Trigonometric Function

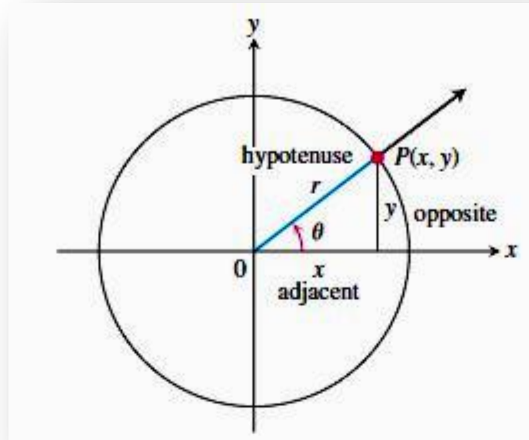


Figure 2.1

Consider the circle (see Figure 2.1)

$$x^2 + y^2 = r^2, \quad r > 0$$

We define

$$\sin\theta = \frac{y}{r} \quad \text{and} \quad \cos\theta = \frac{x}{r}$$

$$\Longrightarrow y = r\sin\theta \quad \text{and} \quad x = r\cos\theta$$

Now, since $x^2 + y^2 = r^2$

$$\Longrightarrow r^2\sin^2\theta + r^2\cos^2\theta = r^2, \quad r \neq 0$$

$$\Longrightarrow \quad \boxed{\sin^2\theta + \cos^2\theta = 1} \quad \dots (1)$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} \quad \text{and} \quad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{x}{y}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x} \quad \text{and} \quad \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$$

$$\sin(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta \Longrightarrow \quad \boxed{\sin(-\theta) = -\sin\theta}$$

$$\cos(-\theta) = \frac{x}{r} = \cos\theta \Longrightarrow \quad \boxed{\cos(-\theta) = \cos\theta}$$

Divide equation (1) by $\cos^2\theta$ we get:

$$\boxed{\tan^2\theta + 1 = \sec^2\theta}$$

Divide equation (1) by $\sin^2\theta$ we get:

$$\boxed{1 + \cot^2\theta = \csc^2\theta}$$

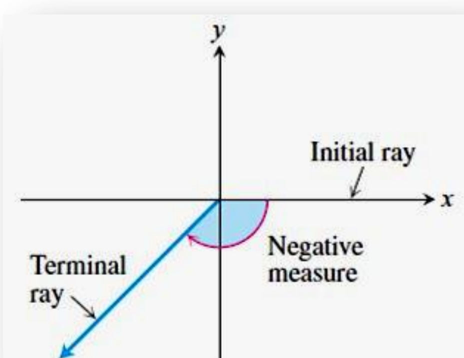


Figure 2.2

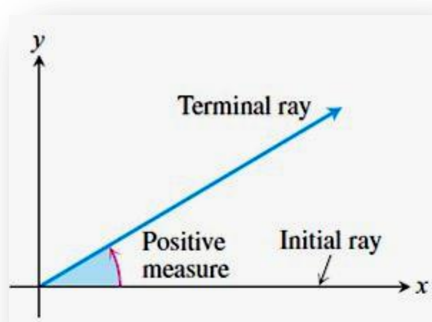
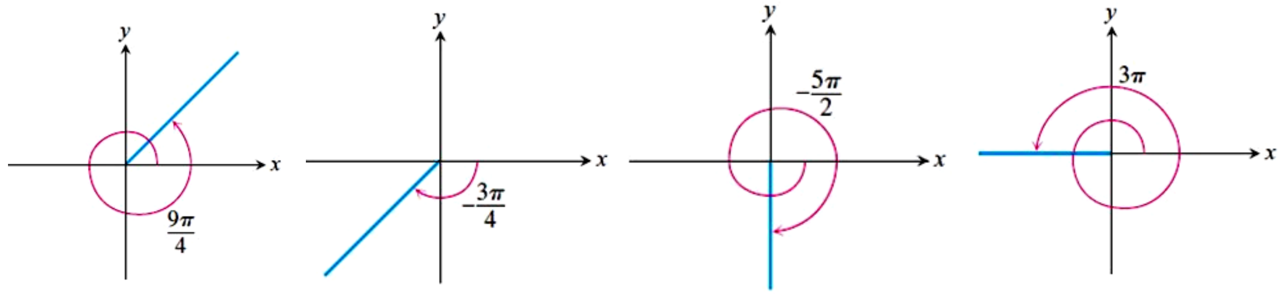


Figure 2.3



Since the circumference of the circle is 2π and one complete revolution of a circle is $360°$, the relation between radians and degrees is given by

$$\pi \text{ radians} = 180°.$$

For example, $45°$ in radian measure is

$$45 \cdot \frac{\pi}{180} = \frac{\pi}{4} \text{ rad},$$

and $\pi/6$ radians is

$$\frac{\pi}{6} \cdot \frac{180}{\pi} = 30°.$$

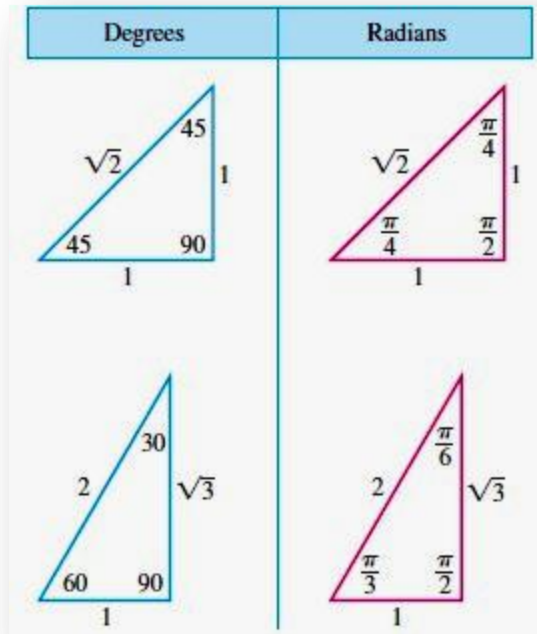


***Figure* 2.4** *The angles of two common triangles, in degrees and radians.*

Conversion Formulas

1 degree $= \frac{\pi}{180} (\approx 0.02)$ radians

Degrees to radians: multiply by $\frac{\pi}{180}$

1 radian $= \frac{180}{\pi} (\approx 57)$ degrees

Radians to degrees: multiply by $\frac{180}{\pi}$

Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

Periods of Trigonometric Functions

Period π: $\tan(x + \pi) = \tan x$
$\cot(x + \pi) = \cot x$

Period 2π: $\sin(x + 2\pi) = \sin x$
$\cos(x + 2\pi) = \cos x$
$\sec(x + 2\pi) = \sec x$
$\csc(x + 2\pi) = \csc x$

Some Important Identities

1. $\sin(\alpha \mp \beta) = \sin\alpha\cos\beta \mp \sin\beta\cos\alpha$

2. $\cos(\alpha \mp \beta) = \cos\alpha\cos\beta \pm \sin\alpha\sin\beta$

3. $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

4. $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

5. $\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$

6. $\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$

7. $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)]$

8. $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

9. $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$

10. $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

11. $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

12. $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

13. $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

14. $\sin 2\alpha = 2\sin\alpha\cos\beta$

15. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

16. $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

1. Sine Function

It is a function $f: \mathbb{R} \to [-1, 1]$ defined by $f(x) = \sin x$.

$\sin x = 0$ iff $x = 0, \mp\pi, \mp 2\pi, \mp 3\pi, \ldots = n\pi, n \in \mathbb{Z}$.

$\sin x = \mp 1$ iff $x = \mp\frac{\pi}{2}, \mp\frac{3\pi}{2}, \ldots = \left(n + \frac{1}{2}\right)\pi, n \in \mathbb{Z}$.

Since $\sin(-x) = -\sin x$, therefore the sine is an odd function.

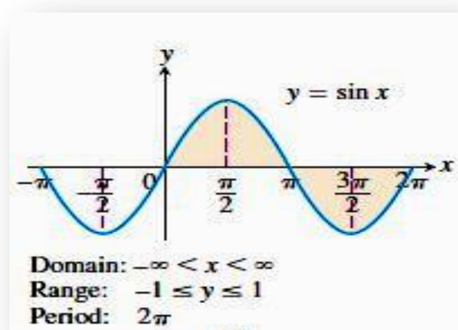


Figure 2.5

It's a one to one function if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. So $\sin: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ is a bijection function.

2. Cosine Function

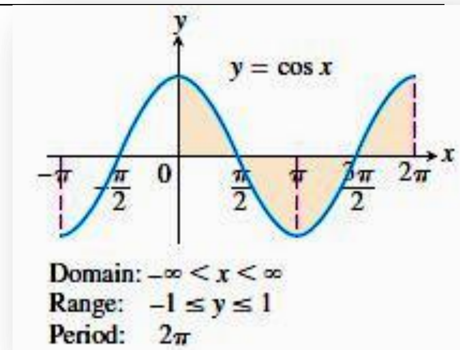


Figure 2.6

It is a function $f: \mathbb{R} \to [-1, 1]$ defined by $f(x) = \cos x$.

$\cos x = 0$ iff $x = \mp\frac{\pi}{2}, \mp\frac{3\pi}{2}, \ldots = \left(n + \frac{1}{2}\right)\pi, n \in \mathbb{Z}$.

$\cos x = \mp 1$ iff $x = 0, \mp\pi, \mp 2\pi, \mp 3\pi, \ldots = n\pi, n \in \mathbb{Z}$

Since $\cos(-x) = \cos x$, therefore the cosine is an even function.

It's a one to one function if $0 \leq x \leq \pi$ or $-\pi \leq x \leq 0$. Therefore $\cos: [0, \pi] \to [-1, 1]$ is a bijection function.

3. Tangent Function

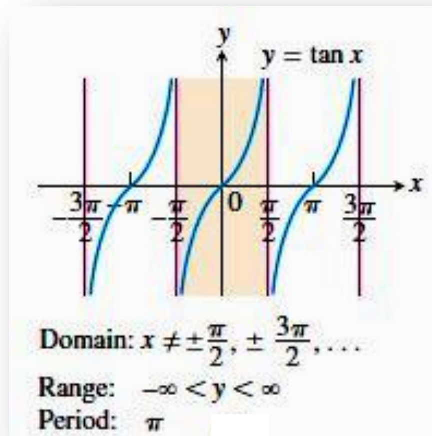


Figure 2.7

It is defined by $\tan: D_{\tan} \to \mathbb{R}$,

$f(x) = \tan x = \frac{\sin x}{\cos x}, \cos x \neq 0$.

$\cos x = 0$ if $x = \left(n + \frac{1}{2}\right)\pi, n \in \mathbb{Z}$,

$D_{\tan} = \mathbb{R}/\{x: x = \left(n + \frac{1}{2}\right)\pi, n \in \mathbb{Z}\}$

$= \{x \in \mathbb{R}: x \neq \left(n + \frac{1}{2}\right)\pi, n \in \mathbb{Z}\}$.

$\tan x = 0$ if $\sin x = 0 \implies x = n\pi, n \in \mathbb{Z}$.

$f(x) = \tan x$ is not one to one function. It becomes one to one if $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

4. Cotangent Function

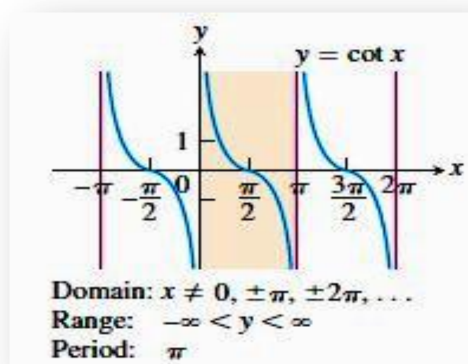


Figure 2.8

It is defined by $\cot: D_{\cot} \to \mathbb{R}$,

$f(x) = \cot x = \frac{\cos x}{\sin x}, \sin x \neq 0$.

$\sin x = 0$ if $x = n\pi, n \in \mathbb{Z}$,

$\implies D_{\cot} = \{x \in \mathbb{R}: x \neq n\pi, n \in \mathbb{Z}\}$.

$\cot x = 0$ if $\cos x = 0 \implies x = \left(n + \frac{1}{2}\right)\pi, n \in \mathbb{Z}$.

$f(x) = \cot x$ is not one to one function. It becomes one to one if $0 < x < \pi$.

5. Secant Function

It is a function $f: D_{\sec} \to (-\infty, -1] \cup [1, \infty)$, and

defined by $f(x) = \sec x = \frac{1}{\cos x}, \cos x \neq 0$.

$\cos x = 0$ if $x = \left(n + \frac{1}{2}\right)\pi, n \in \mathbb{Z}$,

$\Longrightarrow D_{\sec} = \{x \in \mathbb{R}: x \neq \left(n + \frac{1}{2}\right)\pi, n \in \mathbb{Z}\}$.

$f(x) = \sec x$ is not one to one function.

It becomes one to one if $x \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.

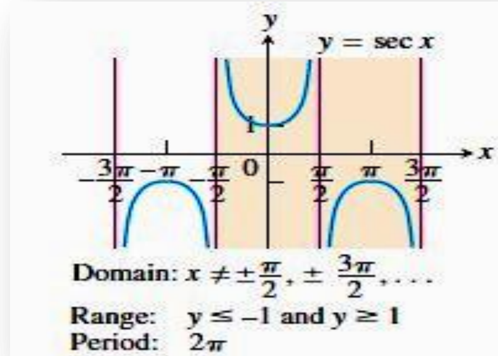


Figure 2.9

6. Cosecant Function

It is a function $f: D_{\csc} \to (-\infty, -1] \cup [1, \infty)$,

and defined by $f(x) = \csc x = \frac{1}{\sin x}, \sin x \neq 0$.

$\sin x = 0$ if $x = n\pi, n \in \mathbb{Z}$,

$\Longrightarrow D_{\csc} = \{x \in \mathbb{R}: x \neq n\pi, n \in \mathbb{Z}\}$.

$f(x) = \sec x$ is not one to one function.

It becomes one to one if $x \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.

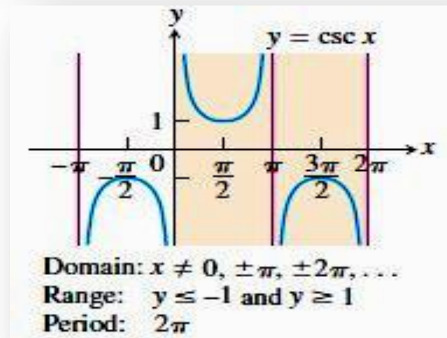


Figure 2.10

The Inverse of Trigonometric Functions

1. The inverse of sine function

It's denoted by $\sin^{-1}$, and defined to be the inverse of the sine function for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

$f(x) = \sin^{-1} x, \sin^{-1}: [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

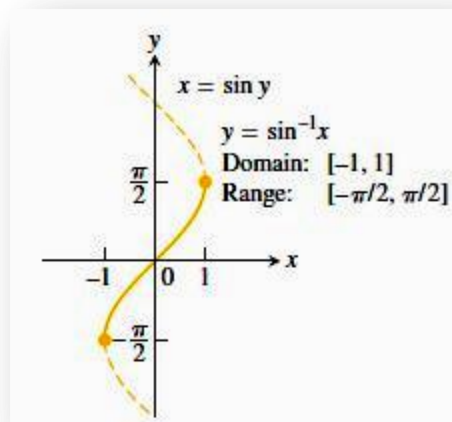


Figure 2.11

2. The inverse of cosine function

It's denoted by $\cos^{-1}$, and defined to be the inverse of the cosine function for $0 \leq x \leq \pi$,

$f(x) = \cos^{-1} x, \cos^{-1}: [-1, 1] \to [0, \pi]$

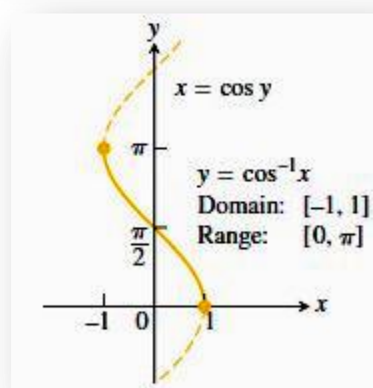


Figure 2.12

3. The inverse of tangent function

It's denoted by $\tan^{-1}$, and defined to be the inverse of the tangent function

$f(x) = \tan^{-1} x,\ \tan^{-1}: \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

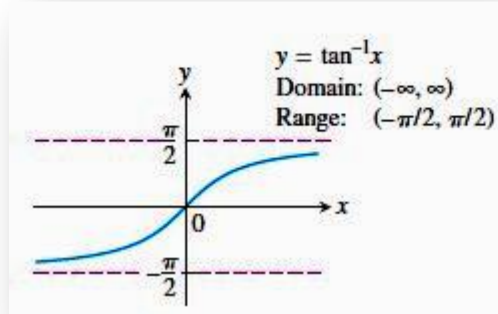


Figure 2.13

4. The inverse of cotangent function

It's denoted by $\cot^{-1}$, and defined to be the inverse of the cotangent function

$f(x) = \cot^{-1} x, \cot^{-1}: \mathbb{R} \to (0, \pi)$

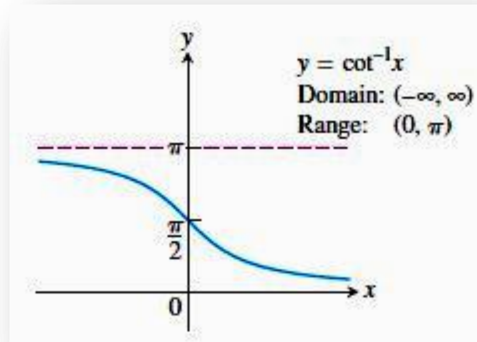


Figure 2.14

5. The inverse of secant function

It's denoted by $\sec^{-1}$, and defined to be the inverse of the secant function

$f(x) = \sec^{-1} x, \sec^{-1}: (-\infty, -1] \cup [1, \infty) \to [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

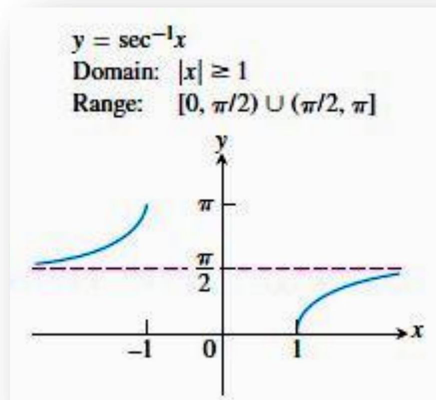
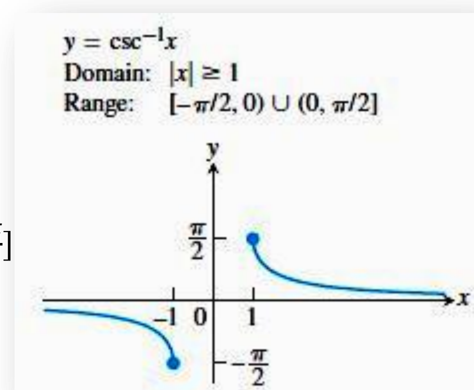


Figure 2.15

6. The inverse of cosecant function

It's denoted by $\csc^{-1}$, and defined to be the inverse of the cosine function

$f(x) = \csc^{-1} x, \csc^{-1}: (-\infty, -1] \cup [1, \infty) \to [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$



Logarithmic Functions

1. The General Logarithmic Functions

Let $a > 0$, $a \neq 1$ be any real number, the general logarithmic from $\text{Log}_a: \mathbb{R}^+ \to \mathbb{R}$ where a is the base of logarithmic function.

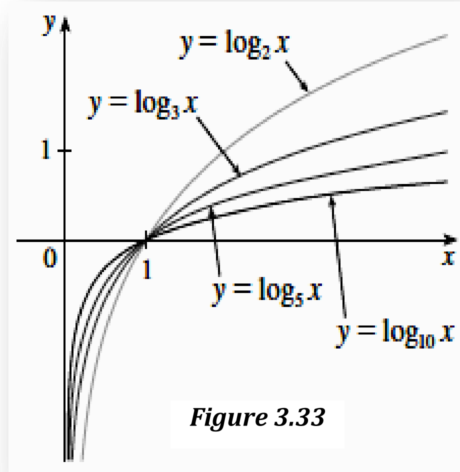


Figure 3.33

Some Properteis

If x and y are positive numbers, then

1. $\text{Log}_a(xy) = \text{Log}_a x + \text{Log}_a y$

2. $\text{Log}_a\left(\frac{x}{y}\right) = \text{Log}_a x - \text{Log}_a y$

3. $\text{Log}_a x^r = r\text{Log}_a x$ (where r is any real number)

4. $\text{Log}_a 1 = 0$

5. $\text{Log}_a a = 1$

6. For $0 < x < 1$, $\text{Log}_a x < 0$

7. For $x \geq 1$, $\text{Log}_a x \geq 0$

8. $\lim_{x \to 0^+} \text{Log}_a x = -\infty, \lim_{x \to \infty} \text{Log}_a x = \infty$

9. It is one to one and onto function, so it is bijective function.

-If $a = 10$, then we denote this function by $f(x) = \text{Log}\, x$.

-If $a = e$ (e is the Euler's number and $e = 2.718281828 \ldots$), we denote this function by $f(x) = \ln x$ and it is called the natural logarithmic function.

2. The Natural Logarithmic Functions

It is the logarithmic function with the base $a = e$.

$i.e.\ f(x) = \text{Log}_a x = \ln x$, $\ln: \mathbb{R}^+ \to \mathbb{R}$.

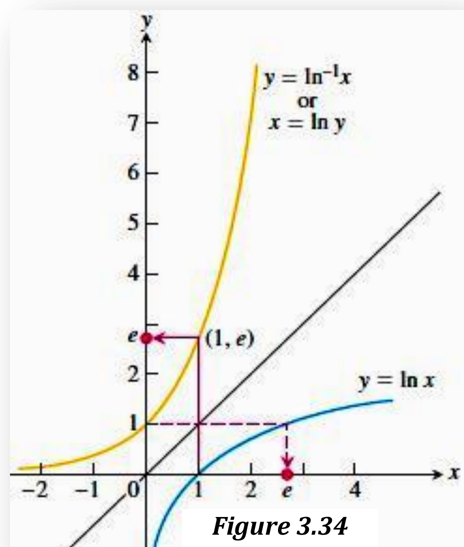


Figure 3.34

Some Properteis

If x and y are positive numbers, then

1. $\ln(xy) = \ln x + \ln y$

2. $\frac{\ln x}{\ln y} = \ln x - \ln y$

x	$\ln x$
0	undefined
0.05	−3.00
0.5	−0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

3. $\ln 1 = 0, \ln e = 1$

4. $\ln x^r = r \ln x$

5. $\ln \frac{1}{a} = \ln a^{-1} = -\ln a$

6. For $0 < x < 1,\ \ln x < 0$

7. For $x \geq 1, \ln x \geq 0$

8. $\lim_{x \to 0^+} \ln x = -\infty, \lim_{x \to \infty} \ln x = \infty$

9. It is one to one and onto function, so it is bijective function.

Exponential Functions

1. The Natural Exponential Functions

Since the natural logarithmic function is a bijective function, so it has an inverse, which's the natural exponential, hence $\exp: \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = e^x, \forall x \in \mathbb{R}$.

Some Properteis

The natural exponential e^x obeys the following laws:

1. $e^x\, e^y = e^{x+y}\ \ \forall\, x, y \in \mathbb{R}$

2. $e^{-x} = \frac{1}{e^x}$

3. $\frac{e^x}{e^y} = e^{x-y}$

4. $(e^x)^y = e^{xy} = (e^y)^x$

5. $e^0 = 1$

6. $e^{\ln x} = x$

7. $\ln e^x = x$

Typical values of e^x

x	e^x (rounded)
-1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

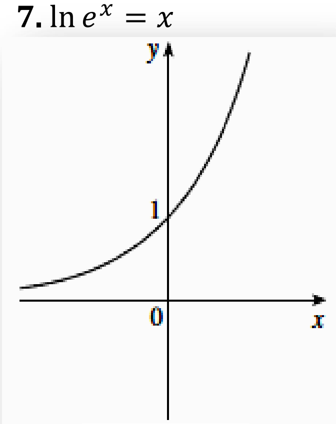


Figure 3.35
(a) $y = e^x$

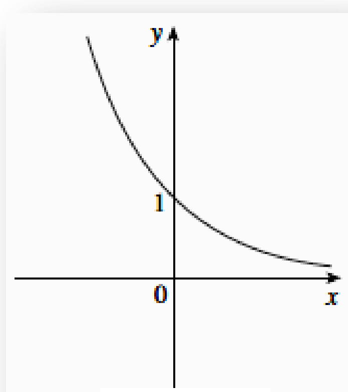


Figure 3.36
(b) $y = e^{-x}$

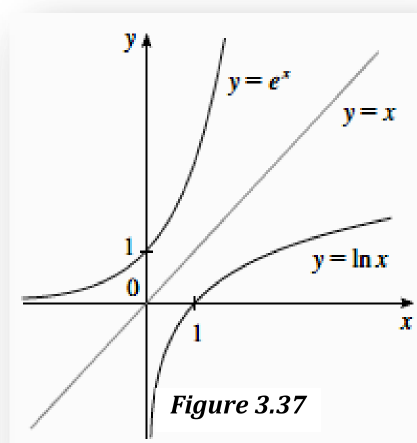


Figure 3.37

2. The General Exponential Functions

It is defined $f: \mathbb{R} \to \mathbb{R}^+$ by

$$f(x) = a^x, \forall x \in \mathbb{R}, a > 0, a \neq 1$$

It is the inverse function of the logarithmic function.

Since $e^{\ln x} = x \implies e^{\ln a} = a$

$$\implies \left(e^{\ln a}\right)^x = a^x$$

$$\implies \boxed{a^x = e^{x \ln a}}$$

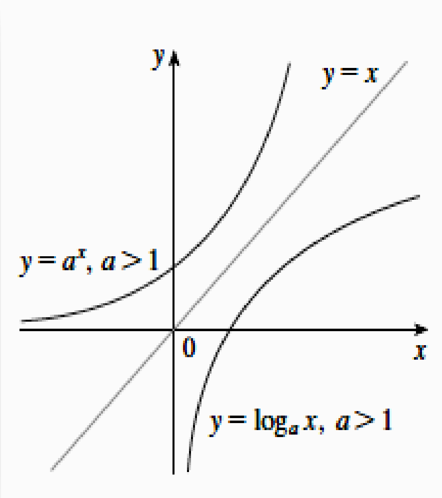


Figure 3.38

$$\boxed{\log_a x = \frac{1}{\ln a} \cdot \ln x = \frac{\ln x}{\ln a}}$$

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0)$$

$$\log_a (a^x) = x \qquad (\text{all } x)$$