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

Hence (by theorem 4.1) we get that $f \in R(\alpha)[a, b]$. .

Theorem 4.4: The Riemann – Stieltjes Integral has the following properties:

(a) **(Linear property)** If $f_1, f_2 \in R(\alpha)[a,b]$, then $c_1 f_1 + c_2 f_2 \in R(\alpha)[a,b]$ for all a real numbers c_1, c_2 , and

$$\int_a^b (c_1 f_1 + c_2 f_2) d\alpha = c_1 \int_a^b f_1 \, d\alpha + c_2 \int_a^b f_2 \, d\alpha \, .$$

(b) **(Order property)** If $f_1, f_2 \in R(\alpha)[a,b]$ and $f_1(x) \leq f_2(x)$ on $[a,b]$, then

$$\int_a^b f_1 \, d\alpha \leq \int_a^b f_2 \, d\alpha \, .$$

(c) **(Additivity)** If $f \in R(\alpha)[a,b]$ and $a < c < b$, then $f \in R(\alpha)[a,c]$ and $f \in R(\alpha)[c,b]$; moreover,

$$\int_a^b f \, d\alpha = \int_a^c f \, d\alpha + \int_c^b f \, d\alpha \, . \qquad \text{.........} (1)$$

Conversely, if $a < c < b$ and if $f \in R(\alpha)[a,c]$ and $f \in R(\alpha)[c,b]$, then $f \in R(\alpha)[a,b]$ and (1) holds.

(d) **(Positive combination)** If $f \in R(\alpha_1)[a,b]$ and $f \in R(\alpha_2)[a,b]$ and k_1, k_2 are nonnegative constants, then $f \in R(k_1\alpha_1 + k_2\alpha_2)[a,b]$, and

$$\int_a^b f \, d(k_1\alpha_1 + k_2\alpha_2) = k_1 \int_a^b f \, d\alpha_1 + k_2 \int_a^b f \, d\alpha_2 \, .$$

(e) **(Absolute integrability)** If $f \in R(\alpha)[a,b]$ **,** then $|f| \in R(\alpha)[a,b]$, and

$$\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f| \, d\alpha$$

Theorem 4.5: (Mean Value Theorem).

Let f be a continuous real – valued function on $[a,b]$ and α a monotone increasing function on $[a,b]$. Then there exists a $c \in [a,b]$ such that

$$\int_a^b f \, d\alpha = f(c)[\alpha(b) - \alpha(a)]$$

Proof: Let m and M denote the minimum and maximum of f on $[a,b]$ respectively. Then

$$m[\alpha(b) - \alpha(a)] \leq \int_a^b f \, d\alpha \leq M[\alpha(b) - \alpha(a)]$$

If $\alpha(b) - \alpha(a) = 0$ then any $c \in [a,b]$ will work. If $\alpha(b) - \alpha(a) \neq 0$ then by the Intermediate Value Theorem there exists a $c \in [a,b]$ such that

$$f(c) = \frac{1}{\alpha(b) - \alpha(a)} \int_a^b f \, d\alpha \, .$$

Theorem 4.6: (Integration by Parts).

Suppose f and α are monotone increasing functions on $[a,b]$.

a) Then $f \in R(\alpha)$ on $[a,b]$ if and only if $\alpha \in R(f)$ on $[a,b]$.

b) If this is the case,

$$\int_a^b f\, d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha\, df.$$

Proof: Proof of part (a). For any partition P of $[a,b]$

$$U(P,f,\alpha) = f(b)\alpha(b) - f(a)\alpha(a) - L(P,\alpha\,,f) \quad \ldots\ldots.. (1)$$

And

$$L(P,f,\alpha) = f(b)\alpha(b) - f(a)\alpha(a) - U(P,\alpha\,,f) \quad \ldots\ldots.. (2)$$

By the subtraction of these 2 equations.

$$U(P,f,\alpha) - L(P,f,\alpha) = U(P,\alpha,f) - L(P,\alpha,f)$$

If $f \in R(\alpha)$ then

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

And due to the fact that these (2) quantities are equal if

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

Then

$$U(P,\alpha,f) - L(P,\alpha,f) < \varepsilon$$

As well. This means both are Riemann Stieltjes Integrable.

Proof: Proof of part (b). Furthermore if $\alpha \in R(f)$ then given $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that

$$L(P,\alpha,f) > \int_a^b \alpha\, df - \varepsilon$$

Hence

$$\int_a^b f\, d\alpha \le U(P,f,\alpha) < f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha\, df + \varepsilon$$

Since the above holds for any $\varepsilon > 0$

$$\int_a^b f\, d\alpha \le f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha\, df\ .$$

A similar argument using the lower sum proves the reverse inequality.

5- Conclusion

The definition and properties of the Riemann-Stieltjes Integral have been given, with an explanation of how to calculate the integral. The Riemann-Stieltjes Integral is a useful mathematical tool when working with discrete and continuous random variables simultaneously.

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