CHAPTER THREE

Differentiation of Functions

In this chapter, we discuss the notions of sequences and convergence of sequences to give the definition of derivative in a space $\mathbb{R}$ with usual metric space.

Definition 3.1: Let I be an open interval in $\mathbb{R}$ and $f: I \to \mathbb{R}$ be a real valued function. *Let*$\{x_n\}_{n\in\mathbb{N}}$ be a sequence in I such that $x_n \to x_0$ for some $x_0 \in I$ and $x_n \neq x_0$.Then we say that f is differentiable at x_0 or it has a derivative at x_0 if and only if the sequence $\{\frac{f(x_n)-f(x_0)}{x_n-x_0}\}_{n\in\mathbb{N}}$ is convergent to the same value λ in $\mathbb{R}$, for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in I where $x_n \to x_0$ and is denoted by $\left.\frac{df}{dx}\right|_{x=x_0}$ or $f'(x_0)$, so usually we can write

$$\lambda = f'(x_0) = \lim_{x\to x_0} \frac{f(x_n) - f(x_0)}{x_n - x_0}$$

Definition 3.2: Let $f: I \to \mathbb{R}$ be a real valued function define on an open interval$I$.Then we say that f is differentiable on I, if f is differentiable at each $x \in I$.

Example 3.3: By using the definition of differentiation if $f(x) = x^2$, then show that $\frac{df}{dx} = 2x \ \forall x \in D_f$.

Solution: Let $\{x_n\}$ be any sequence in D_f such that $x_n \to x$ as $n \to \infty$. Then the sequence

$\{\frac{f(x_n)-f(x)}{x_n-x}\}_{n\in\mathbb{N}} = \left\{\frac{(x_n)^2-x^2}{x_n-x}\right\} = \{x_n + x\}$.

Since $x_n \to x$ as $n \to \infty$, therefore $\{x_n + x\} \to 2x$ as $n \to \infty$.
Then $f'(x) = 2x$.

Theorem 3.4: If f is differentiable at x_0, then f is continuous at x_0.

Proof: *Let f be a differentiable at x_0. Then the sequence $\{\frac{f(x_n)-f(x_0)}{x_n-x_0}\}_{n\in\mathbb{N}}$ is convergent to the same value λ in R_f, for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in D_f where $x_n \to x_0$, by using some properties of sequence we get that the multiplication of both $\{\frac{f(x_n)-f(x_0)}{x_n-x_0}\}_{n\in\mathbb{N}}$ and $\{x_n - x_0\}_{n\in\mathbb{N}}$ is also convergent* sequence.

Then $\frac{f(x_n)-f(x_0)}{x_n-x_0}(x_n - x_0) = f(x_n) - f(x_0)$, *is converges.*
Which mean that the function $f(x_n)$ is converges to $f(x_0)$, by theorem 1.6 we get that f is continuous at x_0.

Remark 3.5: *The converse of theorem 3.4 is not true in general.*

For Example:
Let $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$, discuss continuity and differentiability of the function f at $x = 0$.

Solution:

Since $f(x) = |x|$, is continuous for all $x \in \mathbb{R}$, then $f(x) = |x|$ is continuous at $x = 0$ but is not differentiable at $x = 0$, here we have to show that there exists at least two sequence $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$, where $x_n \to 0$ and also $y_n \to 0$.
While $\{\frac{f(x_n)-f(0)}{x_n-0}\}_{n\in\mathbb{N}} \to \lambda_1$ and $\{\frac{f(y_n)-f(0)}{y_n-0}\}_{n\in\mathbb{N}} \to \lambda_2$ *and* $\lambda_1 \neq \lambda_2$.

Put $\{x_n\}_{n\in\mathbb{N}} = \{\frac{1}{n}\}_{n\in\mathbb{N}}$ *and* $\{y_n\}_{n\in\mathbb{N}} = \{-\frac{1}{n}\}_{n\in\mathbb{N}}$.

So it is clear that $\frac{1}{n} \to 0$ and $-\frac{1}{n} \to 0$ as $n \to \infty$.

While $\frac{f(x_n)-f(0)}{x_n-0} = \frac{\left|\frac{1}{n}\right|}{\frac{1}{n}} = \frac{\frac{1}{n}}{\frac{1}{n}} = 1$, and $\frac{f(y_n)-f(0)}{y_n-0} = \frac{\left|\frac{-1}{n}\right|}{\frac{-1}{n}} = \frac{\frac{1}{n}}{\frac{-1}{n}} = -1 \neq 1 = \frac{f(x_n)-f(0)}{x_n-0}$, *which mean that* f *is not differentiable at* $x = 0$.

Differentiation and Arithmetic Operation (H.W)

The following theorem should be familiar from calculus.

Theorem 3.6: *If* f *and* g *are differentiable at* x_0*, then so are* $f+g$, $f-g$, *and* fg*, with*

a) $(f+g)'(x_0) = f'(x_0) + g'(x_0)$.
b) $(f-g)'(x_0) = f'(x_0) - g'(x_0)$.
c) $(fg)' = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.
d) The quotients f/g is differentiable at x_0 if $g(x_0) \neq 0, with$

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}.$$

Lemma 3.7: If *f is differentiable* at x_0 then

$$f(x) = f(x_0) + [f'(x_0) + E(x)](x - x_0)$$

Where E is defined on a neighborhood of x_0 *and*

$$\lim_{x \to x_0} E(x) = E(x_0) = 0$$

Proof: Define

$$E(x) = \begin{cases} \dfrac{f(x)-f(x_0)}{(x-x_0)} - f'(x_0), & x \in D_f \text{ and } x \neq x_0, \\ 0 & x = x_0 \end{cases}$$

If $x \neq x_0$ *then* $E(x) = \frac{f(x)-f(x_0)}{(x-x_0)} - f'(x_0)$,

then $f(x) = f(x_0) + [f'(x_0) + E(x)](x - x_0)$ *and (3) is obvious if* $x = x_0$.

Then $\lim_{x\to x_0} E(x) = 0$. *We define* $E(x_0) = 0$, *to make* E *continuous at* x_0.

Theorem 3.8: (The Chain Rule)

Suppose that g is differentiable at x_0 and f is differentiable at $g(x_0)$.then the composite function $h = f o\ g$,$defined\ by\ h = f\big(g(x)\big)$. Is differentiable at x_0, with $h'(x_0) = f'\big(g(x_0)\big)g'(x_0)$.

Proof

since f differentiable at $g(x_0)$ $by\ lemma$ 3.7 $implies\ that$

$$f(t) - f\big(g(x_0)\big) = \big[f'\big(g(x_0)\big) + E(t)\big](t - g(x_0))$$

Where

$$\lim_{x \to x_0} E(t) = E(g(x_0)) = 0$$

Letting $t = g(x)$,

$then$ $f(g(x)) - f\big(g(x_0)\big) = \big[f'\big(g(x_0)\big) + E(g(x))\big](g(x) - g(x_0))$

$Since$ $h(x) = f(g(x))$, $this\ implies\ that$

$$\frac{h(x) - h(x_0)}{x - x_0} = \big[f'\big(g(x_0)\big) + E(g(x))\big]\frac{g(x) - g(x_0)}{x - x_0}$$

since g is differentiable at x_0 then g is continuous at x_0 (by theorem 3.4) and Theorem1 12. imply that

$$\lim_{x \to x_0} E(g(x)) = E(g(x_0)) = 0$$

Therefore $\lim\limits_{x \to x_0} \frac{h(x)-h(x_0)}{x-x_0} = f'\big(g(x_0)\big)g'(x_0)$.

Then $h'(x_0) = f'\big(g(x_0)\big)g'(x_0)$.

Applications of Differentiation

Definition 3.9: *A function f has an absolute maximum (or global maximum) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f .the number $f(c)$ is called the maximum value of f on D. Similarly, f has an absolute minimum at c if $f(c) \leq f(x)$ for all x in D and the number $f(c)$ is called the minimum value of f on D. The maximum and minimum values of f are called the extreme values of f.*

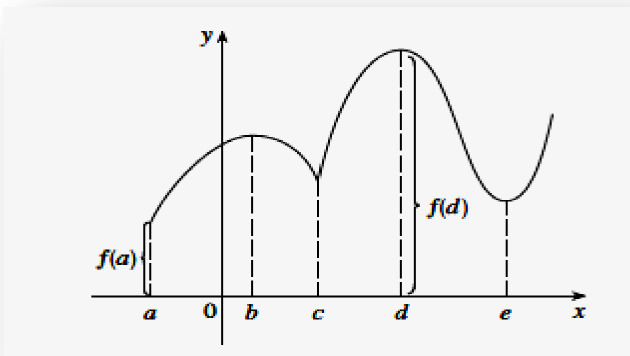


FIGURE 1
Minimum value $f(a)$,
maximum value $f(d)$

Definition 3.10: *A function f has a local maximum (or relative maximum) at c if $f(c) \geq f(x)$ where x near c [this means that $f(c) \geq f(x)$ for all x in some open interval containing c] .the number $f(c)$ is called the maximum value of f on D. Similarly, f has a local minimum at c if $f(c) \leq f(x)$ when x near c.*

The Extreme Value Theorem 3.11:

If f is continuous on a closed interval $[a, b]$ then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some number c and d in $[a, b]$.

Fermat's Theorem 3.12:

If f has a local maximum or minimum at c and if $f'(c)$ exist, then $f'(c) = 0$.

Proof:

suppose that f has local maximum at c, then according to Definition $f(c) \geq f(x)$ if x is sufficiently close to c, and if h is sufficiently close to 0, with h being positive or negative ,then

$$f(c) \geq f(c+h)$$

And therefore

$$f(c+h) - f(c) \leq 0 \qquad \dots\dots\dots\dots\dots 1$$

We can divide both sides of an inequality by a positive number .thus, if $h > 0$ and h is sufficiently small , we have

$$\frac{f(c+h) - f(c)}{h} \leq 0$$

Taking the right-hand limit of both sides of this inequality, we get

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq \lim_{h \to 0^+} 0 = 0$$

But since f'(c) exist, we have

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

And so we have shown that $f'(c) \leq 0$.

If $h < 0$, then the direction of the inequality(1) is reversed when we divided by h:

$$\frac{f(c+h) - f(c)}{h} \geq 0 \qquad h < 0$$

So taking the left-hand limit ,we have

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0$$

We have shown that $f'(c) \geq 0$ and also that $f'(c) \leq 0$.Then $f'(c) = 0$.

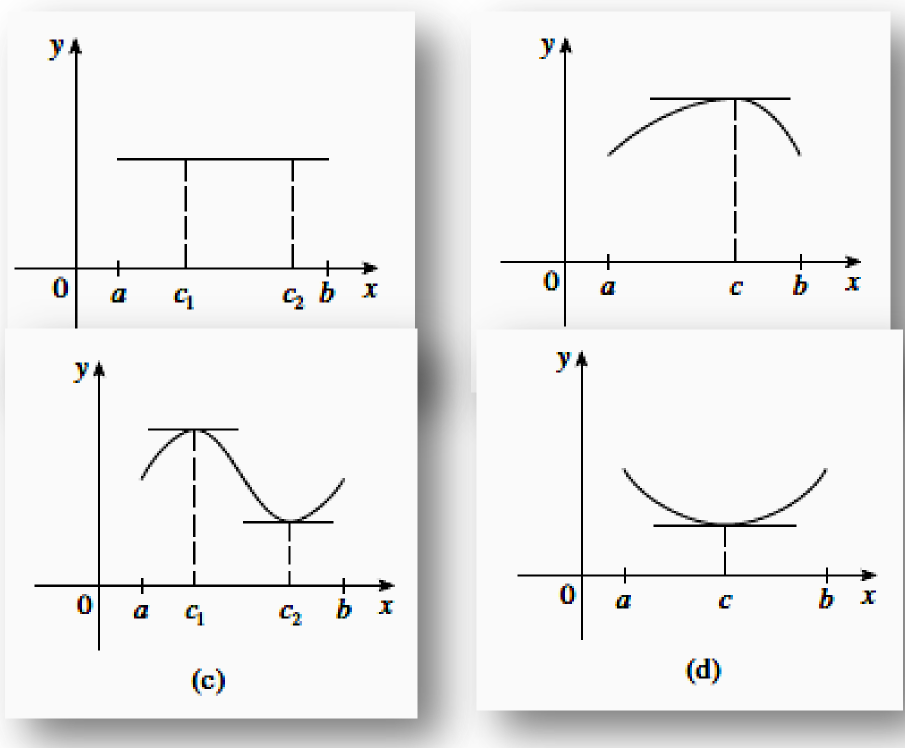
Definition 3.12: *A critical number of a function f is a number c in the domain of* f *such that either* $f'(c) = 0$ *or* $f'(c)$ *does not exist.*

Mean Value Theorems

Rolle's Theorem 3.13: *Let* f *be a function that satisfies the following three hypotheses :*

1. *f is continuouse on the closed interval [a,b].*
2. *f is differentiable on the open interval (a,b).*
3. $f(a) = f(b)$

Then there exists c in (a,b) such that $f'(c) = 0$.



(Figure 2)

Proof: *There are three cases*

CaseI: If $f(x) = k$ *, a constant. Then* $f'(x) = 0$*,*

so the number c can be taken to be any number in (a, b).

Case II: If $f(x) > f(a)$ for some x *in* (a, b) *[as in figure 2 (b) or (c)]*
By the Extreme Value Theorem (which we can apply by hypothese 1), f has a maximum value somewhere in [a,b]. since $f(a) = f(b)$*, it must attain this maximum value at a number c in the open interval (a,b). Then f has a local maximum at c and, by hypothesis 2, f is differentiable on* (a, b) *and c in* (a, b) *them f is differentiable at c .Therefore,* $f'(c) = 0$ *by fermat's Theorem.*
Case III: *If* $f(x) < f(a)$ for some x *in* (a, b) *[as in figure 2(c) or (d)]. By the Extreme Value Theorem, f has a mnimum value in [a,b] and since* $f(a) = f(b)$ *it attains this minimum value at a number c in (a,b). Then f has a local minimum at c and by hypothesis 2, f is differentiable on* (a, b) *and c in* (a, b) *them f is differentiable at c .Therefore,* $f'(c) = 0$ *by fermat's Theorem.*

Theorem 3.14:(Lagrange's Mean Value Theorem Or First Mean Value Theorem)

Let f be contiuous on [a,b], and differentiable on (a,b). Then there exists c in (a, b) *such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad \text{or equivalently} \quad f(b) = f(a) + (b - a)f'(c).$$

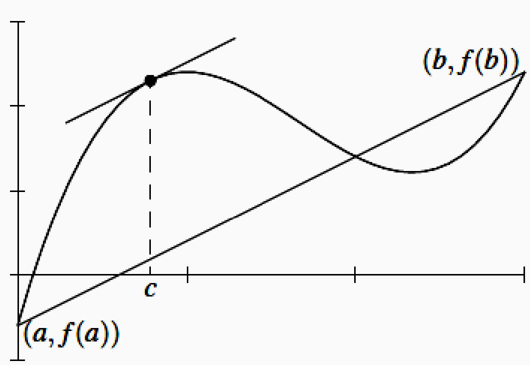


Figure 4.3: Graphical interpretation of the mean value theorem.

Proof

The theorem following easily from Rolle's Theorem.

Define the function $g: [a, b] \to R$

$$g(x) = f(x) - f(b) + (f(b) - f(a))\frac{b-x}{b-a}.$$

Then we know that g *is a differentiable function on* (a, b) *and continuous on* $[a, b]$ *such that*

$g(a) = 0$ *and* $g(b) = 0$ *. Thus there exist* $c \in (a, b)$ *such that* $g'(c) = 0$.

$$0 = g'(c) = f'(c) + \big(f(b) - f(a)\big)\frac{-1}{b-a}.$$

Then $f(b) - f(a) = (b - a)f'(c)$.

For a geometric interpretation of the mean value theorem see figure 3 .The idea is that the value $\frac{f(b)-f(a)}{b-a}$ *is the slop of the line between the point* $\big(a, f(a)\big)$*and* $\big(b, f(b)\big)$. *Then* c *is the point such that* $f'(c) = \frac{f(b)-f(a)}{b-a}$, *that is the tangent line at the point* $(c, f(c))$ *has the same slope as the line between* $\big(a, f(a)\big)$*and* $\big(b, f(b)\big)$.

Example 3.15:

Verify Lagrange's Mean Value Theorem for the function $f(x) = x(x - 1)(x - 2)$ *in* $[0, \frac{1}{2}]$.

Solution

Since f *is a polynomial then* f *is continuous in* $[0, \frac{1}{2}]$ *and differentiable in* $(0, \frac{1}{2})$.

Thus there exists c *in* $[0, \frac{1}{2}]$ *such that* $\frac{f\left(\frac{1}{2}\right)-f(0)}{\frac{1}{2}-0} = f'(c)$. $\qquad \ldots\ldots\ldots (1)$

Now $\quad f'(x) = (x - 1)(x - 2) + x(x - 1) + x(x - 2)$

then $\quad f'(x) = 3x^2 - 6x + 2, \quad f(0) = 0$ *and* $f\left(\frac{1}{2}\right) = \frac{3}{8}$.

From(1), $\quad \frac{3}{8} = \frac{1}{2}(3c^2 - 6c + 2)$ *then* $12c^2 - 24c + 5 = 0$

Then $\quad c = (6 \pm \frac{\sqrt{21}}{6})$, *only* $1 - \frac{\sqrt{21}}{6}$ *lies in* $(0, \frac{1}{2})$.

Hence we get $\quad c = 1 - \frac{\sqrt{21}}{6}$ *and the theorem is verified.*

Some Useful Deduction From The Mean Value Theorem

Proposition 3.16:

Let I be an interval and let $f: I \to \mathbb{R}$ *be a differentiable function such that* $f'(x) = 0$ *for all* x *in* I. *Then* f *is constant.*

Proof

By contrapositive. Suppose that f is not constant, then there exist x and y in I such that $x < y$ *and* $f(x) \neq f(y)$ *then f restricted to* $[x, y]$ *satisfies the hypotheses of the first mean value theorem .Therefore there is a* $c \in (x, y)$ *such that* $f'(c) = \frac{f(y)-f(x)}{y-x}$,

then $f(y) - f(x) = f'(c)(y - x)$.
Since $y \neq x$ *and* $f(y) \neq f(x)$ *then* $f'(c) \neq 0$.

Proposition 3.17:

Let $f: I \to \mathbb{R}$ *be a differential function and f is increasing if and only if* $f'(x) \geq 0$ *for all* $x \in I$.

***Proof**:*

Suppose that f is increasing and differential function then for all x and c in I
we have $\frac{f(x)-f(c)}{x-c} \geq 0$
Then $\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c) \geq 0$
Hence $f'(c) \geq 0$.
Conversely
Suppose that $f'(x) \geq 0$ *for all* $x \in I$. *Let* $x > y$ *in I. Then by the first mean value theorem there is some* $c \in (x, y)$ *such that*

$$f(x) - f(y) = f'(c)(x - y)$$

Since $f'(c) \geq 0$ *and* $x - y > 0$, *then* $f(x) - f(y) \geq 0$.
So f *is increasing.*

Proposition 3.18 (H.W)

Let $f: I \to \mathbb{R}$ *be a differential function and f is decreasing if and only if* $f'(x) \leq 0$ for *all* $x \in I$.

Theorem3.19:(Cauchy Mean Value Theorem OR Second Mean Value Theorem)

If f and g are continuous on the closed interval [a ,b] and differentiable on the open interval (a, b) *,then*

$$[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c)$$

for *some* c *in* (a, b).

Proof *The function*

$$h(x) = [g(b) - g(a)]f(x) - [f(b) - f(a)]g(x)$$

Is continuous on $[a, b]$ *and* $differentiable$ *on* (a, b),

and $h(a) = h(b) = g(b)f(a) - f(b)g(a)$.

Therefore, Rolle's Theorem implies that $h'(c) = 0$ $for\ some\ c\ in\ (a, b)$.

since $h'(c) = [g(b) - g(a)]f'(c) - [f(b) - f(a)]g'(c)$

then $0 = [g(b) - g(a)]f'(c) - [f(b) - f(a)]g'(c)$

therefore, $[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c)$.

Example 3.20:

Verify the Cauchy's mean value theorem *for the functions* x^2 *and* x^3 *in* $[1,2]$ *and also find c of this theorem.*

Solution

Let $f(x) = x^2$ *and* $g(x) = x^3$. *Since* $f(x)$ *and* $g(x)$ *are both polynomial functions, so they are continuous on* $[1,2]$ *and differentiable on* $(1,2)$.

Also $g'(x) = 3x^2 \neq 0$ for *any point in* $(1,2)$.

Hence by Cauchy's mean value theorem there exists at least one number c in $(1,2)$ *such that* $[g(2) - g(1)]f'(c) = [f(2) - f(1)]g'(c)$

Then $(8 - 1)2c = (4 - 1)3c^2$, *there fore* $c = \frac{14}{9}$ *lies in* $(1,2)$.

Hence Cauchy's mean value theorem *is verified.*

The Intermediate Value Theorem 3.21: (W.H)

Suppose that f is continuouse on the closed interval [a,b] and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$. Then there exist a number c in (a, b) such that $f(c) = N$.

Example 3.22:

Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

Solution *First we use the Intermediate Value Theorem to show that a root exist .*
Let $f(x) = x^3 + x - 1$. Then $f(0) = -1 < 0$ and $f(1) = 1 > 0$.
Since f is polynomial ,it is continuous, so the Intermediate Value Theorem state that there is a number c between 0 and 1 such that $f(c) = 0$. Thus, the given equation has a root.
To show that the equation has no another real root, we use Rolle's Theorem and by contraduction suppose that it had two roots a and b .Then $f(a) = 0 = f(b)$ and since f is polynomia, it is differentiable on (a,b) and continuous on [a,b], Thus by Rolle's Theorem
there is a number c between a and b such that $f'(c) = 0$, But

$$f'(x) = 3x^2 + 1 \geq 1 \quad \text{for all } x$$

(since $x^2 \geq 0$) so $f'(x)$ can never be 0 .This gives a contraduction .Therefore the equation can't have two real root.

Exercises:

Q1:

Test if Lagrange's Mean Value Theorem holds for the function $f(x) = |x|$ *in the interval* $[-1,1]$.

Q2:

Applying Lagrange's Mean Value Theoremto the function defined by $f(x) = tan^{-1}x$ *in* $[u, v]$, *show that* $\frac{v-u}{1+v^2} < tan^{-1}v - tan^{-1}u < \frac{v-u}{1+u^2}$, if $0 < u < v$.

Q3:

Discuss the applicability of Rolle's Theorem to $f(x) = 2 + (x-1)^{\frac{2}{3}}$ *in* $[0,2]$.

Q4:

Verify whether the function $f(x) = sinx$ *in* $[0, \pi]$ *satisfies the condition of Rolle's Theorem and hence find c as prescribed by the theorem.*

Q5:

Show that $\frac{sin\alpha - sin\beta}{cos\beta - cos\alpha} = cot\theta$, *where* $0 < \alpha < \theta < \beta < \frac{\pi}{2}$.

By Cauchy's Mean Value Theorem.