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Mathematics II
Matrices

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What is Matrix?

- A matrix is a rectangular array of elements arranged in ***horizontal rows*** and ***vertical columns***, that is used in a great variety of ways, such as to solve linear systems, and model linear behavior.
- Matrix usually enclosed in brackets. We use capital letters (A, B, C, . . .) to denote a matrix. The notation $A_{m\times n} = [a_{ij}]_{m\times n}$ means that the element in the i–th row and j–th column of the matrix A equals a_{ij}.
- Matrices come in various shaped depending on the number of rows and columns. A matrix having m rows and n columns has size m by n, written $m \times n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \dots & \boxed{a_{ij}} & \dots & a_{in} \\ \vdots & \vdots & \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

What is Matrix?

- A matrix with only one row is called a ***row matrix***, A = [a_{11} a_{12} a_{13} ... a_{1n}], It is a row matrix with n columns. So, it is of type $1 \times n$

- A matrix with only one column is called a ***column matrix***. $A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix}$

It is a column matrix with m rows. So, it is of type $m \times 1$.

- Let $A_{m \times n}$ be any matrix. Then A is said to be ***square matrix***, if $m = n$,

i = 1, 2, 3, ..., n; j = 1, 2, 3, ..., n.

What is Matrix?

A square matrix in which all the nondiagonal elements are zero is called a diagonal matrix.

A is a diagonal matrix of order m or n (m=n).

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & a_{mn} \end{bmatrix}$$

What is Matrix?

In a diagonal matrix, if all the diagonal elements are equal to 1, then it is called a ***Unit matrix*** or ***identity matrix***.

$$[1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What is Matrix?

In a matrix (rectangular or square), if all the entries are equal to 0, then it is called a ***zero matrix*** or ***null matrix***.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above matrices A, and B are zero matrices of types 2×2 and 2×4 respectively.

Matrix

EXAMPLE 1

For the matrix A, find a_{12}, a_{53}, a_{32}, a_{44}, a_{14}, a_{25}.

$$A = \begin{bmatrix} -1 & 3 & 23 & -3 & -5 \\ -2 & 4 & -2 & 11 & 23 \\ -7 & 0 & -9 & 17 & 35 \\ -9 & 8 & 11 & 18 & -7 \\ 11 & 1 & 20 & 85 & 92 \end{bmatrix}$$

EXAMPLE 2

What is the size of the following matrices:

$$A = \begin{bmatrix} 11 & -2 & -1 \\ 21 & 22 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 2 & 3 & 9 \\ 0 & 9 & 1 & 6 \\ 1 & 6 & 2 & 5 \end{bmatrix}$$

Matrix

Transpose:

If a given matrix A, we interchange the rows and the corresponding columns, the new matrix obtained is called the ***transpose of the matrix*** A and denoted by A' or A^T.

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \\ 6 & 7 & 8 \end{bmatrix}, \quad A' = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 0 & 7 \\ 4 & 5 & 8 \end{bmatrix}$$

A square matrix will called ***symmetric matrix***, if for all values of i and j, $a_{ij} = a_{ji}$ or $A=A'$

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

A square matrix is called ***skew-symmetric matrix***, if :

1) $a_{ij} = -a_{ji}$ for all values of I and j, or $A' = -A$
2) All Diagonals elements are zero

$$\begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$$

Matrix

A square matrix, all of whose elements *below* the leading diagonal are zero is called ***upper triangular matrix***

A square matrix, all of whose elements *Above* the leading diagonal are zero is called ***lower triangular matrix***

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 6 & 7 \end{bmatrix}$$

Upper triangular matrix Lower triangular matrix

Matrix operations

Addition and Subtraction of Matrices

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same type $m \times n$. Then $A + B = [c_{ij}]$, where $c_{ij} = a_{ij} + b_{ij}$ for all i and j and $A + B$ is of type $m \times n$.

To be added or subtracted, two matrices must be of the same order. The sum or difference is then determined by adding or subtracting corresponding elements.

$$C = [c_{ij}] = A + B = [a_{ij}] + [b_{ij}]$$
$$= [a_{ij} + b_{ij}]$$

Thus, $c_{ij} = a_{ij} + b_{ij};\ i = 1, 2, 3, \ldots m$

$j = 1, 2, 3, \ldots n$

$$C = [c_{ij}] = A - B = [a_{ij}] - [b_{ij}]$$
$$= [a_{ij} - b_{ij}]$$

Thus, $c_{ij} = a_{ij} - b_{ij};\ i = 1, 2, 3, \ldots m$

$j = 1, 2, 3, \ldots n$

The order of the new matrix C is same as that of A and B.

Matrix operations

Properties of Addition and Subtraction of Matrices

If A, B, C are matrices of the same type, then

(i) $A + B = B + A$

(ii) $A + (B + C) = (A + B) + C$

(iii) $A + 0 = A$

(iv) $A + (-A) = 0$

(v) $\boldsymbol{\alpha} (A + B) = \boldsymbol{\alpha} A + \boldsymbol{\alpha} B$

(vi) $(\boldsymbol{\alpha} + \boldsymbol{\beta})A = \boldsymbol{\alpha} A + \boldsymbol{\beta} A$

(vii) $\boldsymbol{\alpha} (\boldsymbol{\beta} A) = (\boldsymbol{\alpha\beta})A$ for any scalars $\boldsymbol{\alpha}$,and $\boldsymbol{\beta}$

Matrix operations

Scalar Multiplication

Let $A = [a_{ij}]$ be an $m \times n$ matrix and k be a scalar, then $kA = [ka_{ij}]$.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \text{ then } kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \end{bmatrix}$$

$$\text{In particular if } k = -1, \text{ then } -A = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix}$$

Matrix operations

EXAMPLE 2

Find:

i) $A+B$

ii) $B-A$

iii) $A'+B'$

$$A = \begin{bmatrix} 4 & 2 & 5 \\ 1 & 3 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$$

EXAMPLE 3

Find:

i) $2A$

ii) $-3B$

iii) $-4A+5B$