CHAPTER ONE

THE REAL NUMBER SYSTE

Introduction:

In this chapter, we shall introduce the system of real numbers as a complete ordered field structure. While we shall give a precise description of what is mean by a complete ordered field, we shall in this manner actually only define the system of real numbers. This implies that we are laying down this definition of the system of real numbers on an axiomatic basis for discussion and development of Real Analysis. It will be seen that the sets of Natural numbers, Integers and Rational numbers will arise as sub-sets of the sets of Real numbers.

Note. *The system of real numbers in an axiomatic manner starting from the system of rational numbers has been constructed by different mathematicians. We mention in this connection the names of Georg Cantor (1845-1918), Richard Dedekind (1831-1916) and Karl Weierstrass (1815-1897).*

Real Numbers and the Number Line

Real numbers

Real Numbers are made up of rational numbers and irrational numbers.

The Number Line

We may use the number line to represent all the real numbers graphically; each real number corresponds to exactly one point on the number line. ∞ *and* -∞ *are not real numbers because there is no point on the number line corresponding to either of them.*

Important Sets of Real Numbers

- *The natural numbers are* $1, 2, 3, 4, 5, \ldots$, *so that the set of natural numbers is define by* $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$.
- The set $\mathbb{Z}$ of integers consists of the numbers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$, *so that the set* $\mathbb{Z}$ *of integers is define by*

  $\mathbb{Z} = \{m : -m \in \mathbb{N}; or\ m = 0; or\ m \in \mathbb{N}\}$.

- *The rational numbers are of the form* $\frac{p}{q}$ *where* p, q *are arbitrary integers with* $q \neq 0$, *so that the set* $\mathbb{Q}$ *of rational numbers is define by* $\mathbb{Q} = \{\frac{p}{q}:\ p \in \mathbb{Z}, q \in \mathbb{N}\}$.
- *A real number which is not rational is called irrational. For example* $\sqrt{2}, \sqrt{8}, \pi$ *and e, etc. Are irrational numbers.*
- *The set of real numbers is denoted by* $\mathbb{R}$.

Field Properties

A set $\{a, b, c, \dots\}$ *on which the operations of addition and multiplication are defined so that every pair of real numbers has a unique sum and product, both real numbers, with the following properties:*

(A) $a + b = b + a$ *and* $ab = ba \ \ \forall\, a, b \in \mathbb{R}$. ***(Commutative laws)***

(B) $(a + b) + c = b + (a + c)$ *and* $(ab)c = a(bc) \ \ \forall\, a, b, c \in \mathbb{R}$. ***(associative laws)***

(C) $a(b + c) = (ab + ac) \ \ \forall\, a, b, c \in \mathbb{R}$. ***(Distributive law)***

(D) *There are distinct real numbers* 0 *and* 1 *such that* $a + 0 = a$ *and* $a1 = a$ *for all a.*

(E) *For each* $a \in \mathbb{R}$ *there is a real number* $-a$ *such that* $a + (-a) = 0$, *and if* $a \neq 0$, *there is a real number* $\frac{1}{a}$ *such that* $a\,\frac{1}{a} = 1$.

The manipulative properties of the real numbers, such as the relations:

$$(a + b)^2 = a^2 + 2ab + b^2,$$

$$(3a + 2b)(4c + 2d) = 12ac + 6ad + 8bc + 4bd,$$

$$-a = (-1)a, \ \ a(-b) = (-a)b = -ab,$$

and

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad (bd \neq 0),$$

*all follow from **(A)–(E)**. We assume that you are familiar with these properties. A set on which two operations are defined so as to have properties **(A)–(E)** is called a field. The real number system is by no means the only field.*

The Order Relation

The real number system is ordered by the relation $<$*, which has the following properties:*

(F) *For each pair of real numbers* a *and* b*, exactly one of the following is true:*

$$a = b, \quad a < b, \quad \text{or} \quad b < a.$$

(G) *If* $a < b$ *and* $b < c$*, then* $a < c$*. (The relation* $<$ *is transitive)*

(H) *If* $a < b$*, then* $a + c < b + c$ *for any c, and if* $c > 0$ *then* $ac < bc$*.*

A field with an order relation satisfying ***(F)–(H)*** *is an ordered field. Thus, the real numbers form an ordered field. The rational numbers also form an ordered field.*

Upper and Lower Bounds

1.1 Definition: *Let* $A \subset \mathcal{F}$*, where* $\mathcal{F}$ *is an ordered field.*

- ***a)*** *We say that* $\alpha \in \mathcal{F}$*, is an* ***upper bound*** *of A if for all* $x \in A$ *then* $\alpha \geq x$*.*
- ***b)*** *We say that* $\beta \in \mathcal{F}$*, is an* ***lower bound*** *of A if for all* $x \in A$ *then* $\beta \leq x$*.*
- ***c)*** *We say that A is* ***bounded above*** *If there exists* $\alpha \in \mathcal{F}$ *and* α *is an* ***upper bound*** *of A.*
- ***d)*** *We say that A is* ***bounded below*** *If there exists* $\beta \in \mathcal{F}$ *and* β *is a* ***lower bound*** *of A.*
- ***e)*** *A set is said to be* ***bounded*** *if it is both* ***bounded above*** *and* ***bounded below****. A set is said to be* ***unbounded*** *if it is not bounded.*

Maximum and Minimum Elements

1.2 Definition: *Let* $A \subseteq \mathbb{R}$*.*

-We say that $u \in \mathbb{R}$ *is a* ***greatest element*** *or* ***maximum*** *of A if*

(i) $u \in A$ *and*

(ii) $x \leq u$ *for all* $x \in A$ *. In this case we write* $u = \boldsymbol{max}\, A$*.*

-We say that $v \in \mathbb{R}$ *is a* ***least element*** *or* ***minimum*** *of A if*

(i) $v \in A$ *and*

(ii) $v \leq x$ *for all* $x \in A$*. In this case we write* $v = \boldsymbol{min}\, A$*.*

1.3 Example: *1 is the maximum of* $(0,1]$ *but there is no minimum for this set.*

1.4 Proposition: *A maximum (if it exists) is unique. Similarly, a minimum is unique.*

Supremum and Infimum

1.5 Definition *Let* $A \subset \mathcal{F}$, *where* $\mathcal{F}$ *is an ordered field.*

- We say that $\alpha \in \mathcal{F}$ *is a **supremum** or a **least upper bound** of A if it satisfies the conditions:*

i. α *is an upper bound of A, and*
ii. *If* u *is any upper bound of A, then* $\alpha \leq u$, *for all* u. *We write* $\boldsymbol{sup} A = \alpha$.

- We say that $\beta \in \mathcal{F}$ *is a **infimum** or a **greatest lower bound** of A if it satisfies the conditions:*

i. β *is a lower bound of A, and*
ii. *If* $\boldsymbol{v}$ *is any lower bound of A, then* $\beta \geq v$, *for all* v. *We write* $\boldsymbol{inf} A = \beta$.

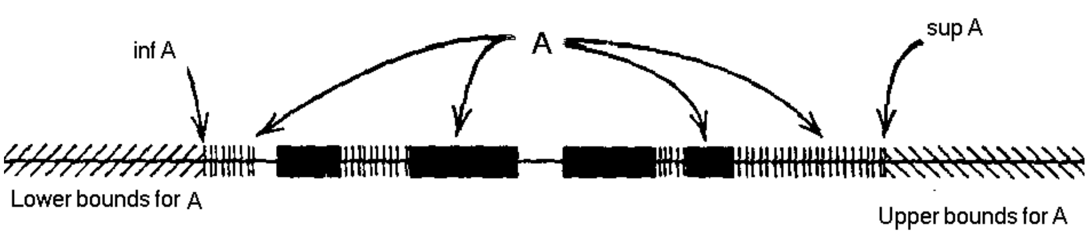

Figure -1- *Supremum and infimum*

1.6 Examples

(1) *If* $S = \{\frac{4n+3}{n} : n \in \mathbb{N}\}$ *then any* $\beta \leq 4$ *is a lower bound,* $inf S = 4 \notin S$ *and* $\alpha \geq 7$ *is an upper bound,* $sup S = 7 \in S$. *The set* S *is bounded above and below*

(2) *If* $S = \mathbb{N}$ *then any* $\beta \leq 1$ *is a lower bound,* $inf S = 1 \in S$. *There is no upper bound for S. The set S is bounded below but not above.*

(3) *If* $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right\}$ *then* $inf S = 0 \notin S$ *and* $sup S = 1 \in S$. *The set* S *is bounded above and below.*

1.7 Lemma *An upper bound* u *of a nonempty set* S *in* $\mathbb{R}$ *is the supremum of* S *if and only if for every* $\varepsilon > 0$ *there exists an* $s_\varepsilon \in S$ *such that* $u - \varepsilon < s_\varepsilon$.

Proof: *Suppose that* $u = \mathbf{sup}\, S$ *and let* $\varepsilon > 0$. *Since* $u - \varepsilon < u$, *then* $u - \varepsilon$ *is not an upper bound of* S. *Therefore, some element* s_ε *of* S *must be greater than* $u - \varepsilon$*; that is,* $u - \varepsilon < s_\varepsilon$.

Conversely, if u *is an upper bound of* S *that satisfies the stated conditions and if* $v < u$, *assume that* v *is another upper bound of* S *.To show that* $u = sup S$ *. We have to show that* $v \in S$ *,then we put* $\varepsilon = u - v$. *Then* $\varepsilon > 0$, *so there exists* $s_\varepsilon \in S$ *such that* $v = u - \varepsilon < s_\varepsilon$. *Therefore,* v *is not an upper bound of* S, *and we conclude that* $u = \mathbf{sup}\, S$.

The Completeness Axiom (Supremum property)

Suppose A is a nonempty set of real numbers which is

1. *A bounded above, then A has a supremum (or l.u.b.) in* $\mathbb{R}$*. and*
2. *A bounded below, then A has infimum (or g.l.b.'s) in* $\mathbb{R}$.

- *The real number system is a complete ordered field.*
- *Is every nonempty set that is bounded above has a supremum? We will see in example 1.6(1, 2).*

The Archimedean Property

The property of the real numbers described in the next theorem is called the Archimedean property. Intuitively, it states that it is possible to exceed any positive number, no matter how large, by adding an arbitrary positive number, no matter how small, to itself sufficiently many times.

***1.8 Theorem** (**The Archimedean Property**)*

For any x and y are positive, then there exists a natural number n such that $nx > y$.

Proof *The proof is by contradiction. If the statement is false then $nx \leq y \ \ \forall n \in \mathbb{N}$, that is y an upper bound of the set $A = \{nx : n \in \mathbb{N}\}$. Then A is bounded above by Completeness Axiom, the set A has a supremum α. Therefore $nx \leq \alpha \ \ \forall\, n \in \mathbb{N}$*

Since $n + 1 \in \mathbb{N}$ then $(n+1)x \leq \alpha$, therefore, $nx \leq \alpha - x \ \ \forall\, n \in \mathbb{N}$.

Hence $\alpha - x$ is an upper bound of A.

Since $\alpha - x < \alpha$, we contradict the statement that α is the least upper bound of A.

***1.9 Theorem** (**Archimedean Property of the Natural Numbers**) Let $\alpha \in \mathbb{R}$. Then $\alpha < n$ for some $n \in \mathbb{N}$ (i. e. $\mathbb{N}$ is not bounded above).*

Proof *If not true, then $n \leq \alpha$ for all $n \in \mathbb{N}$; therefore, α is an upper bound of $\mathbb{N}$. Therefore by The Completeness Axiom, the non empty set $\mathbb{N}$ has a supremum $u \in \mathbb{R}$. So $u - 1 < u$ of $\mathbb{N}$. Therefore $u - 1$ is not an upper bound of $\mathbb{N}$. So there exists $m \in \mathbb{N}$ with $u - 1 < m$. Adding 1 gives $u < m + 1$, and since $m + 1 \in \mathbb{N}$, this inquality contradicts the fact that u is an upper bound of $\mathbb{N}$.*

1.10 Corollary *For any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \varepsilon$.*

Proof *Assume there is no natural number n such that $0 < \frac{1}{n} < \varepsilon$. Then for every $n \in \mathbb{N}$ it follows that $\frac{1}{n} \geq \varepsilon$ and hence $n \leq \frac{1}{\varepsilon}$. Hence $\frac{1}{\varepsilon}$ is an upper bound for $\mathbb{N}$, contradicting the previous Theorem. Hence there is a natural number n such that $0 < \frac{1}{n} < \varepsilon$.*

1.11 Corollary *If $y > 0$, there exists $m \in \mathbb{N}$ such that $m - 1 \leq y < m$.* ***Proof:*** *[H.W]*

1.12 Theorem (The Archimedean Property)

If x and y are two given real numbers with $x > 0$, then there exists a natural number n such that $nx > y$.

Proof: *[H.W]*

Characteristic of supremum and infimum

1- *The necessary and sufficient condition for a real number α to be the supremum of a bounded above set A is that α must satisfy the following conditions:*

(i) $\alpha \geq x \ \ \forall\, x \in A$

(ii)For each positive real number ε, there exists a real number $a \in A$ such that $\alpha - \varepsilon < a \leq \alpha$.

2- *The necessary and sufficient condition for a real number β to be the infimum of a bounded below set A is that β must satisfy the following conditions:*

(i) $\beta \leq x \ \ \forall\, x \in A$

(ii)For each positive real number ε, there exists a real number $a \in A$ such that $\beta \leq a < \beta + \varepsilon$.

1.13 Example: *Let $S = \left\{\frac{1}{n} : \ n \in \mathbb{N}\right\}$. Show that $\boldsymbol{sup}S = 1$.*

Solution: *Since $\frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$, then 1 is an upper bound. To show that 1 is the supremum, it must be shown that for each $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $1 - \frac{1}{n} > 1 - \varepsilon$. By Archimedean Property we have $\frac{1}{n} < \varepsilon$, hence $1 - \varepsilon < 1 - \frac{1}{n} \in S$, so by Characteristic of supremum and infimum (1) we get the result.*

1.14 Example: *Let $S = \left\{1 - \frac{1}{3^n} : n \in \mathbb{N}\right\}$. Show that $\boldsymbol{infS} = \frac{2}{3}$.*

Solution: *By Archimedean Property we have for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$, then $\frac{2}{3} + \frac{1}{n} < \frac{2}{3} + \varepsilon$. Since $\frac{1}{n} > \frac{1}{3^n}$ therefore $1 - \frac{3^{n-1}+1}{3^n} < \frac{2}{3} + \frac{1}{n} < \frac{2}{3} + \varepsilon$, hence $\frac{2}{3} + \varepsilon > 1 - \frac{3^{n-1}+1}{3^n} \in S$, by Characteristic of supremum and infimum (2) we get $\boldsymbol{infS} = \frac{2}{3}$.*

***1.15**: **Lemma** Let $A \subseteq \mathbb{R}$ be a bounded above .If A possess a maximum element α $Sup(A) = \alpha$.*

Proof *Let $\alpha = max(A)$ .Then by definition $\alpha \geq x\ \forall x \in A$ and $\alpha \in A$,by contradiction if*

$Sup(A) \neq \alpha \Rightarrow \beta \neq \alpha$ *and* $Sup(A) = \beta \Rightarrow \beta \geq x\ \forall x \in A$, *and* $\beta < u.b(A)$ *for all* $u.b(A)$

$\Rightarrow \alpha < \beta < u.b(A) \Rightarrow x \leq \alpha < \beta < u.b(A)\quad \forall x$ *then this is contraduction so that* $\beta = Sup(A)$.

Rational and Irrational Numbers

*Elements of $\mathbb{R}$ (real numbers) that can be written as $r = \frac{p}{q}$, where p and $q \in \mathbb{Z}$ and $q \neq 0$ are called **rational numbers**. The set of all rational numbers in $\mathbb{R}$ will be denoted by the standard notation $\mathbb{Q}$. the sum and product of two rational numbers is again a rational number.*

*One consequence is that elements of $\mathbb{R}$ that are not in $\mathbb{Q}$ became known as **irrational numbers**, meaning that they are not ratios of integers.*

1.16 Theorem *There is no element $r \in \mathbb{Q}$ such that $r^2 = 2$.*

Proof *The proof is by contradiction. There is a rational number **r**, then we could write $r = \frac{p}{q}$, where p and $q \in \mathbb{Z}$ and $q \neq 0$. Further we could assume that $g.c.d(p,q) = 1$, such that $\frac{p^2}{q^2} = 2$, so $p^2 = 2q^2$. As p^2 is even then p is also even. We can then write $p = 2m$ and hence $q^2 = 2m^2$. That means q is also even so we can write $q = 2n$.*

Hence $r = \frac{2n}{2m}$, therefore $g.c.d(p,q) \neq 1$, this is the required contradiction.

1.17 Proposition *If $x \in \mathbb{Q}$ and $y \in \boldsymbol{I}rr$, then*

- *i)* $(x + y) \in \boldsymbol{I}rr$.
- *ii)* $xy \in \boldsymbol{I}rr$ *such that* $x \neq 0$. ***[H.W]***

Proof: *Suppose that $x \in \mathbb{Q}$ and $y \in \boldsymbol{I}rr$, we must to prove that $(x + y) \in \boldsymbol{I}rr$.*

Assume that $(x+y) \notin \boldsymbol{I}rr$ *then* $(x+y) \in \mathbb{Q}$ *, we can write* $x+y=z \in \mathbb{Q}$

then $y=z-x$*, since* $x \in \mathbb{Q}$ *and* $z \in \mathbb{Q}$ *then* $y \in \mathbb{Q}$*, this is contradicts the fact that* y *is an irrational number.*

Density of the Rationales and Irrationals

1.18 Theorem(*$\mathbb{Q}$ is dense in $\mathbb{R}$) If* $x, y \in \mathbb{R}$ *and* $x<y$*, then there exists an* $r \in \mathbb{Q}$ *such that* $x<r<y$.

Proof *Assume that* $x>0$*. Since* $y-x>0$*, it follows from Corollary 1.10 that there exists* $n \in \mathbb{N}$ *such that* $\frac{1}{n}<y-x$*. Therefore, we have* $nx+1<ny$*. Since* $x>0$ *and* $n \in \mathbb{N}$*, then* $nx>0$*. If we apply Corollary 1.11 to* $nx>0$*, we obtain* $m \in \mathbb{N}$ *with* $m-1 \leq nx<m$*. Therefore,* $m \leq nx+1<ny$*, hence* $nx<m<ny$*, then* $x<\frac{m}{n}<y$*. Thus the rational number* $r=\frac{m}{n}$ *satisfies* $x<r<y$.

Now assume that $x<0$*.* ***(H. W)***

1.19 Theorem *(The set of irrational numbers is dense in* $\mathbb{R}$*) If* $x, y \in \mathbb{R}$ *with* $x<y$*, then there exists an irrational number* z *such that* $x<z<y$.

Proof *If we apply the Theorem 1.18 to the real numbers* $\frac{x}{\sqrt{2}}$ *and* $\frac{y}{\sqrt{2}}$*, we obtain a rational number* $r \neq 0$ *such that* $\frac{x}{\sqrt{2}}<r<\frac{y}{\sqrt{2}}$*, then* $x<r\sqrt{2}<y$.

Thus $z=r\sqrt{2}$ *is irrational (why?) and satisfies* $x<z<y$.

Exercises:

Q1: *Let $\phi \neq A \subseteq \mathbb{R}$ be a bounded above and below then the supremum and infimum are unique.*

Q2: *Which of the following sets are bounded above, bounded below or otherwise? Also Find the supremum and infimum of the following sets if they exists:*

- $A = \{x \in \mathbb{R}: 2 \leq x < 3\}$.
- $B = \left\{\sqrt{n+1} - \sqrt{n}, n \in \mathbb{N}\right\}$.
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.
- $S = \left\{1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{2^2}, \dots, 1+\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^{n-1}}, \dots\right\}$.
- $H = \{x: x \in \mathbb{Q}, x \geq 0 \text{ and } x^2 < 0\}$.

Q3: *Suppose that A and B are nonempty subsets of $\mathbb{R}$ that satisfy the property: $x \leq y$ for all $x \in A$ and $y \in B$. Show that $\sup A \leq \inf B$.*

Q4: *Define $-A = \{-x: x \in A\}$, prove that if A is bounded set of $\mathbb{R}$, then $-A$ is bounded.*

Q5: *Show that there exists no rational number whose square is 8.*

Q6: *show that If x^2 is an irrational number then x is also irrational.*

Q7: *Show that:*

- $\sup \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\} = 1$.
- $\inf \{r \in \mathbb{Q} : r^2 < 3\} = -\sqrt{3}$.

Q8: *let A and B be bounded subsets of $\mathbb{R}$, then*

1. $\sup(A + B) = \sup A + \sup B$,
2. $\inf(A + B) = \inf A + \inf B$,
3. $\sup(A - B) = \sup A - \inf B$,
4. $\inf(A - B) = \inf A - \sup B$.

Q9: *Let $\phi \neq A, B \subseteq \mathbb{R}$ be a bounded then:*

1. $Sup(A \cup B) = Max\{Sup(A), Sup(B)\}$.
2. $Inf(A \cup B) = Min\{Inf(A), Inf(B)\}$.