CHAPTER THEER

Sequence and Compactness

Sequences in Metric Spaces

3.1 Definition *Let (X, d) be a metric space, a sequence $\{x_n\}$ in X is a function $f: \mathbb{N} \to X$, where $f(n) = x_n$ then $\{x_1, x_2, \dots\}$ is called a sequence in $X$and x_n is called the nth term of the sequence. We also write $\{x_n\}_{n=1}^{\infty}$, or just $\{x_n\}$, for the sequence.*

Convergence of Sequences

3.2 Definition *Let (X, d) be a metric space, a sequence $\{x_n\} \subset X$ is said to be converges to a point $x \in X$, written $x_n \to x$, if for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $\forall n \geq k$ then $d(x_n, x) < \varepsilon$, the point x is called the limit of $\{x_n\}$, and sometimes we write $\lim_{n\to\infty} x_n = x \iff \lim_{n\to\infty} d(x_n, x) = 0$.*

Thus, the sequence $\{x_n\}$ converges to x, if for every $\varepsilon > 0$ there exists the open ball $B_\varepsilon(x)$ includes all terms of x_n, except a finite number of its terms.

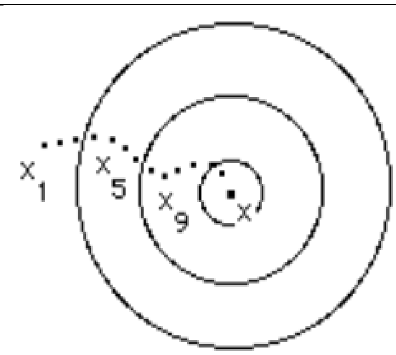

Figure 3.1

3.3 Example *The sequence* $\{\frac{1}{n^2}\}$ *is convergent to zero.*

Solution: *For all* $\varepsilon > 0$, *to find* $k \in \mathbb{N}$ *such that* $\forall n \geq k$ *then* $d(x_n, x) < \varepsilon$.

Fix $\varepsilon > 0$, *then* $\exists k \in \mathbb{N}$ *such that* $k > \frac{1}{\sqrt{\varepsilon}}$. *Now, if* $n \geq k$ *and* $k > \frac{1}{\sqrt{\varepsilon}}$ *then* $n > \frac{1}{\sqrt{\varepsilon}} \implies -\varepsilon < 0 < \frac{1}{n^2} < \varepsilon$ *then* $\left|\frac{1}{n^2} - 0\right| < \varepsilon$.

3.4 Example *The sequence* $\{(-1)^n\}$ *is divergent.*

Solution: *To show that whether* $|x_n - x| < \varepsilon \ \ \forall \varepsilon, n \geq k$ $fixed\ k\ s.t\ x_k = 1$ *and* $\varepsilon = \frac{1}{2}$, *Now for starting with n=k we have* $|x_n - x| < \varepsilon \Rightarrow |1 - x| < \varepsilon = \frac{1}{2} \ldots .(1)$ *, the first n after k is n+1 and*

$$x_{n+1} = -1 \ so \ |x_n - x| < \varepsilon \Rightarrow |-1 - x| < \varepsilon = \frac{1}{2} \ldots (2)$$

$|1 - x| + |1 + x| = |1 - x| + |-1 - x| < \frac{1}{2} + \frac{1}{2} = 1$, *but* $2 = |1 - x + (1 + x)| \leq |1 - x| + |1 + x| < 1$ *and that is a contradiction.*

3.5 Theorem *A convergent sequence has a unique limit.*

Proof: *Suppose* (X, d) *is a metric space, a sequence* $\{x_n\} \subset X$, $x, y \in X$, *such that* $x_n \longrightarrow x$ *as* $n \longrightarrow \infty$, *and* $x_n \longrightarrow y$ *as* $n \longrightarrow \infty$. *Supposing* $x \neq y$, *let* $d(x, y) = r > 0$. *From the definition of convergence there exist* k_1 *and* $k_2 \in \mathbb{N}$ *such that*

$$n \geq k_1 \implies d(x_n, x) < \frac{r}{4},$$

$$n \geq k_2 \implies d(x_n, y) < \frac{r}{4}.$$

Let $k = max\{k_1, k_2\}$. *Then*

$d(x, y) \leq d(x, x_n) + d(x_n, y) < \frac{r}{4} + \frac{r}{4} = \frac{r}{2}$, *i.e.* $d(x, y) < \frac{r}{2}$, *which is a contradiction.*

3.6 Remark *If a sequence* $\{x_n\}$ *converges to* x*, then* x *is the only limit point of the sequence.*

3.7 Theorem *A set is closed if and only if it includes the limit of every sequence in it.*

Proof: *Suppose that* $\mathcal{A}$ *is a closed set and that* $\{x_n\}$ *is a sequence in* $\mathcal{A}$ *with* $x_n \to x$. *To prove that* $x \in \mathcal{A}$*, by contradiction. If* $x \notin \mathcal{A} \Longrightarrow x \in \mathcal{A}^c$*, since* $\mathcal{A}^c$ *is open* $\Longrightarrow \exists\ \varepsilon > 0$ *s. t.* $B_\varepsilon(x) \subset \mathcal{A}^c \Longrightarrow B_\varepsilon(x)$ *contain all, except finite number of the terms of* $\{x_n\}$ *contradiction to that* $x_n \in \mathcal{A}$.

Conversely, suppose that $\mathcal{A}$ *is includes the limit of every sequence in it. To prove that* $\mathcal{A}$ *is closed, if not* $\Longrightarrow \mathcal{A}^c$ *is not open* $\Longrightarrow \exists\ x \in \mathcal{A}^c$ *s.t.* $B_r(x) \cap \mathcal{A}$ *has at least one point* $\forall\ r > 0 \ \Longrightarrow \forall\ n \in \mathbb{N},\ \exists\ x_n \in \mathcal{A}\ \ s.t\ \ d(x_n, x) < \frac{1}{n} \Longrightarrow$ $x_n \in \mathcal{A}$ *and convergent to* x *and* $x \notin \mathcal{A}$ $($*or* $x \in \mathcal{A}^c)$.

3.8 Definition *Let* $\{x_n\}$ *be a sequence in a metric space* (X, d) *and* $\{n_k\}$ *be a strictly increasing sequence. i.e.* $n_1 < n_2 < n_3 < \cdots$. *Then the sequence* $\{x_{n_k}\}$ *is called a* ***subsequence*** *of* $\{x_n\}$.

3.9 Example *Let* $\{x_n\} = \left\{\frac{1}{n}\right\}, n \in \mathbb{N}$.

1. *The sequence* $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \dots\}$ *is a subsequence of* $\{x_n\}$. *For we take* $n_k = 2k$ *then the sequence is* $\{x_{n_k}\} = \{\frac{1}{2k}\}\ for\ \ k = 1, 2, 3, \dots$
2. *The sequence* $\{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \frac{1}{15}, \dots\}$ *is a subsequence of* $\{x_n\}$.
If we take $n_k = 2k + 1$ *then the sequence is* $\{x_{n_k}\} = \left\{\frac{1}{2k+1}\right\}$ for $k = 1, 2, 3, \dots$.
3. *The sequence* $\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{1}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \dots\}$ *is not a subsequence of* $\{x_n\}$.
4. $n_k = k! \Rightarrow$ *the sequence* $\{x_{nk}\} = \left\{\frac{1}{k!}\right\} = \left\{1, \frac{1}{2}, \frac{1}{6}, \dots\right\}\ for\ k = 1,2,3, \dots$ *is a subsequence of* $\{x_n\}$.

5. $n_k = 2^k \Rightarrow$ *the sequence* $\{x_{nk}\} = \left\{\frac{1}{2^k}\right\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right\}$ *for* $k = 1,2,3, \dots$ *is a subsequence of* $\{x_n\}$.

3.10 Lemma *If* n_k *is a strictly increasing sequence of natural numbers then* $n_k \geq k$ *for all* $k \in \mathbb{N}$.

Proof: *We shall prove this by induction. When* $k = 1$, *since* $n_k \in \mathbb{N}$, *it follows that* $n_k \geq 1 = k$. *Now suppose that* $n_k \geq k$ *for a certain* $k \in \mathbb{N}$. *Then, since the sequence is strictly increasing,* $n_{k+1} > n_k$. *However, since the* n_k *are natural numbers, this means that* $n_{k+1} \geq n_k + 1$, *and so*

$$n_{k+1} \geq n_k + 1 \geq k + 1,$$

by the induction hypothesis.

3.11 Theorem *If the sequence* $\{x_n\}$ *converges to* x *if and only if all its subsequences also converge to* x.

Proof: *Suppose that the sequence* $\{x_n\}$ *converges to* x *and* $\{x_{n_k}\}$ *is a subsequence of* $\{x_n\}$. *We must show that the sequence* $\{x_{n_k}\}$ *is also convergent to x,*

this means that $\lim_{n\to\infty} x_{n_k} = x$.

Since $\{x_n\}$ *converges to* x *then* $\forall \varepsilon > 0$ *there exists* $k \in \mathbb{N}$ *such that* $d(x_n, x) < \varepsilon$ *for all* $n \geq k$ *and since* $\{n_k\}$ *is a strictly increasing sequence then (by Lemma 3.10) we get* $n_k \geq m$ $\forall m \in \mathbb{N}$ *and if* $m \geq k \implies k \leq m \leq n_k$ *then* $n_k \geq k$. *It follows that* $\forall \varepsilon > 0$ $\exists k \in \mathbb{N}$ *such that* $d(x_{n_k}, x) < \varepsilon$ *for all* $n_k \geq k$.

3.12 Example *The sequence* $\{x_n\} = \{(-1)^n \left(1 - \frac{1}{n}\right)\}$ *diverges.*

Solution: *The sequence* $\{x_{2k}\} = \{(-1)^{2k} \left(1 - \frac{1}{2k}\right)\}$ *is a subsequence of* $\{x_n\}$, *then* $x_{2k} = 1 - \frac{1}{2k}$ *converges to* 1, *and* $\{x_{2k+1}\} = \{(-1)^{2k+1} \left(1 - \frac{1}{2k+1}\right)\}$

is a subsequence of $\{x_n\}$*, then* $x_{2k+1} = \frac{1}{2k+1} - 1$ *converges to* -1*. Thus, (by Theorem 3.11) we get* $\{x_n\}$ *does not converge.*

3.13 Definition *A sequence* $\{x_n\}$ *of* ***real numbers*** *is said to be* ***bounded*** *if there exists a real number* $M > 0$ *such that* $|x_n| \le M$ *for all* $n \in \mathbb{N}$.

3.14 Example *A sequence* $\{x_n\} = \{(-1)^n\} \in \mathbb{R}$ *is bounded since there exists* $M = 1 \in \mathbb{R}$ *such that* $|(-1)^n| \le 1 \quad \forall n \in \mathbb{N}$.

3.15 Remark *Every convergent sequence is bounded but the converse is not true.*

3.16 Theorem (Bolzano-Weierstrass Theorem) *Every bounded sequence in* $\mathbb{R}$ *has a convergent subsequence. [H.W]*

Cauchy Sequences

3.17 Definition *Let* (X, d) *be a metric space. A sequence* $\{x_n\} \subset X$ *is said to be a* ***Cauchy sequence*** *if, for every* $\varepsilon > 0$*, there exists a natural number* k *such that* $d(x_n, x_m) < \varepsilon$ *whenever* $m, n \ge k$*.*

We sometimes write this as $d(x_n, x_m) \longrightarrow 0$ *as* $m, n \longrightarrow \infty$*.*

3.18 Example *In* $\big((0,1), d\big)$*, the sequence* $\{\frac{1}{n}\}$ *is a Cauchy sequence.*

Solution: *Let* $\varepsilon > 0$ *be given. Take* $k > \frac{2}{\varepsilon}$*. Then for* $n \ge k$ *we have that* $\frac{1}{n} < \frac{\varepsilon}{2}$*.*

Therefore, for all $n, m \ge k$ *we have*

$$d(x_n, x_m) = \left|\frac{1}{n} - \frac{1}{m}\right| \le \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

3.19 Theorem *A convergent sequence in a metric space is a Cauchy sequence.*

Proof: *Let* $\{x_n\}$ *be a sequence in a set* X *with metric* d*, and let* x *be an element of* X *such that* $\lim_{n\to\infty} x_n = x$*. Given any* $\varepsilon > 0$*, there exists some natural number* k *such that* $d(x_n, x) < \frac{\varepsilon}{2}$ *whenever* $n \geq k$*. Consider any natural numbers* n *and* m *such that* $n \geq k$ *and* $m \geq k$*. Then* $d(x_n, x) < \frac{\varepsilon}{2}$ *and* $d(x_m, x) < \frac{\varepsilon}{2}$*.*

Therefore $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

But the converse of this theorem is not hold. See example 3.20

3.20 Example *Let* $(\mathbb{R} - \{0\}, d)$ *be a usual metric space, then the sequence* $\left\{\frac{1}{n}\right\}$ *is a Cauchy sequence in* $\mathbb{R} - \{0\}$*, but not convergent in* $\mathbb{R} - \{0\}$*, since* $\frac{1}{n}$ *is convergent to* 0 *and* $0 \notin \mathbb{R} - \{0\}$*.*

This example tells every Cauchy sequence is not convergent sequence in (X, d) *in general.*

3.21 Remark *In* $\mathbb{R}$ *a sequence converges if and only if it is a Cauchy sequence. But in* (X, d) *every convergent sequence is a Cauchy sequence.*

3.22 Example *Let* $(\mathbb{R}, d)$ *be a metric space and* $\left\{\frac{n+1}{n}\right\} \subset \mathbb{R}$*. prove that the sequence* $\left\{\frac{n+1}{n}\right\}$ *is a Cauchy sequence.*

Solution: *Let* $\varepsilon > 0$ *be given, to find* $k \in \mathbb{N}$ *such that* $\forall\, n, m \geq k$ *then* $d(x_n, x_m) < \varepsilon$*. (By Archimedean property)* $\varepsilon > 0$ *and* $2 \in \mathbb{R}$ *then there exists* $k \in \mathbb{N}$ *such that* $k\varepsilon > 2 \Longrightarrow k > \frac{2}{\varepsilon}$ *Now, if* $n, m \geq k$ *then* $\frac{1}{n} < \frac{\varepsilon}{2}$ *and* $\frac{1}{m} < \frac{\varepsilon}{2}$*.*

Therefore $d(x_n, x_m) = \left|\frac{n+1}{n} - \frac{m+1}{m}\right| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

Complete metric space

3.23 Definition *A metric space (X, d) is said to be* ***complete*** *if every Cauchy sequence in X is convergent to a point in X.*

3.24 Example *Any set X with the discrete forms a complete metric space.*

Solution: *Let $\{x_n\}$ be a Cauchy sequence in the discrete metric space X.*

Then $d(x_n, x_m) = 0$, if $x_n = x_m$ and $d(x_n, x_m) = 1$, if $x_n \neq x_m$. On taking $0 < \varepsilon \leq 1$ in the definition of Cauchy sequence, we obtain $d(x_n, x_m) < \varepsilon \quad \forall\, n, m \geq k \qquad \dots 1$

Using the definition of the discrete metric space. (1) yields $d(x_n, x_k) = 0$ for each n then $x_n \to x_k$ as $n \to \infty$. Hence discrete metric space X is complete.

3.25 Example *Let $X =]0,1[$ and let $d(x, y) = |x - y| \quad \forall\, x, y \in X$, then show that (X, d) is an incomplete metric space.*

Solution: *Let $\{x_n\}$ be a Cauchy sequence in X defined by $x_n = \frac{1}{n} \quad \forall n \in \mathbb{N}$.*

Then as shown in example 3.18 and 3.20, $\{x_n\}$ is a Cauchy sequence in X and $\{x_n\}$ cannot convergent to any point of X. Hence (X, d) is an incomplete metric space.

3.26 Theorem *If (X, d) is complete and $D \subset X$ is closed then (D, d) is a complete metric space.*

Proof: *Since (X, d) is complete metric space then any Cauchy sequence in X is convergent to a point in X, and since D is closed then by (**Theorem 2.37)** we get D contains all its limit points $\Longrightarrow \forall$ Cauchy sequence in D its convergent to a point in D. Hence (D, d) is complete metric space.*

3.27 Theorem *Every Euclidean space is complete. [H.W]*

“Compactness”

History and motivation

The history of what today is called the Heine–Borel theorem starts in the 19th century, with the search for solid foundations of real analysis. Central to the theory was the concept of uniform continuity and the theorem stating that every continuous function on a closed interval is uniformly continuous. Dirichlet was the first to prove this and implicitly he used the existence of a finite sub-cover of a given open cover of a closed interval in his proof. He used this proof in his 1862 lectures, which were published only in 1904. Later Eduard Heine, Karl Weierstrass and Salvatore Pincherle used similar techniques. Émile Borel in 1895 was the first to state and prove a form of what is now called the Heine–Borel theorem. His formulation was restricted to countable covers. Pierre Cousin (1895), Lebesgue (1898) and Schoenflies (1900) generalized it to arbitrary covers. The term compact was introduced by Fréchet in 1906.

3.28 Definition (Cover of a set) *Let* (X, d) *be a metric space and* $\mathcal{A} \subseteq X$*. Let* $G = \{A_\alpha\}_{\alpha \in \Lambda}$ *be a collection of open sets in* X*, then* $G = \{A_\alpha\}_{\alpha \in \Lambda}$ *is said to be an "open cover" or an "open covering" of* $\mathcal{A}$ *if* $\mathcal{A} \subseteq \bigcup_{\alpha \in \Lambda} A_\propto$.

3.29 Definition *A metric space* (X, d) *is said to be compact if every open covering* G *of* X *has a finite sub covering, that is, there is a finite sub collection* $\{A_1, A_2, \dots, A_n\} \subseteq G$ *such that* $X = \bigcup_{i=1}^{n} A_i$.

A set $\mathcal{A} \subseteq X$ of a metric space (X, d) is compact if every open cover of $\mathcal{A}$ has a finite sub-cover of $\mathcal{A}$.

i.e. if $\mathcal{A} \subseteq \bigcup_{i\in\wedge} A_i \longrightarrow \mathcal{A} \subseteq \bigcup_{i=1}^{n} A_i$.

3.30 Example $(0,1) \subseteq \mathbb{R}$ *is not compact in a usual metric space $(\mathbb{R}, d)$.*

Solution: *Let $G = \left\{A_\propto = \left(\frac{1}{\propto}, 2\right) : \propto \in \mathbb{N}\right\} = \left\{(1,2), \left(\frac{1}{2}, 2\right), \left(\frac{1}{3}, 2\right), \dots\right\}$ be an open cover of $(0,1)$. Hence $(0,1) \subseteq \bigcup_{\propto\in\mathbb{N}} \left(\frac{1}{\propto}, 2\right)$. We need to show that this open cover G of $(0,1)$ has no a finite sub-cover of $(0,1)$. By contradiction assume that $(0,1) \subseteq \bigcup_{\propto=1}^{k} \left(\frac{1}{\propto}, 2\right) = (1,2) \cup \left(\frac{1}{2}, 2\right) \cup \dots \cup \left(\frac{1}{k}, 2\right) = \left(\frac{1}{k}, 2\right)$, but from 0 up to $\frac{1}{k}$ there are several (in fact infinite number s.t $\frac{1}{k+3}, \dots < \frac{1}{k+1} < \frac{1}{k}$, there fore $(0,1) \nsubseteq \left(\frac{1}{k}, 1\right) \Longrightarrow (0,1) \nsubseteq \bigcup_{\propto=1}^{k} \left(\frac{1}{\propto}, 2\right)$.*

Which mean that $(0,1)$ is not compact.

3.31 Example *In any metric space (X, d). Every finite sub-set of X is compact.*

Solution: *Let $\mathcal{A} = \{a_i : 1 \leq i \leq n\}$ be a finite sub-set of X, and $G = \{A_i\}_{i\in\Lambda}$ be an open cover of $\mathcal{A}$ this means that $\mathcal{A} \subseteq \bigcup_{i\in\wedge} A_i$. We must to show $\mathcal{A} \subseteq \bigcup_{i=1}^{n} A_i$.*

Then for each $a_i \in \mathcal{A}$, there exist an open set $A_i \in G$ such that $a_i \in A_i \;\; \forall\, i = 1, 2, \dots, n$. Then the family $\{A_i : 1 \leq i \leq n\}$ is clearly a finite open cover of $\mathcal{A}$ consisting of member of G. Then $\mathcal{A} \subseteq A_1 \cup A_2 \cup \dots \cup A_n \Longrightarrow \mathcal{A} \subseteq \bigcup_{i=1}^{n} A_i$.

Hence $\mathcal{A}$ is a compact set.

3.32 Example *$\mathbb{R}$ is not compact with a usual metric.*

Solution: *Let $\{A_\alpha\}_{\alpha\in\Lambda} = \{(-\alpha, \alpha)\}_{\alpha\in\mathbb{N}}$ be an open cover of $\mathbb{R}$,* *this means that* $\mathbb{R} \subseteq \bigcup_{\alpha\in\mathbb{N}}(-\alpha,\alpha)$. *We need to show that this open cover* $\{A_\alpha\}_{\alpha\in\Lambda}$ *of* $\mathbb{R}$ *has no a finite sub-cover of* $\mathbb{R}$. *By contradiction suppose this open cover* $\{A_\alpha\}_{\alpha\in\Lambda}$ *has a finite sub-cover* $\{A_\alpha\}_{\alpha\in\Lambda}$ *when* $\Lambda = \{1, 2, \dots, \mathrm{k}\}$ *then* $\mathbb{R} \subseteq \bigcup_{\propto=1}^{k} A_\alpha$. *But* $\bigcup_{\propto=1}^{k}(-\alpha,\alpha) = (-1,1) \cup (-2,2) \cup \dots \cup (-k,k) = (-k,k)$*, since* $k \in \mathbb{R}$ *and* $k \notin (-k,k)$ *then* $\mathbb{R} \nsubseteq \bigcup_{\propto=1}^{k}(-\alpha,\alpha)$. *Then the finite sub-cover is not cover* $\mathbb{R}$*. Hence* $\mathbb{R}$ *is not compact.*

3.33 Definition *Let* (X,d) *be a metric space. We say that* $\mathcal{A} \subseteq X$ *is bounded if there is an open ball* $\mathcal{A} \subseteq B_r(x)$*. (i.e if* $\exists k > 0$ $s.t$ $d(x,y) \le k$ $\forall\, x,y \in \mathcal{A}$).

3.34 Theorem *Every compact set is bounded.*

Proof: *Let* $\mathcal{A}$ *be compact, for each* $x \in \mathcal{A}$*, let* B_x *be a ball of radius 1 with center x. Then the collection* $\{B_x : x \in \mathcal{A}\}$ *of open sets that cover* $\mathcal{A}$*. Hence there must be a finite sub-cover say as,* $B_{x_1}, B_{x_2}, \dots, B_{x_n}$ *that covers* $\mathcal{A} \Longrightarrow \mathcal{A} \subseteq \bigcup_{i=1}^{n} B_{x_i} \Longrightarrow \mathcal{A}$ *is bounded.*

3.35 Proposition: *Let* (X,d) *be a metric space and* $\mathcal{A} \subseteq X$*. Then a point* x *in a metric space is a limit point of* $\mathcal{A}$ *if and only if there is a sequence of distinct points* $\{x_n\}_{n\in\mathbb{N}}$ *in* $\mathcal{A}$ *such that* $x_n \to x$ *as* $n \longrightarrow \infty$*. [H.W]*

3.36 Remark *Any compact metric space is complete.*

3.37 Theorem *Let* $\mathcal{A}$ *be a subset of the metric space* (X,d)*. If* $\mathcal{A}$ *is compact, then* $\mathcal{A}$ *is a closed subset of* (X,d).

Proof: *Let $x \in X$ be an arbitrary limit point of $\mathcal{A}$. Then by* ***proposition 3.35,*** *there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}$ converging to x. In order to prove that $\mathcal{A}$ is closed we must show that $x \in \mathcal{A}$. Since $\{x_n\}_{n\in\mathbb{N}}$ is convergent sequence in $\mathcal{A}$, it is a Cauchy sequence in $\mathcal{A}$. Since $\mathcal{A}$ is compact, then $\mathcal{A}$ is complete. Hence $\{x_n\}_{n\in\mathbb{N}}$ converges to a point in $\mathcal{A}$ and this point is clearly x. Thus $x \in \mathcal{A}$. Since x is any limit point of $\mathcal{A}$, so all limit points of $\mathcal{A}$ belong to $\mathcal{A}$ and hence $\mathcal{A}$ is closed.*

3.38 Theorem *Every closed subset of compact set is compact.*

Proof: *Let Y be a compact subset of a metric space (X, d) and $\mathcal{A}$ be a closed subset of Y, relative to X. To show that $\mathcal{A}$ is compact. Let $G = \{A_\alpha : \alpha \in \wedge\}$ be any open cover of $\mathcal{A}$, then then $G^* = \{A_\alpha : \alpha \in \wedge\} \cup \{X - \mathcal{A}\}$ is an open cover of Y. Since Y is compact, hence it has a finite sub- subcover. Say $A_{\alpha_1}, A_2, \dots A_{\alpha_n}$, $X - \mathcal{A}$, so that $A_{\alpha_1} \cup A_2 \cup \dots A_{\alpha_n} \cup (X - \mathcal{A}) = X$ and so $\mathcal{A} \subseteq \cup \{A_{\alpha_i} : i = 1, 2, \dots n\}$ $\Longrightarrow \{A_{\alpha_i} : i = 1, 2, \dots n\}$ is a finite sub-cover of $\mathcal{A}$. Hence $\mathcal{A}$ is compact.*

3.39 Theorem (Sequentially compact) *Let $(\mathbb{R}, d)$ be a metric space and $\mathcal{A} \subseteq \mathbb{R}$ is said to be compact if and only if every sequence in $\mathcal{A}$ has a convergent subsequence in $\mathcal{A}$.* ***[H.W]***

3.40 Example *Show that weather the open interval $(-1,1) \subseteq \mathbb{R}$ is compact or not?*

Solution: *If $\{x_n\} = \{1 - \frac{1}{n}\}$ is a sequence of $(-1,1)$ and $\{n_k\} = \{2k\}$ is increasing sequence then $\{x_{n_k}\} = \{1 - \frac{1}{2k}\}$ is a subsequence of $\{x_n\}$.*

Since $x_{n_k} = 1 - \frac{1}{2k} \longrightarrow 1 \notin (-1, 1)$ *as* $k \longrightarrow \infty$

then the subsequence $\{x_{n_k}\}$ *of* $\{x_n\}$ *has no convergent in* $(-1, 1)$. *Thus by above theorem we get* $(-1, 1)$ *is not compact.*

3.41 Theorem (Cantor nested interval theorem)

For each $n \in \mathbb{N}$, *let* $I_n = [a_n, b_n]$ *be a non-empty closed bounded interval of real numbers such that* $I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots$ *and* $\lim_{n\to\infty} b_n - a_n = \lim_{n\to\infty} l(I_n) = 0$, *where* $l(I_n)$ *denotes the length of the interval* I_n. *Then* $\bigcap_{n=1}^{\infty} I_n$ *contains precisely one point.*

3.42 Theorem (Heine-Borel theorem)

Every closed and bounded interval on $\mathbb{R}$ *is compact.*

Proof: *Let* $I_1 = [a, b] = [a_1, b_1]$ *be a closed and bounded interval on* $\mathbb{R}$. *Suppose that* I_1 *is not compact. Then there exists an open cover* $G = \{A_\alpha : \alpha \in \curlywedge\}$ *of* I_1 *which has no finite sub cover. Bisect* I_1 *into two equal closed intervals* $I_{11} = [a_1, \frac{a_1+b_1}{2}]$ *and* $I_{12} = [\frac{a_1+b_1}{2}, b_1]$. *Then by our hypothesis, at least one of these two intervals must have no finite sub-cover of the open cover* G. *Let that particular interval be denoted by* I_2 *as follows:* $I_2 = [a_2, b_2]$, *where* $I_2 = I_{11}$ *or* I_{12} . *As before, Bisect* I_2 *into two equal closed intervals* $I_{21} = [a_2, \frac{a_2+b_2}{2}]$ *and* $I_{22} = [\frac{a_2+b_2}{2}, b_2]$ *at least one of these two intervals which has no finite sub cover and obtain* $I_3 = [a_3, b_3]$, *where* $I_3 = I_{21}$ *or* I_{22}. *On repeating the above process an infinite number of times, we arrive at a sequence of closed intervals* $I_1, I_2, I_3, \ldots, I_n, \ldots$ *and satisfying the following conditions:*

i. $I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots$ *i.e* $I_n \supset I_{n+1} \quad \forall\, n \in \mathbb{N}$

ii. $I_n = [a_n, b_n]$ *is a closed interval* $\quad \forall\, n \in \mathbb{N}$

iii. $\lim_{n\to\infty} l(I_n) = 0$, *where* $l(I_n)$ *denotes the length of the interval* I_n

iv. I_n *is not covered by any finite sub-family of G.*

We obtain the sequence of closed intervals $\{I_n\}_{n\in\mathbb{N}}$ *satisfies all the conditions of Cantor nested interval theorem, and hence there exists a real number* $x \in \cap \{I_n : n \in \mathbb{N}\}$. *Then* $x \in I_n \subset I_1 \subset \cup\{A_\alpha : \alpha \in \land\}$ *so that* $x \in A_{\alpha_0}$ *for some* $\alpha_0 \in \land$. *Since* A_{α_0} *is open, there exists* $\varepsilon > 0$ *such that* $(x-\varepsilon, x+\varepsilon) \subset A_{\alpha_0}$. *Now we can take n so large that* $I_n \subset (x-\varepsilon, x+\varepsilon) \subset A_{\alpha_0}$ *, whence* I_n *is covered by a single member* A_{α_0} *of* G. *But this contradicts the fact that* G *has no finite subcover of* I_1. *The theorem is thus proved.*

Exercises:

Q1: *Show that the following sequances are convergent with a usual metric*

1. $\{\frac{1}{2^n}\}_{n\in\mathbb{N}}$
2. $\{1 + (\frac{(-1)^n}{n})\}_{n\in\mathbb{N}}$
3. $\{\frac{1}{n+1}\}_{n\in\mathbb{N}}$
4. $\{\sqrt{n+1} - \sqrt{n}\}_{n\in\mathbb{N}}$

Q2: *Show that* $(\mathbb{R}, d)$ *is complete metric space.*

Q3: *Show that* $(\mathbb{Q}, d)$ *is not complete metric space.*

Q4: *Show that* $\mathcal{S} = \left\{\frac{1}{n}, n \in \mathbb{N}\right\} \cup \{0\}$ *is compact in a usual metric space*$(\mathbb{R}, d)$*.*

Q5: *Show that* $\mathbb{Z}$ *and* $\mathbb{R}$ *are not compact with a usual metric.*

Q6: *Show that weather the closed interval* $[0, 9] \subseteq \mathbb{R}$ *is compact or not?*

Q7: *Show that* $[1, \infty) \subseteq \mathbb{R}$ *is not compact with a usual metric.*