Chapter 2 - Integration

Definition: $F(x)$ is an antiderivative of $f(x)$ if $\frac{d}{dx}[F(x)] = f(x)$ and we write $\int f(x)dx = F(x)+c$ and we say that the integral of $f(x)$ with respect to x is equal $F(x)+c$ with respect to x.

Example1: $\int x^3 dx = \frac{1}{4}x^4 + c$ because $\frac{d}{dx}\left[\frac{1}{4}x^4\right] = x^3$

In fact, $\frac{1}{4}x^4, \frac{1}{4}x^4+2, \frac{1}{4}x^4-4, \frac{1}{4}x^4+\frac{1}{2}$ are all antiderivatives of x^3, because they all differentiate to x^3.

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1. $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r \quad , \quad r \neq -1 : \quad \int x^r\, dx = \left[\frac{x^{r+1}}{r+1}\right] + c$
2. $\frac{d}{dx}\left[e^x\right] = e^x : \quad \int e^x\, dx = e^x + c$
3. $\frac{d}{dx}\left[e^{ax}\right] = ae^x : \quad \int e^{ax}\, dx = \frac{1}{a}e^x + c$
4. $\frac{d}{dx}\left[b^x\right] = (\ln b)b^x, \ \mathrm{b} > 0,\ \mathrm{b} \neq 1 : \quad \int b^x = \frac{b^x}{(\ln b)} + c$
5. $\frac{d}{dx}\left[b^{ax}\right] = a(\ln b)b^x, \ \mathrm{b} > 0,\ \mathrm{b} \neq 1 : \quad \int b^{ax} = \frac{b^x}{a(\ln b)} + c$
6. $\frac{d}{dx}\left[\ln|x|\right] = \frac{1}{x} : \quad \int \frac{1}{x}\, dx = \ln|x| + c$
7. $\frac{d}{dx}\left[\ln|f(x)|\right] = \frac{f'(x)}{f(x)},\ f(x) \neq 0 : \quad \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$
8. $\frac{d}{dx}[\sin x] = \cos x : \quad \int \cos x\, dx = \sin x + c$
9. $\frac{d}{dx}[\cos x] = -\sin x : \quad \int \sin x\, dx = -\cos x + c$
10. $\frac{d}{dx}[\tan x] = \sec^2 x : \quad \int \sec^2 x\, dx = \tan x + c$
11. $\frac{d}{dx}[\cot x] = -\csc^2 x : \quad \int \csc^2 x\, dx = -\cot x + c$
12. $\frac{d}{dx}[\sec x] = \sec x \tan x : \quad \int \sec x \tan x\, dx = \sec x + c$
13. $\frac{d}{dx}[\csc x] = -\csc x \cot x : \quad \int \csc x \cot x\, dx = -\csc x + c$

14. $\frac{d}{dx}\left[\sin^{-1}x\right]=\frac{1}{\sqrt{1-x^2}}$ **:** $\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+c$

15. $\frac{d}{dx}\left[\tan^{-1}x\right]=\frac{1}{1+x^2}$ **:** $\int\frac{dx}{1+x^2}=\tan^{-1}x+c$

16. $\frac{d}{dx}\left[\sec^{-1}x\right]=\frac{1}{|x|\sqrt{x^2-1}}$ **:** $\int\frac{dx}{|x|\sqrt{x^2-1}}=\sec^{-1}x+c$

Teorem: Suppose that $f(x)$ and $g(x)$ has antiderivatives. Then for any constants a and b,

$$\int\left[a\,f(x)\pm b\,g(x)\right]dx=a\int f(x)\,dx\pm b\int g(x)\,dx$$

Example 2: Evaluate $\int\frac{1}{2x^3}\,dx$.

Solution $\int\frac{1}{2x^3}dx=\frac{1}{2}\int x^{-3}dx=\frac{1}{2}\left[\frac{x^{-2}}{-2}\right]+c=-\frac{1}{4x^2}+\text{c}$

Example 3: Evaluate $\int\left(x^3-2x+7\right)dx$.

Solution

$$\int\left(x^3-2x+7\right)dx=\int x^3dx-2\int xdx+7\int dx=\left[\frac{x^4}{4}\right]-2\left[\frac{x^2}{2}\right]+7x+c=\frac{1}{4}x^4-x^2+7x+c$$

Example 4: Evaluate $\int\left(x^{2/3}-4x^{-1/5}+4\right)dx$.

Solution

$$\int\left(x^{2/3}-4x^{-1/5}+4\right)dx=\int x^{2/3}dx-4\int x^{-1/5}dx+4\int dx=\left[\frac{x^{5/3}}{5/3}\right]-4\left[\frac{x^{4/5}}{4/5}\right]+4x+c$$

$$=\frac{3}{5}x^{5/3}-5x^{4/5}+4x+c$$

Example 5: Evaluate $\int\left(\frac{7}{y^{3/4}}-\sqrt[3]{y}+4\sqrt{y}\right)dy$.

Solution

$$\int\left(\frac{7}{y^{3/4}}-\sqrt[3]{y}+4\sqrt{y}\right)dy=7\int y^{-3/4}dy-\int y^{1/3}dy+4\int y^{1/2}=7\left[\frac{y^{1/4}}{1/4}\right]-\left[\frac{y^{4/3}}{4/3}\right]+4\left[\frac{y^{3/2}}{3/2}\right]+c$$

$$=28y^{1/4}-\frac{3}{4}y^{4/3}+\frac{8}{3}y^{3/2}+c$$

Example 6: Evaluate $\int x^{1/3}\left(2-x^2\right)dx$.

Solution

$$\int x^{1/3}\left(2-x^{2}\right)dx=\int\left(2x^{1/3}-x^{7/3}\right)dx=2\int x^{1/3}dx-\int x^{7/3}dx=2\left[\frac{x^{4/3}}{4/3}\right]-\left[\frac{x^{10/3}}{10/3}\right]+c$$

$$=\frac{3}{2}x^{4/3}-\frac{3}{10}x^{10/3}+c$$

Example 7: Evaluate $\int\frac{x^{5}+2x^{2}-1}{x^{4}}dx$.

Solution

$$\int\frac{x^{5}+2x^{2}-1}{x^{4}}dx=\int\left(x+\frac{2}{x^{2}}-\frac{1}{x^{4}}\right)dx=\int xdx+2\int x^{-2}dx-\int x^{-4}dx$$

$$=\left[\frac{x^{2}}{2}\right]+2\left[\frac{x^{-1}}{-1}\right]-\left[\frac{x^{-3}}{-3}\right]+c=\frac{1}{2}x^{2}-\frac{2}{x}+\frac{1}{3x^{3}}+c$$

Example 8: Evaluate $\int\left(3\cos x-\frac{1}{x}\right)dx$.

Solution $\int\left(3\cos x-\frac{1}{x}\right)dx=3\int\cos xdx-\int\frac{1}{x}dx=3\sin x-\ln|x|+c$

Example 9: Evaluate $\int\frac{4x}{x^{2}+4}dx$.

Solution $\int\frac{4x}{x^{2}+4}dx=2\int\frac{2x}{x^{2}+4}dx=2\ln\left|x^{2}+4\right|+c=2\ln\left(x^{2}+4\right)+c$

Example 10: Integrate $\int\sqrt{e^{x}}dx$.

Solution $\int\sqrt{e^{x}}dx=\int\left(e^{x}\right)^{1/2}dx=\int e^{x/2}dx=\frac{1}{1/2}e^{x/2}+c=2\sqrt{e^{x}}+c$

Example 11: Evaluate $\int\left(\frac{e^{x}}{e^{x}+3}\right)dx$.

Solution: $\int\left(\frac{e^{x}}{e^{x}+3}\right)dx=\ln\left|e^{x}+3\right|+c=\ln\left(e^{x}+3\right)+c$

Example 12: Evaluate $\int\cot xdx$.

Solution $\int\cot xdx=\int\frac{\cos x}{\sin x}dx=\ln|\sin x|+\mathrm{c}$

Example 13: Evaluate $\int e^{2\ln x}dx$.

Solution $\int e^{2\ln x}dx=\int e^{\ln x^{2}}dx=\int x^{2}dx=\frac{1}{3}x^{3}+c$

Example 14: Evaluate $\int\frac{x+1}{x^{2}+2x+5}dx$.

Solution

$$\int \frac{x+1}{x^2+2x+5}dx = \frac{1}{2}\int \frac{2(x+1)}{x^2+2x+5}dx = \frac{1}{2}\ln\left|x^2+2x+5\right|+c$$

Example 15: Evaluate $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$.

Solution $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} = \ln\left|e^x + e^{-x}\right| = \ln\left(e^x + e^{-x}\right) + c$

Example 16: Evaluate $\int \left[\ln e^x + \ln e^{-x}\right]dx$.

Solution

$$\int \left[\ln e^x + \ln e^{-x}\right]dx = \int \ln\left(e^x . e^{-x}\right)dx = \int \ln\left(e^0\right)dx = \int \ln(1)dx = \int 0\, dx = c$$

Example 17: Evaluate $\int [4\sin x + 2\cos x]dx$.

Solution

$$\int [4\sin x + 2\cos x]\, dx = 4\int \sin x dx + 2\int \cos x dx = -4\cos x + 2\sin x + c$$

Example 18: Evaluate $\int \sec x\,(\tan x + \cos x)\, dx$.

Solution

$$\int \sec x\,(\tan x + \cos x)\, dx = \int \sec x\, \tan x dx + \int \sec x\, \cos x dx = \int \sec x\, \tan x dx + \int \frac{1}{\cos x}\cos x dx$$

$$= \int \sec x\, \tan x dx + \int dx = \sec x + x + c$$

Example 19: Evaluate $\int \frac{\sin 2x}{\cos x}\, dx$.

Solution: $\int \frac{\sin 2x}{\cos x}\, dx = \int \frac{2\sin x\, \cos x}{\cos x}\, dx = 2\int \sin x dx = -2\cos x + c$

Example 20: Evaluate $\int \frac{\sin x}{\cos^2 x}dx$.

Solution $\int \frac{\sin x}{\cos^2 x}dx = \int \left(\frac{\sin x}{\cos x}\right)\left(\frac{1}{\cos x}\right)dx = \int \tan x\, \sec x dx = \sec x + c$

Example 21: Evaluate $\int \left[1+\sin^2\theta\csc\theta\right] d\theta$.

Solution

$$\int \left[1+\sin^2\theta\csc\theta\right] d\theta = \int d\theta + \int \sin^2\theta\left(\frac{1}{\sin\theta}\right)d\theta = \theta + \int \sin\theta d\theta = \theta - \cos\theta + c$$

Example 22: Evaluate $\int \frac{1}{1+\sin x}\, dx$.

Solution

$$\int \frac{1}{1+\sin x}\,dx = \int \frac{1-\sin x}{(1+\sin x)(1-\sin x)}\,dx = \int \frac{1-\sin x}{1-\sin^2 x}\,dx = \int \frac{1-\sin x}{\cos^2 x}\,dx = \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right) dx$$

$$= \int \left(\sec^2 x - \sec x \tan x\right) dx = \tan x - \sec x + c$$

Example 23: Evaluate $\int \tan^2 x \, dx$.

Solution

$$\int \tan^2 x \, dx = \int \left(\sec^2 x - 1\right) dx = \tan x - x + c$$

Example 24: Evaluate $\int \sin^2\left(\frac{x}{2}\right) dx$.

Solution

$$\int \sin^2\left(\frac{x}{2}\right) dx = \int \frac{1}{2}(1-\cos x)\, dx = \frac{1}{2}[x - \sin x] + c$$

Example 25: Evaluate $\int \frac{4}{\sqrt{1-x^2}} dx$.

Solution

$$\int \frac{4}{\sqrt{1-x^2}} dx = 4\int \frac{1}{\sqrt{1-x^2}} dx = 4\sin^{-1}(x) + c$$

Example 26: Evaluate $\int \frac{3}{4x^2+4} dx$.

Solution

$$\int \frac{3}{4x^2+4} dx = \frac{3}{4}\int \frac{1}{x^2+1} dx = \frac{3}{4}\tan^{-1} x + c$$

Integration by Parts

Integration by Parts is based on the Product Rule

$$[f(x)\bullet g(x)]' = f(x)\bullet g'(x) + g(x)\bullet f'(x)$$

So, the Integration by Parts formula can be written as:

$$\int u\ dv = uv - \int v\ du$$

Example 1) Solve the integral $F(x) = \int x\cos xdx$

Solution

$F(x) = \int x\cos xdx$

$u = x \qquad dv = \cos xdx$

$du = dx \qquad v = \sin xdx$

$F(x) = x\sin x - \int \sin xdx$

$F(x) = x\sin x + \cos x + C$

Sample 2) Solve the integral $F(x) = \int x^2 e^x dx$

Solution

$F(x) = \int x^2 e^x dx$

$u = x^2 \qquad dv = e^x dx$

$du = 2xdx \qquad v = e^x dx$

$F(x) = x^2 e^x - \int 2xe^x dx$

$$u = 2x \qquad dv = e^x dx$$
$$du = 2dx \qquad v = e^x dx$$

$$F(x) = x^2 e^x - 2xe^x + \int 2e^x dx$$

$$F(x) = x^2 e^x - 2xe^x + 2e^x + C$$

Example #3

$$\int \ln x \, \partial x$$

This seems a bit more confusing since it only looks like there is one "part" to the integration. However, the one part we do see would certainly be easier to deal with if differentiated. Here is the trick to use:

let $u = \ln x$, $\frac{\partial u}{\partial x} = \frac{1}{x}$; $\frac{\partial v}{\partial x} = 1$, $v = x$

$$\int \ln x \, \partial x = x \ln x - \int x \cdot \frac{1}{x} \partial x$$
$$= x \ln x - \int \partial x$$
$$= x \ln x - x + C$$

Integration by Partial Fractions

Problem Statement: Evaluate $\int \frac{p(x)}{q(x)}dx$, where p and q are polynomials with no common factors. So $p(x)=a_n x^n + a_{n-1}x^{n-1}+\cdots+a_0, a_n \neq 0$, and $q(x)=b_m x^m + b_{m-1}x^{m-1}+\cdots+b_0, b_m \neq 0$.

Step 0: If $n \geq m$, divide the fraction out (using long division), obtaining a quotient Q and a remainder R for which $\frac{p(x)}{q(x)}=Q(x)+\frac{R(x)}{q(x)}$, and the ***degree of R is smaller than the degree of q***.

Notice that Q is a polynomial and therefore easy to integrate. The problem is how to integrate $\frac{R(x)}{q(x)}$. Partial Fractions is a technique that applies to this situation.

Linear Factor Rule: For each factor of the form $(ax+b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_m}{(ax+b)^m}$$

where $A_1, A_2, \ldots, A_m$ are constants to be determined.

Quadratic Factor Rule: For each factor of the form $(ax^2+bx+c)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_mx+B_m}{(ax^2+bx+c)^m}$$

where $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$ are constants to be determined.

Example 1: Evaluate $\int \frac{4x^3+2x^2+3x-4}{2x^2+x-1}dx$

Solution: (*The key to recognition is that the integrand is a ratio of polynomials. Next compare the degrees of the numerator and denominator to ascertain if Step 0 is necessary.*)

Step 0: $\dfrac{4x^3+2x^2+3x-4}{2x^2+x-1}=2x+\dfrac{5x-4}{2x^2+x-1}$ (*Algebra*)

(*In terms of the above notes,* $p(x)=4x^3+2x^2+3x-4$ *and* $q(x)=2x^2+x-1$. *So* $Q(x)=2x$ *and* $R(x)=5x-4$.)

$$\int\frac{4x^3+2x^2+3x-4}{2x^2+x-1}dx=\int\left[2x+\frac{5x-4}{2x^2+x-1}\right]dx$$

$$=\int 2x\,dx+\int\frac{5x-4}{2x^2+x-1}dx=x^2+\int\frac{5x-4}{2x^2+x-1}dx$$

(*This last integral is the integral that the technique of Partial Fractions can be applied. Notice that the degree of the numerator* $R(x)=5x-4$ *is one and the degree of the denominator* $q(x)=2x^2+x-1$ *is two. That the degree of the denominator is larger than the degree of the numerator is the critical issue.*)

Step 1: $2x^2+x-1=(2x-1)(x+1)$ (*Factoring*)

Step 2: $\dfrac{5x-4}{2x^2+x-1}=\dfrac{5x-4}{(2x-1)(x+1)}=\dfrac{A}{2x-1}+\dfrac{B}{x+1}$ (*Most general PF decomposition*)

Step 3: $5x-4=A(x+1)+B(2x-1)$ (*Clearing the fractions*)

$5x-4=(A+2B)x+(A-B)$ (*Algebra*)

$A+2B=5$

$A-B=-4$ (*Set corresponding coefficients equal.*)

(*Now solve this system of equations using Algebra. You may need to review how to do this. This system is solved below to refresh your memory, but in the following examples, the system will not be explicitly solved since you should know how to do this.*)

$$\begin{cases}A+2B=5---Eq.1\\A-B=-4---Eq.2\end{cases}$$

$3B=9$ (*Subtract Eq.2 from Eq.1to eliminate the variable A*)

$B=3$ (*Algebra*)

$A=B-1=3-4=-1$ (*Substitute into one of the two equations.*)

(*Write the Partial Fractions decomposition by applying the results of Step 3 to Step 2.*)

$$\frac{5x-4}{(2x-1)(x+1)}=\frac{-1}{2x-1}+\frac{3}{x+1}$$

Step 4: (Returning to the integral, *substitute the PF decomposition into the integral.*)

$$\int\frac{5x-4}{2x^2+x-1}dx=\int\left[\frac{3}{x+1}-\frac{1}{2x-1}\right]dx$$

$$=3\int\frac{dx}{x+1}-\int\frac{dx}{2x-1}$$ (*Properties of the integral*)

Use the substitutions $\begin{cases} u = x+1 \\ du = dx \end{cases}$ and $\begin{cases} v = 2x-1 \\ dv = 2dx \end{cases}$.

$$= 3\int \frac{du}{u} - \frac{1}{2}\int \frac{dv}{v}$$

$$= 3\ln|u| - \frac{1}{2}\ln|v| + C \qquad \text{(\textit{Integration formulae})}$$

$$= 3\ln|x+1| - \frac{1}{2}\ln|2x-1| + C \qquad \text{(\textit{Substitute for u and v})}$$

(*This is just the evaluation of the integral* $\int \frac{5x-4}{2x^2+x-1}dx$. *To complete the solution, return to Step 0.*)

$$\int \frac{4x^3+2x^2+3x-4}{2x^2+x-1}dx = x^2 + \int \frac{5x-4}{2x^2+x-1}dx$$

$$= x^2 + 3\ln|x+1| - \frac{1}{2}\ln|2x-1| + C$$

Example 2: Evaluate $\int \frac{x-12}{x^3+4x^2}dx$

Solution: (*Since the degree (1) of the numerator is smaller than the degree (3) of the denominator, there is no Step 0.*)

Step 1: $x^3+4x^2 = x^2(x+4)$

Step 2: $\frac{x-12}{x^3+4x^2} = \frac{x-12}{x^2(x+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+4}$

Step 3: $x-12 = Ax(x+4) + B(x+4) + Cx^2$ (*Clearing fractions*)

$= Ax^2 + 4Ax + Bx + 4B + Cx^2$ (*Algebra*)

$= (A+C)x^2 + (4A+B)x + 4B$ (*Algebra*)

$$\begin{cases} A+C=0 \\ 4A+B=1 \\ 4B=-12 \end{cases} \qquad \text{(\textit{Set corresponding coefficients equal.})}$$

$B=-3, A=1, C=-1$ (*Solving the system of equations*)

$$\frac{x-12}{x^3+4x^2} = \frac{x-12}{x^2(x+4)} = \frac{1}{x} + \frac{-3}{x^2} + \frac{-1}{x+4}$$

Step 4: $\int \frac{x-12}{x^3+4x^2}dx = \int \frac{x-12}{x^2(x+4)}dx = \int \left[\frac{1}{x} + \frac{-3}{x^2} + \frac{-1}{x+4}\right]dx$

$$= \int \frac{dx}{x} - 3\int x^{-2}dx - \int \frac{dx}{x+4}$$

Use Substitution $\begin{cases} u = x+4 \\ du = dx \end{cases}$

$$= \ln|x| - 3\left(\frac{x^{-1}}{-1}\right) - \int \frac{du}{u}$$
$$= \ln|x| + 3x^{-1} - \ln|u| + C$$
$$= \ln|x| + 3x^{-1} - \ln|x+4| + C$$

Example 3: Find $\int \frac{3x-17}{x^2-2x-3}\,dx$.

$x^2-2x-3=(x-3)(x+1) \Rightarrow$ using the **Linear Factor Rule**, we get

$\frac{3x-17}{x^2-2x-3} = \frac{3x-17}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \Rightarrow\ 3x-17 = A(x+1)+B(x-3)$ after

multiplying by $(x-3)(x+1)$. If we let $x=3$, then $-8=4A \Rightarrow A=-2$; if we let

$x=-1$, then $-20=-4B \Rightarrow B=5$. Thus, $\int \frac{3x-17}{x^2-2x-3}\,dx =$

$\int \frac{3x-17}{(x-3)(x+1)}\,dx = -2\int \frac{1}{x-3}\,dx + 5\int \frac{1}{x+1}\,dx = -2\ln|x-3| + 5\ln|x+1| + C$.

Example 4: Find $\int \frac{3x-4}{x^2-4x+4}\,dx$.

$x^2-4x-4=(x-2)(x-2)=(x-2)^2 \Rightarrow$ by the **Linear Factor Rule**, we get

$\frac{3x-4}{x^2-4x+4} = \frac{3x-4}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} \Rightarrow 3x-4 = A(x-2)+B$ after multiplying

by $(x-2)^2$. If we let x = 2, then $2=B$; if we let x = 3, then $5=A+B=A+2 \Rightarrow$

$A=3$. Thus, $\int \frac{3x-4}{x^2-4x+4}\,dx = 3\int \frac{1}{x-2}\,dx + 2\int \frac{1}{(x-2)^2}\,dx =$

$3\ln|x-2| - \frac{2}{x-2} + C$.

Example 5: Find $\int \frac{4x^2+x-2}{x^3-x^2}\,dx$.

$x^3 - x^2 = x^2(x-1) \Rightarrow$ using the **Linear Factor Rule** with both x^2 (multiplicity of linear factors) and $x - 1$, we get $\dfrac{4x^2+x-2}{x^3-x^2} = \dfrac{4x^2+x-2}{x^2(x-1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1} \Rightarrow$

$4x^2 + x - 2 = Ax(x-1) + B(x-1) + Cx^2$ after multiplying by $x^2(x-1)$. If we let $x = 0$, then $-2 = B(-1) \Rightarrow B = 2$; if we let $x = 1$, then $C = 3$; and, if we let $x = 2$, then $16 = 2A + B + 4C = 2A + 2 + 12 \Rightarrow 2 = 2A \Rightarrow A = 1$. Thus,

$$\int \frac{4x^2+x-2}{x^3-x^2}\,dx = \int \frac{1}{x}\,dx + 2\int \frac{1}{x^2}\,dx + 3\int \frac{1}{x-1}\,dx = \ln|x| - \frac{2}{x} + 3\ln|x-1| + C.$$

Example 6: Find $\displaystyle\int \frac{7x^2+x+2}{(x-1)(x^2+1)}\,dx$.

Using both the **Linear Factor Rule** and the **Quadratic Factor Rule**, we get

$$\frac{7x^2+x+2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \Rightarrow 7x^2 + x + 2 = A(x^2+1) + (Bx+C)(x-1)$$

after multiplying by $(x-1)(x^2+1)$. If we let $x = 1$, then $10 = 2A \Rightarrow A = 5$;

if we let $x = 0$, then $2 = A - C = 5 - C \Rightarrow C = 3$; and if we let $x = 2$, then 32 = $5A + 2B + C = 25 + 2B + 3 \Rightarrow 2B = 4 \Rightarrow B = 2$. Thus, $\displaystyle\int \frac{7x^2+x+2}{(x-1)(x^2+1)}\,dx =$

$$\int \frac{5}{x-1}\,dx + \int \frac{2x+3}{x^2+1}\,dx = 5\int \frac{1}{x-1}\,dx + \int \frac{2x}{x^2+1} + 3\int \frac{1}{x^2+1}\,dx =$$

$5\ln|x-1| + \ln|x^2+1| + 3\tan^{-1} x + C$.

Example 7: Find $\displaystyle\int \frac{5x^2+11}{(x^2+1)(x^2+4)}\,dx$.

$\frac{5x^2+11}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} \Rightarrow 5x^2+11 = (Ax+B)(x^2+4) +$ $(Cx+D)(x^2+1)$ after multiplying by $(x^2+1)(x^2+4)$.

If we let $x = 0$, then $11 = 4B + D$;
if we let $x = 1$, then $16 = 5A + 5B + 2C + 2D$;
if we let $x = -1$, then $16 = -5A + 5B - 2C + 2D$;
and if $x = 2$, then $31 = 16A + 8B + 10C + 5D$.

If we add the equations $16 = 5A + 5B + 2C + 2D$ and $16 = -5A + 5B - 2C + 2D$, we get
$32 = 10B + 4D \Rightarrow$ combining this equation with $11 = 4B + D$, we get
$B = 2$ and $D = 3$.
If we subtract the equations $16 = 5A + 5B + 2C + 2D$ and $16 = -5A + 5B - 2C + 2D$,
we get $0 = 10A + 4C \Rightarrow 2C = -5A \Rightarrow$ combining this equation with $31 = 16A + 8B + 10C + 5D$, $B = 2$, and $D = 3$,
we get 31 = $16A + 16 - 25A + 15 \Rightarrow A = 0 \Rightarrow C = 0$.
Thus,

$$\int \frac{5x^2+11}{(x^2+1)(x^2+4)}\,dx = 2\int \frac{1}{x^2+1}\,dx + 3\int \frac{1}{x^2+4}\,dx = 2\tan^{-1} x + \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + C.$$

Example 8: Find $\int \frac{1}{x^4+x^2}\,dx$.

$\frac{1}{x^4+x^2} = \frac{1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} \Rightarrow 1 = Ax(x^2+1) + B(x^2+1) +$ $(Cx+D)x^2$ after multiplying by $x^2(x^2+1)$. If we let $x = 0$, then $1 = B$. If we let $x = 1$, then $1 = 2A + 2B + C + D \Rightarrow 2A + C + D = -1$. If we let $x = -1$, then $1 = -2A + 2B - C + D \Rightarrow -2A - C + D = -1 \Rightarrow$ adding this equation to the equation $2A + C + D = -1$, we get $D = -1$. If we let $x = 2$, then $1 = 10A + 5B + 8C + 4D \Rightarrow$ $0 = 10A + 8C$. Taking the equations $0 = 10A + 8C$ and $2A + C = 0$, we obviously get that $A = 0$ and $C = 0$. Thus, $\int \frac{1}{x^4+x^2}\,dx = \int \frac{1}{x^2}\,dx - \int \frac{1}{x^2+1}\,dx =$ $\frac{-1}{x} - \tan^{-1} x + C$.

H.W:- 1- $\int \frac{3x^2 + x + 1}{(x-1)(x^2+4)}\, dx$ 2- $\int \frac{-x^2 + 3x + 4}{x(x+2)^2}\, dx$

Integration by substitution

The method of substitution is a method for algebraically simplifying the form of a function so that its antiderivative can be easily recognized. This method is intimately related to the chain rule for differe- ntiation. For example, since the derivative of e^x is

$$\frac{d}{dx}\left[e^x\right]=e^x$$

it follows easily that

$$\int e^x dx = e^x + c$$

However, it may not be obvious to some how to integrate

$$\int (2x+2)e^{x^2+2x+3}dx$$

Note that the derivative of e^{x^2+2x+3} can be computed using the chain rule and is

$$\frac{d}{dx}\left[e^{x^2+2x+3}\right]=(2x+2)e^{x^2+2x+3}$$

Thus, it follows easily that

$$\int (2x+2)e^{x^2+2x+3}dx = e^{x^2+2x+3}+c$$

This is an illustration of the chain rule "backwards". Now the method of substitution will be illustrated on this same example. Begin with

$$\int (2x+2)e^{x^2+2x+3}dx$$

and let

$$u = x^2+2x+3$$

then the derivative of u is

$$\frac{du}{dx}=2x+2$$

Now "pretend" that the differentiation notation $\frac{du}{dx}$ is an arithmetic fraction, and multiply both sides of the previous equation by dx getting

$$du=(2x+2)dx$$

or

$$\frac{du}{(2x+2)}=dx$$

Make substitutions into the original problem, removing all forms of x , resulting in

$$\int (2x+2)e^{x^2+2x+3}dx = \int (2x+2)e^u \frac{du}{(2x+2)} = \int e^u du = e^u + c = e^{x^2+2x+3}+c\ .$$

Of course, it is the same answer that we got before, using the chain rule "backwards". In essence, the method of substitution is a way to recognize the antiderivative of a chain rule derivative. Here is another illustraion of substitution. Consider

$$\int \frac{x^2+1}{x^3+3x}dx$$

Let

$$u = x^3 + 3x$$

Then Go directly to the du part.

$$du = \left(3x^2 + 3\right)dx = 3\left(x^2 + 1\right)dx$$

so that

$$dx = \frac{du}{3\left(x^2+1\right)}$$

Make substitutions into the original problem, removing all forms of x, resulting in

$$\int \frac{x^2+1}{x^3+3x}dx = \int \frac{x^2+1}{u}\frac{du}{3\left(x^2+1\right)} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + c = \frac{1}{3}\ln\left|x^3+3x\right| + c .$$

Example 1: Integrate $\int \sec 2x \tan 2x dx$.

Solution: Let $u = 2x$, so that $du = 2dx$.

Substitute into the original problem, replacing all forms of x getting

$$\int \sec 2x \tan 2x dx = \frac{1}{2}\int \sec u \tan u du = \frac{1}{2}\sec u + c = \frac{1}{2}\sec 2x + c .$$

Example 2: Integrate $\int \sin^2 x \cos x \; dx$.

Solution: Let $u = \sin x$, so that $du = \cos x \; dx$.

Substitute into the original problem, replacing all forms of x getting

$$\int \sin^2 x \cos x \; dx = \int u^2 \cos x \frac{du}{\cos x} = \int u^2 du = \frac{u^3}{3} + c = \frac{\sin^3 x}{3} + c .$$

Example 3: Integrate $\int t^4 \sqrt[3]{3-5t^5}\, dt$.

Solution: Let $u = 3-5t^5$, so that $du = -25t^4 dt$.

Substitute into the original problem, replacing all forms of x getting

$$\int t^4 \sqrt[3]{3-5t^5}\, dt = \int t^4 \sqrt[3]{u} \frac{du}{-25t^4} = -\frac{1}{25}\int u^{\frac{1}{3}} du = \frac{-1}{25}\frac{u^{\frac{4}{3}}}{4/3} + c = -\frac{3}{100}u^{\frac{4}{3}} + c = -\frac{3}{100}\left(3-5t^5\right)^{\frac{4}{3}} + c$$

Example 4: Integrate $\int \frac{4}{x^{1/3}\sqrt{1+x^{2/3}}}dx$.

Solution: Let $u = 1 + x^{2/3}$, so that $du = \frac{2}{3}x^{-1/3}dx \Rightarrow dx = \frac{3x^{1/3}}{2}du$.

Substitute into the original problem, replacing all forms of x getting

$$\int \frac{4}{x^{1/3}\sqrt{1+x^{2/3}}}dx = \int \frac{4}{x^{1/3}\sqrt{y}}.\frac{3x^{1/3}}{2}du = 6\int \frac{1}{\sqrt{u}}du = 12\sqrt{u} + c = 12\sqrt{1+x^{2/3}} + c .$$

Example 5: Integrate $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$.

Solution: Let $u=\sqrt{x}$, so that $du=\frac{1}{2\sqrt{x}}dx=\frac{1}{2u}dx \Rightarrow dx=2u\ du$.

Substitute into the original problem, replacing all forms of x getting

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx=\int \frac{e^{u}}{u}.2u\,du=2\int e^{u}\,du=2e^{u}+c=2e^{\sqrt{x}}+c .$$

Example 6: Integrate $\int \frac{2}{x^{1/4}+x}dx$.

Solution: First note that

$$\int \frac{2}{x^{1/4}+x}dx=\int \frac{2}{x^{1/4}\left(1+x^{3/4}\right)}dx$$

Let $y=1+x^{3/4}$, so that $dy=\frac{3}{4}x^{-1/4}dx \Rightarrow dx=\frac{4}{3}x^{1/4}dy$.

Substitute into the original problem, replacing all forms of x getting

$$\int \frac{2}{x^{1/4}\left(1+x^{3/4}\right)}dx=\int \frac{2}{x^{1/4}y}.\frac{4}{3}x^{1/4}dy=\frac{8}{3}\int \frac{dy}{y}=\frac{8}{3}\ln|y|+c=\frac{8}{3}\ln\left|1+x^{3/4}\right|+c .$$

Example 7: Integrate $\int \frac{1}{\sqrt{x}+x}dx$

Solution: Let $u=\sqrt{x}$, so that $du=\frac{1}{2\sqrt{x}}dx=\frac{1}{2u}dx$

Substitute into the original problem, replacing all forms of x , getting

$$\int \frac{1}{\sqrt{x}+x}dx=\int \frac{2u}{u+u^{2}}du=\int \frac{2}{1+u}du=2\ln|1+u|+c=2\ln\left|1+\sqrt{x}\right|+c$$

Inverse Trigonometric Functions, Related Integral

Here are the integration formulas for some of the inverse trig functions. We'll giving extended versions of the formulas that you'd get if you use the method of substitution.

$$(1)\int \frac{dx}{\sqrt{a^2-b^2x^2}}=\frac{1}{b}\sin^{-1}\left(\frac{b}{a}x\right)+c \qquad [a>0, b>0]$$

$$(2)\int \frac{dx}{a^2+b^2x^2}=\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}x\right)+c \qquad [a>0, b>0]$$

$$(3)\int \frac{dx}{|x|\sqrt{b^2x^2-a^2}}=\frac{1}{a}\sec^{-1}\left(\frac{b}{a}x\right)+c \qquad [a>0, b>0]$$

For example, you can derive this $\sin^{-1}$ formula from the formula

$\int \frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+c$ as follows:

$$\int \frac{dx}{\sqrt{a^2-b^2x^2}}=\int \frac{dx}{\sqrt{a^2\left(1-\frac{b^2}{a^2}x^2\right)}}=\int \frac{dx}{\sqrt{a^2}\sqrt{\left(1-\left(\frac{bx}{a}\right)^2\right)}}=\int \frac{dx}{|a|\sqrt{\left(1-\left(\frac{bx}{a}\right)^2\right)}}$$

$$=\frac{1}{a}\int \frac{dx}{\sqrt{\left(1-\left(\frac{bx}{a}\right)^2\right)}} \quad \left[\text{use the substitution } y=\frac{bx}{a}\right]$$

$$=\frac{1}{a}\cdot\frac{a}{b}\int \frac{dy}{\sqrt{1-y^2}}=\frac{1}{b}\int \frac{dy}{\sqrt{1-y^2}}=\frac{1}{b}\sin^{-1}(y)+c=\frac{1}{b}\sin^{-1}\left(\frac{b}{a}x\right)+c$$

Example 1:

Using the $\sin^{-1}$ formula with $a=1$ **and** $b=3$

$$\int \frac{dx}{\sqrt{1-9x^2}}=\frac{1}{3}\sin^{-1}(3x)+c$$

Using the $\sin^{-1}$ formula with $a=\sqrt{3}$ **and** $b=2$

$$\int \frac{dx}{\sqrt{3-4x^2}}=\frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right)+c$$

Using the $\sec^{-1}$ formula with $a=\sqrt{5}$ **and** $b=\sqrt{3}$

$$\int \frac{dx}{|x|\sqrt{3x^2-5}}=\frac{1}{\sqrt{5}}\sec^{-1}\left(\frac{\sqrt{3}}{\sqrt{5}}x\right)+c=\frac{1}{\sqrt{5}}\sec^{-1}\left(\sqrt{\frac{3}{5}}x\right)+c$$

Using the $\sin^{-1}$ formula with $a=5$ **and** $b=3$

$$\int \frac{dx}{\sqrt{25-9x^2}}=\frac{1}{3}\sin^{-1}\left(\frac{3}{5}x\right)+c$$

Using the $\tan^{-1}$ formula with $a=7$ **and** $b=2$

$$\int \frac{dx}{4x^2+49}=\frac{1}{14}\tan^{-1}\left(\frac{2}{7}x\right)+c$$

Example 2: Integrate $\int \frac{\sin\theta}{\sqrt{1-\cos^2\theta}}\,d\theta$

Solution: Let $u=\cos\theta$, so that $du=-\sin\theta d\,\theta$ or $d\theta=-\frac{du}{\sin\theta}$

Substitute into the original problem, replacing all forms of θ, getting

$$\int \frac{\sin\theta}{\sqrt{1-\cos^2\theta}}\,d\theta=-\int \frac{\sin\theta}{\sqrt{1-u^2}}\,\frac{du}{\sin\theta}=-\int \frac{du}{\sqrt{1-u^2}}=-\sin^{-1}(u)+c=-\sin^{-1}(\cos\theta)+c$$

Example 3: Integrate $\int \frac{4}{5+2x+x^2}dx$

Solution: Complete the square on the quadratic. This gives,

$$5+2x+x^2=x^2+2x+1+4=(x+1)^2+4$$

After completing the square the integral becomes,

$$\int \frac{4}{5+2x+x^2}dx=\int \frac{4}{(x+1)^2+4}dx\ \ [\text{use the substitution } y=x+1]$$

$$=4\int \frac{dy}{y^2+4}=4\left(\frac{1}{2}\right)\tan^{-1}\left(\frac{y}{2}\right)+c=2\tan^{-1}\left(\frac{x+1}{2}\right)+c$$

Example 4: Integrate $\int \frac{4x}{5+2x+x^2}dx$

Solution: Complete the square on the quadratic. This gives,

$$5+2x+x^2=x^2+2x+1+4=(x+1)^2+4$$

After completing the square the integral becomes,

$$\int \frac{4x}{5+2x+x^2}dx=\int \frac{4x}{(x+1)^2+4}dx\ \ [\text{use the substitution } y=x+1]$$

$$=4\int \frac{y-1}{y^2+4}dy=4\left[\int \frac{y}{y^2+4}dy-\int \frac{1}{y^2+4}dy\right]$$

$$=4\left[\frac{1}{2}\int \frac{2y}{y^2+4}dy-\int \frac{1}{y^2+4}dy\right]=2\ln\left|y^2+4\right|-2\tan^{-1}\left(\frac{y}{2}\right)+c$$

$$=2\ln\left|(x+1)^2+4\right|-2\tan^{-1}\left(\frac{x+1}{2}\right)+c=2\ln\left(x^2+2x+5\right)-2\tan^{-1}\left(\frac{x+1}{2}\right)+c$$

Integrals by Trigonometric Substitution

Integrals Involving the Form $\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$, and $\sqrt{u^2-a^2}$

A. Sometimes, you should make a *u*–substitution before you make a trigonometric substitution.

Example 1: Find $\displaystyle\int \frac{\sqrt{9-4x^2}}{x^2}\,dx$.

Let $u^2=4x^2 \Rightarrow u=2x \Rightarrow du=2dx$. Also, $a^2=9 \Rightarrow a=3$.

$\displaystyle\int \frac{\sqrt{9-4x^2}}{x^2}\,dx = 2\int \frac{\sqrt{9-4x^2}}{4x^2}\cdot 2\,dx = 2\int \frac{\sqrt{9-u^2}}{u^2}\,du$. Let $u=3\sin\theta$

where $-\pi/2 \le \theta \le \pi/2$. $\sqrt{9-u^2}=3\cos\theta$ and $du=3\cos\theta\,d\theta \Rightarrow$

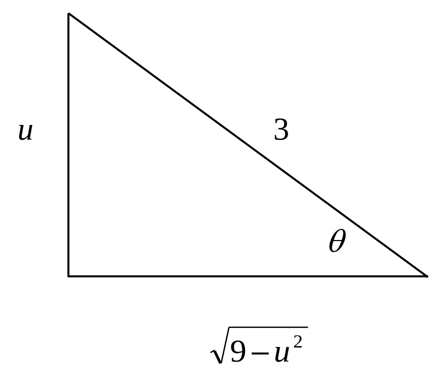
$$2\int \frac{\sqrt{9-u^2}}{u^2}\,du = 2\int \frac{3\cos\theta(3\cos\theta\,d\theta)}{(3\sin\theta)^2} = 2\int \cot^2\theta\,d\theta =$$

$$2\int(\csc^2\theta-1)\,d\theta = -2\cot\theta-2\theta+C =$$

$$-2\left(\frac{\sqrt{9-u^2}}{u}\right)-2\sin^{-1}\left(\frac{u}{3}\right)+C =$$

$$-2\left(\frac{\sqrt{9-4x^2}}{2x}\right)-2\sin^{-1}\left(\frac{2x}{3}\right)+C =$$

$$\frac{-\sqrt{9-4x^2}}{x}-2\sin^{-1}\left(\frac{2x}{3}\right)+C.$$



Example 2: Find $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$.

Let $x=2\sec\theta$ where $0\le\theta<\pi/2$ or $\pi\le\theta<3\pi/2$. Then $\sqrt{x^2-4}=2\tan\theta$ and

$dx=2\sec\theta\tan\theta\,d\theta$. Thus, $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx = \int \frac{(2\tan\theta)(2\sec\theta\tan\theta\,d\theta)}{2\sec\theta} =$

$$2\int \tan^2\theta\,d\theta = 2\int(\sec^2\theta-1)\,d\theta = 2\tan\theta-2\theta+C =$$

$$2\left(\frac{\sqrt{x^2-4}}{2}\right)-2\sec^{-1}\left(\frac{x}{2}\right)+C = \sqrt{x^2-4}-2\sec^{-1}\left(\frac{x}{2}\right)+C.$$

B. Sometimes, you have to **complete the square** before making a u–substitution or a trigonometric substitution.

Example 2: Find $\int \frac{1}{4x^2-8x+13}\,dx$.

Complete the square for $4x^2-8x+13$:

$$4x^2-8x+13 = 4(x^2-2x\quad)+13$$
$$= 4(x^2-2x+1)+13-4$$
$$= 4(x-1)^2+9$$

4

Let $u^2=4(x-1)^2 \Rightarrow u=2(x-1) \Rightarrow du=2dx$. Also, $a^2=9 \Rightarrow a=3$.

Thus, $\int \frac{1}{4x^2-8x+13}\,dx = \int \frac{1}{4(x-1)^2+9}\,dx = \frac{1}{2}\int \frac{1}{u^2+9}\,du =$

$\frac{1}{2}\left\{\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)\right\}+C = \frac{1}{6}\tan^{-1}\left(\frac{2(x-1)}{3}\right)+C$.

Example 2: Find $\int \sqrt{15+6x-9x^2}\,dx$.

Complete the square for $15+6x-9x^2$:

$$15+6x-9x^2 = -9x^2+6x+15$$
$$= -9\left(x^2-\frac{2}{3}x\quad\right)+15$$
$$= -9\left(x^2-\frac{2}{3}x+\frac{1}{9}\right)+15+1$$
$$= 16-(9x^2-6x+1) = 16-(3x-1)^2$$

Let $u^2=(3x-1)^2 \Rightarrow u=3x-1 \Rightarrow du=3dx$. Also, $a^2=16 \Rightarrow a=4$.

Thus, $\int \sqrt{15+6x-9x^2}\,dx = \frac{1}{3}\int \sqrt{16-(3x-1)^2}\,3dx =$

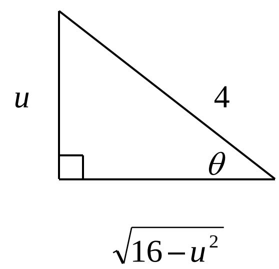
$\frac{1}{3}\int \sqrt{16-u^2}\,du$. Let $u=4\sin\theta$ where $-\pi/2 \le \theta \le \pi/2$. Then,

$\sqrt{16-u^2} = 4\cos\theta$ and $du = 4\cos\theta\,d\theta$. $\frac{1}{3}\int \sqrt{16-u^2}\,du =$

$\frac{1}{3}\int (4\cos\theta)(4\cos\theta\,d\theta) = \frac{16}{3}\int \cos^2\theta\,d\theta = \frac{8}{3}\theta+\frac{8}{3}\sin\theta\cos\theta+C =$

$$\frac{8}{3}\sin^{-1}\left(\frac{u}{4}\right)+\frac{8}{3}\left(\frac{u}{4}\right)\left(\frac{\sqrt{16-u^2}}{4}\right)+C=$$

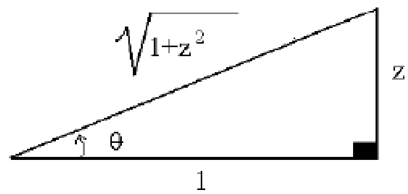
$$\frac{8}{3}\sin^{-1}\left(\frac{3x-1}{4}\right)+\frac{(3x-1)\sqrt{15+6x-9x^2}}{6}+C.$$



Integrating rational functions of trigonometric expressions

Any rational function of trigonometric functions can be integrated by using the substitution

$$z = \tan(\frac{x}{2})$$



$$\sin\theta = \frac{z}{\sqrt{1+z^2}}, \quad \cos\theta = \frac{1}{\sqrt{1+z^2}},$$

so that

$$\sin x = 2\sin\theta\cos\theta = 2\cdot\frac{z}{\sqrt{1+z^2}}\cdot\frac{1}{\sqrt{1+z^2}} = \frac{2z}{1+z^2}.$$

Of course, we obtained $\cos\theta$ as well using this calculation, that is, we found that since $\theta = x/2$,

$$\cos(x/2) = \frac{1}{\sqrt{1+z^2}}.$$

But the trigonometric identity $\cos 2\phi = 2\cos^2\phi - 1$, valid for any angle ϕ, means that we can set $\phi = x/2$ so that

$$\cos x = 2\cos^2(x/2) - 1 = 2\left(\frac{1}{\sqrt{1+z^2}}\right)^2 - 1 = \frac{1-z^2}{1+z^2}.$$

Finally, we need to determine the new "dx" term. This is not difficult since $z = \tan(x/2)$ implies that $dz = (1/2)\sec^2(x/2)\,dx$. But the trig. identity $\sec^2\phi - \tan^2\phi = 1$ valid for any angle ϕ means that we can set $\phi = x/2$ as before. This then gives us

$$\frac{dz}{dx} = (1/2)\sec^2(x/2) = (1/2)(1+\tan^2(x/2)) = (1/2)(1+z^2).$$

From this and the Chain Rule we also get that

$$\frac{dx}{dz} = \frac{2}{1+z^2}.$$

Example *Evaluate the integral* $I \equiv \int \frac{1}{4\cos x - 3\sin x}\, dx.$

Solution: The integrand is a rational function of the sine and cosine function so we can use the substitution in Table 1. Thus,

$$\begin{aligned}
I &= \int \frac{1}{4\left(\frac{1-z^2}{1+z^2}\right) - 3\left(\frac{2z}{1+z^2}\right)} \left(\frac{2\,dz}{1+z^2}\right), \\
&= 2\int \frac{1}{4(1-z^2) - 3(2z)}\, dz, \\
&= \int \frac{1-2z}{z+2}\, dz \\
&= \int \left\{\frac{2/5}{1-2z} + \frac{1/5}{z+2}\right\} dz, \quad \text{(using partial fractions)} \\
&= \int \left\{-\frac{4/5}{z-(1/2)} + \frac{1/5}{z+2}\right\} dz, \\
&= -\frac{4}{5}\log|z - \frac{1}{2}| + \frac{1}{5}\log|z+2| + C, \\
&= -\frac{4}{5}\log|\tan(x/2) - \frac{1}{2}| + \frac{1}{5}\log|\tan(x/2)+2| + C.
\end{aligned}$$

Example *Evaluate* $\int_0^2 \frac{1}{2+\sin x}\, dx.$

Solution: Using the substitution $z = \tan(x/2)$, etc. we find that an antiderivative is given by evaluating

$$\begin{aligned}
\int \frac{1}{2+\sin x}\, dx &= \int \frac{1}{2+\left(\frac{2z}{1+z^2}\right)} \left(\frac{2\,dz}{1+z^2}\right), \\
&= \int \frac{1}{z^2+z+1}\, dz
\end{aligned}$$

$$
\begin{aligned}
&= \int \frac{1}{\left(z+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dz \\
&= \frac{2\sqrt{3}}{3}\text{Arctan}\left(\frac{\sqrt{3}}{3}(2z+1)\right) \\
&= \frac{2\sqrt{3}}{3}\text{Arctan}\left(\frac{\sqrt{3}}{3}(2\tan(x/2)+1)\right).
\end{aligned}
$$

So the value of the given definite integral is given by

$$
\begin{aligned}
\int_0^2 \frac{1}{2+\sin x}\, dx &= \frac{2\sqrt{3}}{3}\text{Arctan}\left(\frac{\sqrt{3}}{3}(2\tan(x/2)+1)\right)\Bigg|_{x=0}^{x=2} \\
&= \frac{2\sqrt{3}}{3}\left(\text{Arctan}\left(\frac{\sqrt{3}}{3}(2\tan(1)+1)\right) - \text{Arctan}\left(\frac{\sqrt{3}}{3}\right)\right) \\
&\approx .7491454107
\end{aligned}
$$