Mathematical Analysis

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Question Bank *1st semester*

Q1-Example: Show that there exists no rational number whose square is 8.

Q2-Example: Prove that every finite set is bounded.

Q3-Example: Prove that every subset of a bounded set is bounded.

Q4-Example: which of the following sets are bounded above, bounded below or otherwise?

Also find the supremum (l.u.b) and infimum (g.l.b), if they exist. Which of these belong to the set?

i) $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.

ii) $S = \{(-1)^n \frac{1}{n} : n \in \mathbb{N}\}$.

iii) $S = \{(-1)^n\, n : n \in \mathbb{N}\}$.

iv) $S = \{1 + (-1)^n \frac{1}{n} : n \in \mathbb{N}\}$.

v) $S = \{\frac{4n+3}{n} : n \in \mathbb{N}\}$.

vi) $S = \{\frac{n}{2n+1} : n \in \mathbb{N}\}$.

vii) $S = \{(-1)^n(\frac{1}{4} - \frac{4}{n}) : n \in \mathbb{N}\}$.

viii) $S = \{\left(1-\frac{1}{n}\right)\sin\frac{n\pi}{2} : n \in \mathbb{N}\}$.

ix) $S = \{x \in \mathbb{I} : x \leq 25\}$.

x) $S = \{m - \frac{1}{n} : m, n \in \mathbb{N}\}$.

Q5-Example: Let $A \subset \mathbb{R}$ and $c \in \mathbb{R}$, we define

$cA = \{y \in R : y = cx$ for some $x \in A\}$. If $sup\ A$ and $inf\ A$ are exist and $c \geq 0$, then prove that

$sup\ cA = c\ sup\ A, \quad inf\ cA = c\ inf\ A.$

If $c < 0$, then

$sup\ cA = c\ inf\ A, \quad inf\ cA = c\ sup\ A,$ provided that $sup\ A$ and $inf\ A$ are exist.

Q6-Example: If $A \subset \mathbb{R}$, $A = \{\frac{n}{n+1}$, for $n \in \mathbb{N}\}$ and $c = 2$, show that $sup\ cA = c\ sup\ A, \quad inf\ cA = c\ inf\ A.$

Q7-Example: Suppose that A, B are nonempty subsets of real numbers such that $x \leq y$ for all $x \in A$ and $y \in B$. Then $sup\ A \leq inf\ B$.

Q8-Example: Using **Example 7** show that if $A \subset \mathbb{R}$, $A = \{\frac{1}{n}$, for $n \in \mathbb{N}\}$ and $B \subset \mathbb{R}$, $B = \{\frac{2n}{n+1}$, for $n \in \mathbb{N}\}$, then $sup\ A \leq inf\ B$.

Q9-Example: Suppose that A, B are subsets of $\mathbb{R}$ such that $A \subset B$. If $sup\ A$ and $sup\ B$ exist, then $sup\ A \leq sup\ B$, and if $inf\ A, inf\ B$ exist, then $inf\ A \geq inf\ B$.

Q10-Example:let $A, B \subset \mathbb{R}$, $A = \{\frac{1}{2n}$, for $n \in \mathbb{N}\}$, $B = \{\frac{1}{n}$, for $n \in \mathbb{N}\}$ by using **Example 9** show that $sup\ A \leq sup\ B$ and $inf\ A \geq inf\ B$.

Q11-Example: let A, B be two nonempty subsets of $\mathbb{R}$. Define

$$A + B = \{z : z = x + y \text{ for some } x \in A, y \in B\},$$

$$A - B = \{z : z = x - y \text{ for some } x \in A, y \in B\}$$

If $sup\ A$, $sup\ B$, $inf\ A$ *and* $inf\ B$ are exist, then,

- $sup(A + B) = sup\ A + sup\ B,$
- $inf(A + B) = inf\ A + inf\ B,$
- $sup(A - B) = sup\ A - inf\ B,$
- $inf(A - B) = inf\ A - sup\ B.$

Q12-Example:

We shall now take a look at an example of a different kind. Assume that we want to send messages in a language of N symbols (letters, numbers, punctuation marks, space, etc.) We assume that all messages have the same length K (if they are too short or too long, we either fill them out or break them into pieces). We let X be the set of all messages, i.e., all sequences of symbols from the language of length K. If $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k)$ are two messages, we define $d(x,y) =$ the number of indices n such that $x_n \neq y_n$.

It is not hard to check that d is a metric. It is usually referred to as the Hamming-metric, and is much used in coding theory where it serves as a measure of how much a message gets distorted during transmission.

Q13-Example:

Let X be a non-empty set, and let $\rho : X \times X \to R$ be a function satisfying:

(i) $\rho(x,y) \geq 0$ with equality if and only if $x = y$.

(ii) $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$ for all $x, y, z \in X$.

Define $d : X \times X \to R$ by

$$d(x,y) = max\{\rho(x,y), \rho(y,x)\}$$

Show that d is a metric on X.(H.W)

Q14-Example:

Let X be a non-empty set and a function $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

1. $d(x,y) = \log(\frac{y}{x})$.
2. $d(x,y) = \ln(1 + \rho(x,y))$ where $\rho(x,y)$ be a metric space on X.
3. $d(x,y) = |x - y + 1|$.

Determine that $d(x,y)$ is a metric space or not.

Q15-Example: By giving examples show that

1. $\bigcap_{\infty}$ open $\neq$ open.
2. $\bigcup_{i=1}^{\infty}$ closed $\neq$ closed.

Q16-Example:

Find all limit points and drive sets of S in **Example14** and determine that S is closed or not.