Chapter 3: Limits and Continuity

Concept of Limit

Definition: We say that the limit of $f(x)$ is L as x approaches a and write this as

$$\lim_{x\to a} f(x) = L.$$

An alternative notation that we will occasionally use in denoting limits is $f(x)\to L$ as $x\to a$ without actually letting $x = a$.

This means that the definition says that as x gets closer and closer to $x = a$ from both sides of course then $f(x)$ must be getting closer and closer to L or, as we move in towards $x = a$ then $f(x)$ must be moving in towards L.

Definition: Right-handed limit is denoted by $\lim_{x\to a+} f(x) = L$ and left-handed limit **is** denoted by $\lim_{x\to a-} f(x) = L$.

Given a function $f(x)$ if, $\lim_{x\to a+} f(x) = L = \lim_{x\to a-} f(x)$ then the limit will exist and $\lim_{x\to a} f(x) = L$

Likewise, if $\lim_{x\to a} f(x) = L$ then, $\lim_{x\to a+} f(x) = L = \lim_{x\to a-} f(x)$

If $\lim_{x\to a+} f(x) \neq \lim_{x\to a-} f(x)$ then the limit does not exist.

Limit Properties

If $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} g(x) = L_2$,then

1. $\lim_{x\to c}(f(x)\pm g(x)) = L_1 \pm L_2$.
2. $\lim_{x\to c}(f(x).g(x)) = L_1.L_2$.
3. $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{L_1}{L_2}, L_2 \neq 0$.
4. $\lim_{x\to c}(af(x)) = aL_1,\ a$ constant.
5. $\lim_{x\to c}(f(x))^n = (\lim_{x\to c} f(x))^n = L_1^{\,n},\ n\in N$.
6. $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = \sqrt[n]{L_1},\ n\in N$, and for n even, we assume that $L_1 > 0$.
7. $\lim_{x\to c} x = c$.
8. $\lim_{x\to c} x^n = c^n$

__**Example 1**__ $\lim_{x\to 3}\dfrac{x^3-2x^2}{x^2+2} = \dfrac{\lim_{x\to 3}\left(x^3-2x^2\right)}{\lim_{x\to 3}\left(x^2+2\right)} = \dfrac{(3)^3-2(3)^2}{(3)^2+2} = \dfrac{9}{11}$

__**Example 2:**__

$$\lim_{x\to -3}\sqrt{\frac{x^2+2x+1}{8+2x}} = \sqrt{\lim_{x\to -3}\left(\frac{x^2+2x+1}{8+2x}\right)} = \sqrt{\frac{\lim_{x\to -3}(x^2+2x+1)}{\lim_{x\to -3}(8+2x)}} = \sqrt{\frac{9-6+1}{8-6}} = \sqrt{\frac{4}{2}} = \sqrt{2}$$

Remark: Avoid common mistakes of the form $\frac{0}{0}$. Typically zero in the denominator means it's undefined. However that will only be true if the numerator isn't also zero. Also, zero in the numerator usually means that the fraction is zero, unless the denominator is also zero.

So, there are three cases to compute $\lim_{x\to a}\frac{f(x)}{g(x)}$.

1. $g(a)\neq 0$ and $f(a) = 0$. In this case $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f(a)}{g(a)}$.
2. $g(a)=0$ and $f(a)\neq 0$. In this case $\lim_{x\to a}\frac{f(x)}{g(x)}$ does not exist.
3. $g(a)=0$ and $f(a)=0$. In this case $\lim_{x\to a}\frac{f(x)}{g(x)}$ can be calculated by using algebraic manipulation .

Case 1: $g(a)\neq 0$

Example 1: Evaluate the following limit.

(1) $\lim_{x\to 2}\frac{x^3+x+1}{x^2+2}$ (2) $\lim_{x\to 1}\frac{x^2-1}{x^2+1}$

Solution: (1) $\lim_{x\to 2}\frac{x^3+x+1}{x^2+2}=\frac{(2)^3+2+1}{(2)^2+2}=\frac{11}{6}$ (2) $\lim_{x\to 1}\frac{x^2-1}{x^2+1}=\frac{0}{2}=0$

Case 2: $g(a)=0$ and $f(a)\neq 0$ **[Limits that equal infinity]**

Definition

We say that $\lim_{x\to a}f(x)=\infty$ if we can make $f(x)$ arbitrarily large for all x sufficiently close to $x=a$, from both sides, without actually letting $x=a$. We say that $\lim_{x\to a}f(x)=-\infty$ if we can make $f(x)$ arbitrarily large and negative for all x sufficiently close to $x=a$, from both sides, without actually letting $x=a$.

Remark: Concider the limit $\lim_{x\to a}\frac{f(x)}{g(x)}$

1. If $\lim_{x\to a+}f(x)=\infty$ and $\lim_{x\to a-}f(x)=-\infty$, then the limit doesn't exist.
2. If $\lim_{x\to a+}f(x)=-\infty$ and $\lim_{x\to a-}f(x)=\infty$, then the limit doesn't exist.
3. If $\lim_{x\to a+}f(x)=\infty$ and $\lim_{x\to a-}f(x)=\infty$, then the limit doesn't exist and $\lim_{x\to a}f(x)=\infty$.
4. If $\lim_{x\to a+}f(x)=-\infty$ and $\lim_{x\to a-}f(x)=-\infty$, then the limit doesn't exist and $\lim_{x\to a}f(x)=-\infty$.

Example 2: Evaluate $\lim_{x\to 0}\frac{1}{x^2}$

Solution: $\lim_{x\to 0+}\frac{1}{x^2}=\infty$ and $\lim_{x\to 0-}\frac{1}{x^2}=\infty$. So $\lim_{x\to 0}\frac{1}{x^2}=\infty$.

Case 3: $g(a)=0$ and $f(a)=0$

Example 3: Evaluate $\lim\limits_{x\to -2}\dfrac{x^3+8}{x^2-4}$?

Solution: take $x=-2$ we get, $\lim\limits_{x\to -2}\dfrac{x^3+8}{x^2-4}=\left(\dfrac{0}{0}\right)$

$$\lim_{x\to -2}\frac{x^3+8}{x^2-4}=\lim_{x\to -2}\frac{(x+2)(x^2-2x+4)}{(x+2)(x-2)}=\lim_{x\to -2}\frac{(x^2-2x-4)}{(x-2)}$$

$$\lim_{x\to -2}\frac{x^3+8}{x^2-4}=\lim_{x\to -2}\frac{(x^2-2x-4)}{(x-2)}=\frac{(-2)^2-2(-2)-4}{-2-2}=\frac{4+4-4}{-4}=-1$$

Example 4: Evaluate $\lim\limits_{x\to 4}\dfrac{x+\sqrt{x}-6}{\sqrt{x}-2}$.

Solution

$$\lim_{x\to 4}\frac{x+\sqrt{x}-6}{\sqrt{x}-2}\left(\frac{0}{0}\right)=\lim_{x\to 4}\frac{(\sqrt{x}+3)(\sqrt{x}-2)}{(\sqrt{x}-2)}=\lim_{x\to 4}(\sqrt{x}+3)=\sqrt{4}+3=2+3=5$$

Example 5: Evaluate the following limit.

$$\lim_{h\to 0}\frac{(h+1)^2-1}{h}$$

Solution

$$\lim_{h\to 0}\frac{(h+1)^2-1}{h}\left(\frac{0}{0}\right)=\lim_{h\to 0}\frac{h^2+2h+1-1}{h}=\lim_{h\to 0}\frac{h(h+2)}{h}=\lim_{h\to 0}(h+2)=2$$

Exercise : Evaluate the following limit. (1) $\lim\limits_{x\to 3}\dfrac{x^3+4x-39}{x^3-27}$ (2) $\lim\limits_{x\to 3}\dfrac{x^2-x-6}{x^2-2x-3}$

(3) $\lim\limits_{x\to 1}\dfrac{x^{\frac{3}{2}}-x}{x^{\frac{1}{2}}-1}$ (4) $\lim\limits_{x\to 0}\dfrac{(x+2)^2-4}{x}$?

Limits at infinity

Definition:
By limits at infinity we mean one of the following two limits.

$$\lim_{x\to\infty} f(x) \qquad\qquad \lim_{x\to-\infty} f(x)$$

Theorem: For $n>0$ we have

$$\lim_{x\to\infty}\frac{1}{x^n}=0 \qquad\qquad \lim_{x\to-\infty}\frac{1}{x^n}=0$$

This fact should make sense if you think about it. We require $n>0$ to make sure the term stays in the denominator and as we increase x then x^n will also increase. So, what we end up with is a constant divided by an increasingly large number so the quotient of the two will become increasingly small. In the limit we will get zero.

Theorem:

$$\lim_{x\to+\infty}(a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0) = \lim_{x\to+\infty} a_n x^n, a_n \neq 0$$

$$\lim_{x\to-\infty}(a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0) = \lim_{x\to-\infty} a_n x^n, a_n \neq 0$$

Remark: You can avoid common mistakes by giving careful consideration to the forms $\frac{\infty}{\infty}$ and $\infty-\infty$ during the computations of the limit. Initially, many students incorrectly conclude that $\frac{\infty}{\infty}$ is equal to 1, or that the limit does not exist, or is $+\infty$ or $-\infty$. Many also conclude that $\infty-\infty$ is equal to 0. In fact, the forms $\frac{\infty}{\infty}$ and $\infty-\infty$ are examples of indeterminate forms. This means that you have not yet determined an answer. Usually, these indeterminate forms can be circumvented by using algebraic manipulation. Such tools as algebraic simplification and conjugates can easily be used to circumvent the forms $\frac{\infty}{\infty}$ and $\infty-\infty$ so that the limit can be calculated.

Example 1: Compute (1) $\lim_{x\to\infty}\left(3x^3-1000x^2\right)$. (2) $\lim_{x\to-\infty}\left(x^4+5x^2+1\right)$ (3) $\lim_{x\to\infty}\frac{100}{x^2+5}$

Solution: (1) $\lim_{x\to\infty}\left(3x^3-1000x^2\right)=\lim_{x\to\infty}\left(3x^3\right)=\infty$

(2) $\lim_{x\to-\infty}\left(x^4+5x^2+1\right)=\lim_{x\to-\infty}\left(x^4\right)=\infty$

(3) $\lim_{x\to\infty}\frac{100}{x^2+5}=\left(\frac{100}{\infty}\right)=0$

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Example 2: Compute (1) $\lim_{x\to-\infty}\frac{x+7}{3x+5}$. (2) $\lim_{x\to\infty}\frac{7x^2+x-100}{2x^2-5x}$ (3) $\lim_{x\to\infty}\frac{x^2-3x+7}{x^3+10x-4}$

Solution: (1) $\lim_{x\to-\infty}\frac{x+7}{3x+5}=\left(\frac{-\infty}{-\infty}\right) = \lim_{x\to-\infty}\frac{\frac{x}{x}+\frac{7}{x}}{\frac{3x}{x}+\frac{5}{x}}=\lim_{x\to-\infty}\frac{1+\frac{7}{x}}{3+\frac{5}{x}}=\frac{1+0}{3+0}=\frac{1}{3}$

(2) $\lim\limits_{x\to\infty}\dfrac{7x^2+x-100}{2x^2-5x}=\left(\dfrac{\infty}{\infty}\right)$ Circumvent it by dividing each term by x^2 .

So $\lim\limits_{x\to\infty}\dfrac{7x^2+x-100}{2x^2-5x}=\lim\limits_{x\to\infty}\dfrac{\dfrac{7x^2}{x^2}+\dfrac{x}{x^2}-\dfrac{100}{x^2}}{\dfrac{2x^2}{x^2}-\dfrac{5x}{x^2}}=\lim\limits_{x\to\infty}\dfrac{7+\dfrac{1}{x}-\dfrac{100}{x^2}}{2-\dfrac{5}{x}}=\dfrac{7+0-0}{2-0}=\dfrac{7}{2}$

(3) Circumvent it by dividing each term by x^3 .

So $\lim\limits_{x\to\infty}\dfrac{x^2-3x+7}{x^3+10x-4}=\lim\limits_{x\to\infty}\dfrac{\dfrac{x^2}{x^3}-\dfrac{3x}{x^3}+\dfrac{7}{x^3}}{\dfrac{x^3}{x^3}+\dfrac{10x}{x^3}-\dfrac{4}{x^3}}=\lim\limits_{x\to\infty}\dfrac{\dfrac{1}{x}-\dfrac{3}{x^2}+\dfrac{7}{x^3}}{1+\dfrac{10}{x^2}-\dfrac{4}{x^3}}=\dfrac{0-0+0}{1+0-0}=0$

Remark: Dividing by x^2, the highest power of x in the numerator, also leads to the correct answer.

Example 3: Compute (1) $\lim\limits_{x\to\infty}\left(x-\sqrt{x^2+7}\right)$. (2) $\lim\limits_{x\to-\infty}\left(x-\sqrt{x^2+7}\right)$

Solution: (1) $\lim\limits_{x\to\infty}\left(x-\sqrt{x^2+7}\right)=(\infty-\infty)$

$\lim\limits_{x\to\infty}\left(x-\sqrt{x^2+7}\right)=\lim\limits_{x\to\infty}\dfrac{\left(x-\sqrt{x^2+7}\right)\left(x+\sqrt{x^2+7}\right)}{\left(x+\sqrt{x^2+7}\right)}=\lim\limits_{x\to\infty}\dfrac{\left(x^2-\left(x^2+7\right)\right)}{\left(x+\sqrt{x^2+7}\right)}=\lim\limits_{x\to\infty}\dfrac{-7}{\left(x+\sqrt{x^2+7}\right)}=\dfrac{-7}{\infty}=0$

(2) $\lim\limits_{x\to-\infty}\left(x-\sqrt{x^2+7}\right)=(-\infty-\infty)$. This is not an indeterminate form. It means $=-\infty$

Example 4: Compute (1) $\lim\limits_{x\to-\infty}\dfrac{e^x}{4+5e^{3x}}$ (2) $\lim\limits_{x\to\infty}\dfrac{2^x}{3^x}$

(4) $\lim\limits_{x\to\infty}\dfrac{\ln\left(2+e^{3x}\right)}{\ln\left(1+e^x\right)}$ (5) $\lim\limits_{x\to\infty}e^{2x-1}$

Solution: (1) $\lim\limits_{x\to-\infty}\dfrac{e^x}{4+5e^{3x}}=\dfrac{0}{4+0}=0$

(2) $\lim\limits_{x\to\infty}\dfrac{2^x}{3^x}\left(\dfrac{\infty}{\infty}\right)=\lim\limits_{x\to\infty}\left(\dfrac{2}{3}\right)^x=0$

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(4)
$$\lim_{x\to\infty}\frac{\ln\left(2+e^{3x}\right)}{\ln\left(1+e^{x}\right)}\left(\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{\ln\left(e^{3x}\left(\frac{2}{e^{3x}}+1\right)\right)}{\ln\left(e^{x}\left(\frac{1}{e^{x}}+1\right)\right)}=\lim_{x\to\infty}\frac{\ln e^{3x}+\ln\left(\frac{2}{e^{3x}}+1\right)}{\ln e^{x}+\ln\left(\frac{1}{e^{x}}+1\right)}$$
$$=\lim_{x\to\infty}\frac{3x+\ln\left(\frac{2}{e^{3x}}+1\right)}{x+\ln\left(\frac{1}{e^{x}}+1\right)}=\lim_{x\to\infty}\frac{3x}{x}=\lim_{x\to\infty}3=3$$

(5) Notice that $\lim_{x\to\infty}(2x-1)=\infty$ and $\lim_{x\to\infty}e^{x}=\infty$. Combining these two results getting $\lim_{x\to\infty}e^{2x-1}=\infty$

Exercise 1: Compute (1) $\lim_{x\to-\infty}\frac{7x^2+x+11}{4-x}$ (2) $\lim_{x\to-\infty}\frac{x+3}{\sqrt{9x^2-5x}}$ (3) $\lim_{x\to\infty}\frac{x+3}{\sqrt{9x^2-5x}}$

(4) $\lim_{x\to\infty}\left(\sqrt{5x^2+x+3}-\sqrt{5x^2+4x+7}\right)$ (5) $\lim_{x\to\infty}\left[\frac{\sqrt{5x+9x^2}}{1+3x}+2\right]$

Exercise 2: Compute (1) $\lim_{x\to\infty}\ln 2x$ (2) $\lim_{x\to0+}e^{-2/x}$ (3) $\lim_{x\to0+}\tan^{-1}(\ln x)$ (4) $\lim_{x\to0+}e^{1/x^2}$

(5) $\lim_{x\to\infty}\frac{5^x}{3^x+2^x}$

Limits of Trigonometric Functions:

Theorem

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Example 1: Evaluate the following limit.

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x}\right)$$

Solution

$$\lim_{x\to 0}\frac{1-\cos x}{x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{1-\cos x}{x}.\frac{1+\cos x}{1+\cos x}=\lim_{x\to 0}\frac{1-\cos^2 x}{x\,(1+\cos x\,)}=\lim_{x\to 0}\frac{\sin^2 x}{x\,(1+\cos x\,)}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}.\lim_{x\to 0}\frac{\sin x}{(1+\cos x\,)}=(1)(\frac{0}{1+1})=(1)(0)=0$$

Example 2: show that.

$$1.\lim_{x\to 0}\frac{\sin ax}{bx}=\lim_{x\to 0}\frac{ax}{\sin bx}=\lim_{x\to 0}\frac{\sin ax}{\sin bx}=\frac{a}{b}$$

$$2.\lim_{x\to 0}\frac{\tan ax}{bx}=\lim_{x\to 0}\frac{ax}{\tan bx}=\lim_{x\to 0}\frac{\tan ax}{\tan bx}=\frac{a}{b}$$

$$3.\lim_{x\to 0}\frac{\sin ax}{\tan bx}=\lim_{x\to 0}\frac{\tan ax}{\sin bx}=\frac{a}{b}$$

Solution

We will show that

$$\lim_{x\to 0}\frac{\sin ax}{bx}=\frac{a}{b}\qquad\qquad \lim_{x\to 0}\frac{\tan ax}{bx}=\frac{a}{b}\qquad\qquad \lim_{x\to 0}\frac{\sin ax}{\tan bx}=\frac{a}{b}$$

and left the rest as an exercise.

Solution 1

$$\lim_{x\to 0}\frac{\sin ax}{bx}=\lim_{x\to 0}\frac{\dfrac{\sin ax}{ax}}{\dfrac{bx}{ax}}=\lim_{x\to 0}\frac{\dfrac{\sin ax}{ax}}{\dfrac{b}{a}}=\lim_{x\to 0}\frac{a}{b}\frac{\sin ax}{ax}$$

let $y=ax$, as $x\to 0, y\to 0$. So

$$\lim_{x\to 0}\frac{\sin ax}{bx}=\frac{a}{b}\lim_{x\to 0}\frac{\sin ax}{ax}=\frac{a}{b}\lim_{y\to 0}\frac{\sin y}{y}=\frac{a}{b}(1)=\frac{a}{b}$$

Solution 2

$$\lim_{x\to 0}\frac{\tan ax}{bx}=\lim_{x\to 0}\left(\frac{\frac{\sin ax}{\cos bx}}{bx}\right)=\lim_{x\to 0}\left(\frac{1}{\cos bx}.\frac{\sin ax}{bx}\right)=\lim_{x\to 0}\left(\frac{1}{\cos bx}\right).\lim_{x\to 0}\left(\frac{\sin ax}{bx}\right)=1.\frac{a}{b}=\frac{a}{b}$$

Solution 3

$$\lim_{x\to 0}\frac{\sin ax}{\tan bx}=\lim_{x\to 0}\frac{\frac{\sin ax}{ax}}{\frac{\tan bx}{ax}}=\frac{\lim\limits_{x\to 0}\frac{\sin ax}{ax}}{\lim\limits_{x\to 0}\frac{\tan bx}{ax}}=\frac{1}{\frac{b}{a}}=\frac{a}{b}$$

Example 3: Evaluate the following limit

$$\lim_{x\to 0}\frac{4x}{\tan 3x+\sin 2x}$$

Solution

$$\lim_{x\to 0}\frac{4x}{\tan 3x+\sin 2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\frac{4x}{x}}{\frac{\tan 3x+\sin 2x}{x}}=\lim_{x\to 0}\frac{4}{\frac{\tan 3x}{x}+\frac{\sin 2x}{x}}=\frac{4}{\frac{3}{1}+\frac{2}{1}}=\frac{4}{5}$$

Example 4: Evaluate the following limit

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta\sin\theta}$$

Solution

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta\sin\theta}\left(\frac{0}{0}\right)=\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta\sin\theta}.\frac{1+\cos\theta}{1+\cos\theta}=\lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta\sin\theta(1+\cos\theta)}=\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta\sin\theta(1+\cos\theta)}$$

$$=\lim_{\theta\to 0}\frac{\sin\theta}{\theta(1+\cos\theta)}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}.\lim_{\theta\to 0}\frac{1}{(1+\cos\theta)}=(1)(\frac{1}{1+1})=\frac{1}{2}$$

Exercise : Evaluate the following limits.

(1) $\lim\limits_{x\to\pi/2}\dfrac{\cos(x)}{\cos(-x)}$

(2) $\lim\limits_{x\to 0}\dfrac{5\sin 3x+\tan 7x}{3x+x^2}$

Theorem: Squeeze Theorem

Suppose that for all x on $[a,b]$ we have,

$$g(x) \le f(x) \le h(x)$$

Also suppose that,

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

for some $a \le c \le b$ .Then,

$$\lim_{x \to c} f(x) = L$$

The Squeeze theorem is also known as the Sandwich Theorem.

Example 14: Evaluate the following limit.

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$$

Solution: since $-1 \le \cos x \le 1$

$$-1 \le \cos\left(\frac{1}{x}\right) \le 1 \quad [\text{for all } x \ne 0]$$

Now if we have the above inequality for our cosine we can just multiply everything by an x^2 and get the following.

$$-x^2 \le x^2 \cos\left(\frac{1}{x}\right) \le x^2$$

In other words we've managed to squeeze the function that we were interested in between two other functions that are very easy to deal with. So, the limits of the two outer functions are.

$$\lim_{x \to 0} x^2 = 0 = \lim_{x \to 0}\left(-x^2\right)$$

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$$

Exercise: Evaluate the following limits.

(1) $\lim\limits_{x \to 0+} \sqrt{x} \sin\left(x + \frac{1}{x}\right)$

(2) $\lim\limits_{x \to \infty} \left(e^{-3x} \cos 2x\right)$

Continuity

Definition

A function $f(x)$ is said to be continuous at $x = c$ if

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ is exist.
3. $\lim_{x \to c} f(x) = f(c)$

A function is said to be continuous on the interval $[a, b]$ if it is continuous at each point in the interval.

Example : discuss the continuity at $x = 3$ for the following functions.

$$f(x) = \frac{x^2 - 9}{x - 3} \qquad g(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, x \neq 3 \\ 4, x = 3 \end{cases} \qquad h(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, x \neq 3 \\ 6, x = 3 \end{cases}$$

Solution

1. the functions is undefined at $x = 3$, and hence is not continuous at that point.

2.

$$f(3) = 4$$

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \to 3} (x + 3) = 6$$

The function value and the limit aren't the same and so the function is discontinuous at $x = 3$.

3.

$$f(3) = 6 \qquad \lim_{x \to 3} f(x) = 6$$

The function is continuous at $x = 3$ since the function and limit have the same value.

Exercise:- Find the points of discontinuity of the function

$$f(x) = \begin{cases} \dfrac{x - 1}{x^3 - 1} & if \quad x < 1 \\ \dfrac{1}{x + 2} & if \quad 1 \leq x < 2 \\ \dfrac{1}{4} & if \quad 2 \leq x < 3 \\ 2x^2 - 4 & if \quad x \geq 3 \end{cases}$$