Mathematical Analysis

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Question Bank 2^{nd} *semester*

Q1-*Theorem*: A mapping $f: X \to Y$ of a metric space (X, d) in to a metric space (Y, ρ) is continuous on X if and only if $f^{-1}(F)$ is closed in X for all closed subsets F of Y.

Q2-Example: Let f be a mapping of $\mathbb{R}$ into $\mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{x} & when \quad x \neq 0 \\ 0 & at \quad x = 0 \end{cases}.$$

Find whether f is continuous with respect to the usual metric for $\mathbb{R}$

Q3-Example: Let f be a mapping of $\mathbb{R}$ into $\mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & where \quad x > 0 \\ 0 & where \quad x = 0. \\ -1 & where \quad x < 0 \end{cases}$$

Find whether f is continuous with respect to the usual metric for $\mathbb{R}$.

Q4-Example: Let f be a mapping of $\mathbb{R}$ into $\mathbb{R}$ defined by

$$f(x) = \begin{cases} 5 & where \quad x \geq 0 \\ -5 & where \quad x < 0 \end{cases}.$$

Find whether f is continuous with respect to the usual metric for $\mathbb{R}$.

Q5- Example; Let f be a mapping of $\mathbb{R}$ into $\mathbb{R}$ defined by $f(x) = x^2$. Find whether f is continuous with respect to the usual metric for $\mathbb{R}$.

Q6-Example:

- Show that $f(x) = \sin x^2$ is not uniform continuous on $[0, \infty)$.
- Show that $f(x) = \sin x$ is uniform continuous on $[0, \infty)$.
- Show that $f(x) = \cos\frac{1}{x}$ is continuous on $(0, 1]$ but it is not uniform continuous on $(0, 1]$.

Q7-Example: Let $f: (0,1) \to \mathbb{R}$ defined by $f(x) = e^x$. Is f is uniform continuous on $(0,1)$?

Q8Example: Is $f(x) = \frac{e^x}{x}$ uniform continuous on [2, 5].

Q9-Example: Is $f(x) = \frac{1}{x^2}$ uniform continuous on (0,1) and (1,∞).

Q10-***Example:***

Show that the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$, where $f_n(x) = nx(1-x)^n$ is not uniformly convergent on $[0, 1]$.(by using M_n Test).

Q11-Example:

Consider the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ of functions defined by $f_n(x) = \frac{x^2+nx}{n}$ converges

Pointwise to $f(x) = x$ on all $x \in \mathbb{R}$ but not uniformly converges on $\mathbb{R}$.

Q12-Example:

Show that the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ $f_n\colon [2,\infty) \to \mathbb{R}$, where $f_n(x) = \frac{1}{1+x^n}$ is uniformly convergent on $[2,\infty)$.

Q13-Example:

Consider the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ of functions defined by $f_n(x) = \frac{nx}{1+n^2x^2}$ converges Pointwise to $f(x) = 0$ but not uniformly converges on $\mathbb{R}$.

Q14-Example:

Consider the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ of functions defined by $f_n(x) = \frac{\sin(nx+3)}{\sqrt{1+n}}$ $f(x) = 0$.

Q15-Example:

Let $f_n(x) = \frac{1}{n(1+x^2)}$ for any $x \in \mathbb{R}$ so that f_n is convergent pointwise to the function $f(x)$. Is f_n uniformly converges on $\mathbb{R}$ or not?

Q16-Example:

Let $f_n\colon [-1,1] \to \mathbb{R}$ defined by $f_n(x) = \cos(\frac{n\pi}{1+x^2})$. To show that f_n is not convergent pointwise on $[-1,1]$.

Q17-Example:

Consider the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ of functions defined by $f_n(x) = \frac{x^2+nx}{n^2}$ converges Pointwise to $f(x) = 0$ but not uniformly converges.

Q18-Example:

Show that the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ where $f_n(x) = nxe^{-nx^2}, x \geq 0$ is not uniform converges on $[0,k], k > 0$.(by using M_n Test).

Q19-Example:

Show that the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ where $f_n(x) = x^{n-1}(1-x)$ converges uniformly in the interval [0,1].(by using M_n Test).

Q20-Example:

- Test the uniform convergence the series $\sum_{n=1}^{\infty} xe^{-nx}$ in the closed interval $[0, 1]$.
- Show that the sequence $\{g_n(x)\}$ where $g_n(x) = \frac{\sin nx}{\sqrt{n}}$ is uniformly converges *on* $[0, \pi]$.
- Test for uniform convergence of the following series on $\mathbb{R}$:
    1) $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$. (Weierstrass's M Test)
    2) $\sum_{n=1}^{\infty} \frac{\sin(x^2+nx^2)}{n(n+1)}$.(absolutely uniform convergent).
    3) $\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}$.(Weierstrass's M Test)
    4) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} |x|^n$ *where* $-1 \leq x \leq 1$.(Abel's Test)
    5) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+x^2)}$.(Dirichlet's Test)
    6) $\sum_{n=1}^{\infty} \frac{(x)^n}{(n+1)}$ where $x \in [-\delta, \delta]$ and δ is any fixed positive number less than unity.(Dirichlet's Test)