THE REAL NUMBER SYSTEM

Having taken calculus, you know a lot about the real number system; however, you probably do not know that all its properties follow from a few basic ones. Although we will not carry out the development of the real number system from these basic properties, it is useful to state them as a starting point for the study of real analysis and also to focus on one property, completeness that is probably new to you.

Real Numbers and the Number Line

Real numbers

Real Numbers are made up of rational numbers and irrational numbers.

The Number Line

We may use the number line to represent all the real numbers graphically; each real number corresponds to exactly one point on the number line. ∞ and -∞ are not real numbers because there is no point on the number line corresponding to either of them.

Important Sets of Real Numbers

We define

$2 = 1 + 1, 3 = 2 + 1, \dots , 9 = 8 + 1,$

$10 = 9 + 1, \dots ,19 = 18 + 1, \dots ,100 = 99+1, \dots$

The set N of natural numbers is define by

$N = \{1, 2, 3, \dots \}$

The set Z of integers is define by

$Z = \{m : -m \in N; \text{ or } m = 0, \text{ or } m \in N\}$

The set Q of rational numbers is define by

$Q = \{\frac{m}{n}: \; m \in Z, \; n \in N\}.$

The set of all real numbers is denoted by R.

A real number is irrational if it is not rational.

Field Properties

The real number system (which we will often call simply the reals) is first of all a set {a, b, c, ...} on which the operations of addition and multiplication are defined so that every pair of real numbers has a unique sum and product, both real numbers, with the following properties.

(A) $a + b = b + a$ *and* $ab = ba$ *(Commutative laws).*

(B) $(a + b) + c = b + (a + c)$ *and* $(ab)c = a(bc)$ *(associative laws).*

(C) $a(b + c) = (ab + ac)$ *(Distributive law).*

(D) *There are distinct real numbers 0 and 1 such that* $a + 0 = a$ *and* $a1 = a$ *for all a.*

(E) *For each a there is a real number* $-a$ *such that* $a + (-a) = 0$, *and if* $a \neq 0$, *there is a real number* $\frac{1}{a}$ *such that* $a\frac{1}{a} = 1$.

The manipulative properties of the real numbers, such as the relations

$$(a + b)^2 = a^2 + 2ab + b^2,$$

$$(3a + 2b)(4c + 2d) = 12ac + 6ad + 8bc + 4bd,$$

$$-a = (-1)a, \quad a(-b) = (-a)b = -ab,$$

and

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \quad (bd \neq 0),$$

all follow from ***(A)–(E)****. We assume that you are familiar with these properties. A set on which two operations are defined so as to have properties* ***(A)–(E)*** *is called a field. The real number system is by no means the only field. The rational numbers (which are the real numbers that can be written as* $r = \frac{p}{q}$, *where* p *and* q *are integers and* $q \neq 0$*).*

The Order Relation

The real number system is ordered by the relation <, which has the following properties.

(F) *For each pair of real numbers a and b, exactly one of the following is true:* $a = b$, $a < b$, *or* $b < a$.

(G) *If* $a < b$ *and* $b < c$*, then* $a < c$*. (The relation < is transitive.)*

(H) *If* $a < b$*, then* $a + c < b + c$ *for any c, and if* $0 < c$*, then* $ac < bc$*.*

A field with an order relation satisfying ***(F)–(H)*** *is an ordered field. Thus, the real numbers form an ordered field. The rational numbers also form an ordered field.*

Remark *For our purpose from we suppose our field (order field) is real field unless we determining the field.*

Greatest and Least Elements

1.1 Definition *Let* $A \subseteq R$*.*

-We say that $u \in R$ *is a* ***least element*** *or* ***minimum*** *of A if*

- (i) $u \in A$ *and*
- (ii) $u \leq x$ *for all* $x \in A$*. In this case we write* $u = A$ *.*

-We say that $v \in R$ *is a* ***greatest element*** *or* ***maximum*** *of A if*

- (i) $v \in A$ *and*
- (ii) $x \leq v$ *for all* $x \in A$ *. In this case we write* $v = A$ *.*

1.2 Example *1 is the minimum of [1, 2) but there is no maximum for this set.*

1.3 Proposition *A maximum (if it exists) is unique. Similarly a minimum is unique.*

Upper and Lower Bounds

1.4 Definition *Let* $A \subset F$*, where F is an ordered field.*

- a) *If there exists* $\alpha \in F$ *such that* $x \leq \alpha$ *for all* $x \in A$*, then we say A is* ***bounded above*** *and* α *is an* ***upper bound*** *of A.*
- b) *If there exists* $\beta \in F$ *such that* $\beta \leq x$ *for all* $x \in A$*, then we say A is* ***bounded below*** *and* β *is an* ***lower bound*** *of A.*
- c) *A set is said to be* ***bounded*** *if it is both bounded above and bounded below. A set is said to be* ***unbounded*** *if it is not bounded.*

Suprema and Infima

1.5 Definition *Let* $A \subset F$*, where F is an ordered field.*

a) *If A is bounded above, then a number* $\alpha_0 \in R$ *is said to be a* ***supremum*** *(or a* ***least upper bound****) of A if it satisfies the conditions:*

1) α_0 *is an upper bound of A, and*

2) *If* α *is any upper bound of A, then* $\alpha_0 \le \alpha$.

We write $supA = \alpha_0$.

b) *If A is bounded below, then a number* $\beta_0 \in R$ *is said to be a* ***infimum*** *(or a* ***greatest lower bound****) of A if it satisfies the conditions:*

1) β_0 *is a lower bound of A, and*

2) *If* β *is any lower bound of A, then* $\beta \le \beta_0$.

We write $inf A = \beta_0$.

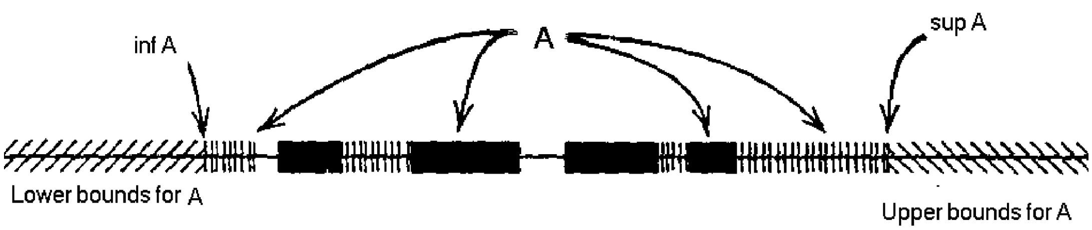

Figure -1- *Suprema and infima*

1.6 Examples

(1) *If S = [1, ∞) then any* $\beta \le 1$ *is a lower bound, and* ***inf*** *S = 1. There is no upper bound for S. The set S is bounded below but not above.*

(2) *If S = [0, 1) then* ***inf*** *S = 0 ∈ S and* ***sup*** *S = 1 ∉ S. The set S is bounded above and below.*

(3) *If* $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right\}$ *then* ***inf*** *S = 0 ∉ S and* ***sup*** *S = 1 ∈ S. The set S is bounded above and below.*

1.7 Lemma *An upper bound* u *of a nonempty set* S *in* R *is the supremum of* S *if and only if for every* $\varepsilon > 0$ *there exists an* $s_\varepsilon \in S$ *such that* $u - \varepsilon < s_\varepsilon$.

Proof *If* u *is an upper bound of* S *that satisfies the stated conditions and if* $v < u$, *assume that* v *is another upper bound of* S *.To show that* $u = supS$ *. We have to show that* $v \in S$ *,then we put* $\varepsilon = u - v$. *Then* $\varepsilon > 0$, *so there exists* $s_\varepsilon \in S$ *such that* $v = u - \varepsilon < s_\varepsilon$. *Therefore,* v *is not an upper bound of* S, *and we conclude that* $u = \sup sup\, S$.

Conversely, suppose that $u = \sup sup\, S$ *and let* $\varepsilon > 0$. *Since* $u - \varepsilon < u$, *then* $u - \varepsilon$ *is not an upper bound of* S. *Therefore, some element* s_ε *of* S *must be greater than* $u - \varepsilon$; *that is,* $u - \varepsilon < s_\varepsilon$.

1.8 Exercise (H. W)

1. *Suppose* A *and* B *is nonempty subsets of* R *that satisfy the property:* $a \le b$ *for all* $a \in A$ *and all* $b \in B$. *Show that* **sup** $A \le$ **inf** B.
2. *Find the supremum and infimum of the following sets if they exists:*
   - ***A***=$\left\{1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{2^2}, \ldots, 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^{n-1}}\right\}$.
   - ***S***=$\left\{m + \frac{1}{n} \;\; m, n \in N\right\}$.
   - ***B***=$\left\{\left(1 - \frac{1}{n}\right) \sin sin \frac{n\pi}{2} \quad , n \in N\right\}$.

The Completeness Axiom (Supremum property)

Suppose A *is a nonempty set of real numbers which is bounded above. Then* A *has a supremum (or l.u.b.) in R.*

A similar result follows for infimum (or g.l.b.'s).

- The real number system is a complete ordered field.

- Is every nonempty set that is bounded above has a supremum? We will see in example 1. 19.

The Archimedean Property

The property of the real numbers described in the next theorem is called the Archimedean property. Intuitively, it states that it is possible to exceed any positive number, *no matter how large, by adding an arbitrary positive number, no matter how small, to itself sufficiently many times.*

1.9 Theorem (The Archimedean Property) *If* ε *and* ρ *are positive, then* $n\varepsilon > \rho$ *for some integer* n.

Proof *The proof is by contradiction. If the statement is false then* $n\varepsilon < \rho\ \forall n$, *that* ρ *is an upper bound of the set* $S = \{x |\ x = n\varepsilon,\ \ n\ \text{ is an integer}\}$.

Therefore, S has a supremum β, *by Completeness Axiom. Therefore,*

$$n\varepsilon \leq \beta \quad \text{for all integers } n. \qquad (*)$$

Since $n + 1$ *is an integer whenever* n *is, (*) implies that*

$$(n + 1)\varepsilon \leq \beta$$

and therefore

$$n\varepsilon \leq \beta - \varepsilon$$

for all integers n. *Hence,* $\beta - \varepsilon$ *is an upper bound of S. Since* $\beta - \varepsilon < \beta$, *this contradicts the definition of* β.

1.10 Theorem (Archimedean Property of the Natural Numbers) *Let* $x \in R$. *Then* $x < n$ *for some* $n \in N$ *(i. e. N is not bounded above).*

Proof *If not true, then* $n \leq x$ *for all* $n \in N$; *therefore,* x *is an upper bound of* N. *Therefore by The Completeness Axiom, the non empty set* N *has a supremum* $u \in R$. *So* $u - 1$ *is smaller than the supremum* u *of* N. *Therefore* $u - 1$ *is not an upper bound of* N. *So there exists* $m \in N$ *with* $u - 1 < m$. *Adding 1 gives* $u < m + 1$, *and since* $m + 1 \in N$, *this inquality contradicts the fact that* u *is an upper bound of* N.

1.11 Corollary *If* $t > 0$, *there exists* $n \in N$ *such that* $0 < \frac{1}{n} < t$.

Proof *Assume there is no natural number* n *such that* $0 < \frac{1}{n} < t$. *Then for every* $n \in N$ *it follows that* $\frac{1}{n} \geq t$ *and hence* $n \leq \frac{1}{t}$. *Hence* $\frac{1}{t}$ *is an upper bound for N, contradicting the previous Theorem. Hence there is a natural number* n *such that* $0 < \frac{1}{n} < t$.

1.12 Corollary (H. W) *If* $y > 0$, *there exists* $m \in N$ *such that* $m - 1 \leq y < m$.

1.13 Example *Show that* **sup** $\{1 - \frac{1}{n} : n \in N\} = 1.$

Solution *Since* $1 - \frac{1}{n} < 1$ *for all* $n \in N$, *1 is an upper bound. To show that 1 is the supremum, it must be shown that for each* $\varepsilon > 0$ *there exists* $n \in N$ *such that* $1 - \frac{1}{n} > 1 - \varepsilon$. *By Archimedean Property we have* $\frac{1}{n} < \varepsilon$, *hence* $1 - \frac{1}{n} > 1 - \varepsilon$, *so we get the result.*

1.14 Example (H. W) *Show that* **inf** $\{\frac{1}{n} : n \in N\} = 0.$

1.15 Theorem *There exists a positive real number* $x \in R$ *such that* $x^2 = 2$.

Proof: *Only for reading.*

1.16: **Lemma** *Let* $A \subseteq R$ *be a bounded above .If* A *possess a maximum element* α $Sup(A) = \alpha$.

Proof *Let* $\alpha = (A)$ *.Then by definition* $\alpha > x \ \forall x \in A$ *and* $\alpha \in A$ *,by contradiction if*

$Sup(A) \neq \alpha \rightarrow \exists \beta \neq \alpha$ *and* $Sup(A) = \beta \rightarrow \beta > x \ \forall x \in A$, *and* $\beta\rho \ \forall \rho$ *are upper bound*

$$\rightarrow \alpha < \beta < \rho \rightarrow x \leq \alpha < \beta < \rho$$

$\forall x$ *then this is contraduction so that* $\beta = Sup(A)$.

Rational and Irrational Numbers

Elements of R (real numbers) that can be written as $r = \frac{p}{q}$ *, where* p *and* $q \in Z$ *and* $q \neq 0$ *are called* ***rational numbers***. *The set of all rational numbers in R will be denoted by the standard notation Q. the sum and product of two rational numbers is again a rational number.*

The fact that there are elements in R that are not in Q is not immediately apparent. In the sixth century B. C. the ancient of Pythagoreans discovered that the diagonal of a square with unit sides could not be expressed as a ratio of integers. In view of the Pythagorean Theorem for right triangles, this implies that the square of no rational *number can equal 2. This discovery had a profound impact on the development of Greek mathematics. One consequence is that elements of R that are not in Q became known as* ***irrational numbers****, meaning that they are not ratios of integers. Although the word "irrational" in modern English usage has a quite deferent meaning, we shall adopt the standard mathematical usage of this term.*

1.17 Theorem *There is no element* $r \in Q$ *such that* $r^2 = 2$.

Proof *If there were such an r, then we could write* $r = \frac{p}{q}$, *where* p *and* $q \in Z$ *and* $q \neq 0$. *Further we could assume that* $g.c.d(p, q) = 1$. *Then* $\frac{p^2}{q^2} = 2$, *so* $p^2 = 2q^2$. *As* p^2 *is even then* p *is also even. We can then write* $p = 2m$ *and hence* $q^2 = 2m^2$. *That means q is also even so we can write* $q = 2n$. *Hence* $r = \frac{2n}{2m}$, *therefore* $g.c.d(p, q) \neq 1$, *this is the required contradiction.*

1.18 Proposition *If* $x \in Q$ *and* $y \in Irr$, *then*

i) $(x + y) \in Irr$.
ii) $xy \in Irr$ *such that* $x \neq 0$.

Proof (H. W)

Density of the Rationales and Irrationals

1.19 Theorem *(Q is dense in R) If* $x, y \in R$ *and* $x < y$, *then there exists an* $r \in Q$ *such that* $x < r < y$.

Proof *Assume that* $x > 0$. *Since* $y - x > 0$, *it follows from Corollary 1.11 that there exists* $n \in N$ *such that* $\frac{1}{n} < y - x$. *Therefore, we have* $nx + 1 < ny$. *If we apply Corollary 1.12 to* $nx > 0$, *we obtain* $m \in N$ *with* $m - 1 \leq nx < m$. *Therefore,* $m \leq nx + 1 < ny$, *hence* $nx < m < ny$. *Thus the rational number* $r = \frac{m}{n}$ *satisfies* $x < r < y$.

Now assume that $x < 0$. ***(H. W)***

1.20 Example

1) $(A) = -(-A)$
2) $A \subseteq B \Rightarrow Sup(A) \leq Sup(B)$

3) $Sup(a_n + b_n) \leq Sup(a_n) + Sup(b_n)$
4) Find supremum and infimum of $\{x \in Q : x^2 < 2\}$ and $\{(-1)^n + \frac{1}{n} : n \in N\}$
5) Discuss boundedness of $\{x^2 + y^2 : xy = 1\}$
6) $Sup(A \cup B) = Max\{Sup(A), Sup(B)\}, Inf(A \cup B) = Min\{Inf(A), Inf(B)\}$
7) $Max(x, y) + Min(x, y) = x + y$

1.21 Example *The rational number system is not complete; that is, a set of rational numbers may be bounded above (by rationals), but not have a rational upper bound less than any other rational upper bound. To see this, let*

$$S = \{\, r \mid r \text{ is rational and } r^2 < 2\}.$$

If $r \in S$, *then* $r < \sqrt{2}$. *Theorem 1.18 implies that if* $\varepsilon > 0$ *there is a rational number* r_0 *such that* $\sqrt{2} - \varepsilon < r_0 < \sqrt{2}$, *so Lemma 1.7 implies that* $\sqrt{2} =$ ***sup*** *S. However,* $\sqrt{2}$ *is irrational; that is, it cannot be written as the ratio of integers. Therefore, if* r_1 *is any rational upper bound of S, then* $\sqrt{2} < r_1$. *By Theorem 1.18, there is a rational number* r_2 *such that* $\sqrt{2} < r_2 < r_1$. *Since* r_2 *is also a rational upper bound of S, this shows that S has no rational supremum.*

1.22 Theorem *(The set of irrational numbers is dense in the reals R) If* $x, y \in R$ *with* $x < y$, *then there exists an irrational number z such that* $x < z < y$.

Proof *If we apply the Theorem 1.18 to the real numbers* $\frac{x}{\sqrt{2}}$ *and* $\frac{y}{\sqrt{2}}$, *we obtain a rational number* $r \neq 0$ *such that* $\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$.

Then $z = r\sqrt{2}$ *is irrational (why?) and satisfies* $x < z < y$.