Differentiation in Calculus | Definition, Formulas, Examples

THE DERIVATIVE

Definition

The derivative of a function f at a point x, written $f'(x)$, is given by:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

if this limit exists.

1. Derivatives of usual functions

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Derivatives

Basic Properties/Formulas/Rules

$\frac{d}{dx}(cf(x)) = cf'(x)$, c is any constant. $\quad (f(x) \pm g(x))' = f'(x) \pm g'(x)$

$\frac{d}{dx}(x^n) = nx^{n-1}$, n is any number. $\quad \frac{d}{dx}(c) = 0$, c is any constant.

$(f\,g)' = f'\,g + f\,g'$ – **(Product Rule)** $\quad \left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$ – **(Quotient Rule)**

$\frac{d}{dx}(f(g(x))) = f'(g(x))\,g'(x)$ **(Chain Rule)**

$\frac{d}{dx}\left(\mathbf{e}^{g(x)}\right) = g'(x)\,\mathbf{e}^{g(x)} \quad \frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{g(x)}$

Common Derivatives

Polynomials

$\frac{d}{dx}(c) = 0 \quad \frac{d}{dx}(x) = 1 \quad \frac{d}{dx}(cx) = c \quad \frac{d}{dx}(x^n) = nx^{n-1} \quad \frac{d}{dx}(cx^n) = ncx^{n-1}$

Trig Functions

$\frac{d}{dx}(\sin x) = \cos x \quad \frac{d}{dx}(\cos x) = -\sin x \quad \frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x \quad \frac{d}{dx}(\csc x) = -\csc x \cot x \quad \frac{d}{dx}(\cot x) = -\csc^2 x$

Exponential/Logarithm Functions

$$\frac{d}{dx}\left(a^x\right)=a^x\ln(a) \qquad \frac{d}{dx}\left(\mathbf{e}^x\right)=\mathbf{e}^x$$

$$\frac{d}{dx}\left(\ln(x)\right)=\frac{1}{x},\ x>0 \qquad \frac{d}{dx}\left(\ln|x|\right)=\frac{1}{x},\ x\neq 0 \qquad \frac{d}{dx}\left(\log_a(x)\right)=\frac{1}{x\ln a},\ x>0$$

Example:-

$$(x^4)' = 4\,x^{4-1} = 4\,x^3$$

$$(x^{1/2})' = 1/2\,x^{\frac{1}{2}-1} = 1/2\,x^{-1/2}$$

$$\frac{1}{x^4} = x^{-4} \quad \rightarrow \quad \left(\frac{1}{x^4}\right)' = (x^{-4})' = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$

$$(3^x)' = 3^x\,ln(3)$$

$$\left(\left(\frac{1}{2}\right)^x\right)' = \left(\frac{1}{2}\right)^x ln\left(\frac{1}{2}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\pi^x = \pi^x\cdot\ln(\pi) \qquad \frac{\mathrm{d}}{\mathrm{d}x}6^{\sin(x)} = 6^{\sin(x)}\ln(6)\cdot\cos(x)$$

Example If $y = xe^{2x}\sin x$ then find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 1(e^{2x}\sin x) + x(2e^{2x})\sin x + xe^{2x}(\cos x)\\ &= e^{2x}[\sin x + 2x\sin x + x\cos x]\end{aligned}$$

Example Find the derivative of $y = \dfrac{\ln x}{x}$

Solution

Here $f(x) = \ln x$ and $g(x) = x$

$$\therefore \quad \frac{df}{dx} = \frac{1}{x} \quad \text{and} \quad \frac{dg}{dx} = 1$$

Hence $\dfrac{dy}{dx} = \dfrac{x\left(\frac{1}{x}\right) - 1(\ln x)}{[x]^2} = \dfrac{1 - \ln x}{x^2}$

Example 1 Differentiate each of the following functions.

(a) $R(w) = 4^w - 5\log_9 w$ **(b)** $f(x) = 3e^x + 10x^3 \ln x$ **(c)** $y = \dfrac{5e^x}{3e^x + 1}$

Solution

(a) This will be the only example that doesn't involve the natural exponential and natural logarithm functions.

$$R'(w) = 4^w \ln 4 - \frac{5}{w\ln 9}$$

(b) use the product rule on the second term.

$$f'(x) = 3e^x + 30x^2 \ln x + 10x^3\left(\frac{1}{x}\right)$$

$$= 3e^x + 30x^2 \ln x + 10x^2$$

(c) We'll need to use the quotient rule on this one.

$$y = \frac{5e^x(3e^x + 1) - (5e^x)(3e^x)}{(3e^x + 1)^2}$$

$$= \frac{15e^{2x} + 5e^x - 15e^{2x}}{(3e^x + 1)^2}$$

$$= \frac{5e^x}{(3e^x + 1)^2}$$

Examples:- 1-

$$\left(lnx - \frac{1}{x^2} + 8\right)' = (lnx)' - (x^{-2})' + (8)'$$

$$= \frac{1}{x} - (-2x^{-3}) + 0$$

$$= \frac{1}{x} + \frac{2}{x^3}$$

2.

$$(3\sqrt{x}lnx)' = (3\sqrt{x})'lnx + 3\sqrt{x}(lnx)'$$

$$= 3\left(x^{\frac{1}{2}}\right)' lnx + 3\sqrt{x}(lnx)'$$

$$= 3\left(\frac{1}{2}x^{\frac{1}{2}-1}\right) lnx + 3\sqrt{x}\frac{1}{x}$$

$$= \frac{3}{2}x^{-\frac{1}{2}}lnx + 3x^{-\frac{1}{2}}$$

3.

$$\left(\frac{3\sqrt{x}}{lnx}\right)' = \frac{(3\sqrt{x})' lnx - 3\sqrt{x}(lnx)'}{(lnx)^2} = \frac{3\left(x^{\frac{1}{2}}\right)' lnx - 3\sqrt{x}(lnx)'}{(lnx)^2}$$

$$= \frac{3\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)lnx - 3\sqrt{x}\frac{1}{x}}{(lnx)^2} = \frac{3x^{-\frac{1}{2}}lnx - 6x^{-\frac{1}{2}}}{2(lnx)^2}$$

$$= \frac{3x^{-\frac{1}{2}}(lnx-2)}{2(lnx)^2}$$

Chain Rule. The derivative of the composition of two functions is computed as a product of their derivatives. To be precise: Let $f(x)$ and $g(x)$ be two functions and let $(g \circ f)(x) = g(f(x))$ be their composition.

Chain rule (version I)
If $h(x) = g(f(x))$, then $h'(x) = g'(f(x))\, f'(x)$

Chain rule (version II)
If $y = g(u)$ and $u = f(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$

Examples.

$$f(x) = \ln(x^4 + 5)$$

$$f'(x) = \frac{1}{x^4 + 5}\frac{d}{dx}(x^4 + 5)$$

$$f'(x) = \frac{1}{x^4 + 5} \cdot 4x^3 = \frac{4x^3}{x^4 + 5}$$

$$f(x) = 4\ln\sqrt{x} = 4\ln x^{1/2} \quad \boxed{= 2\ln x}$$

$$f'(x) = 4\frac{1}{x^{1/2}} \quad \frac{1}{2}x^{-1/2} \quad = \frac{2}{x}$$

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Examples:-

1.

$$[\ln(x^2 + 2x + 1)]' = \frac{1}{x^2 + 2x + 1}(x^2 + 2x + 1)'$$

$$= \frac{1}{x^2+2x+1}(2x + 2)$$

$$= \frac{2x+2}{x^2+2x+1}$$

$$[(lnx + 3x - e^x)^4]' = 4(lnx + 3x - e^x)^3(lnx + 3x - e^x)'$$

$$= 4(lnx + 3x - e^x)^3(\frac{1}{x} + 3 - e^x)$$

2.

$$[e^{xlnx}]' = e^{xlnx}.(xlnx)'$$

$$= e^{xlnx}[(x)'lnx + x(lnx)']$$

$$= e^{xlnx}\left(1.lnx + x.\frac{1}{x}\right)$$

$$= e^{xlnx}(lnx + 1)$$

Example 9. Compute the derivatives of the following functions.

(a) $y = (x^2+1)^{14}$ (b) $y = \sqrt{\sin x + 1}$ (c) $y = \dfrac{1}{e^x+2}$

(d) $y = \sin(x^2+1)$ (e) $y = \cos(\sec x)$ (f) $y = (\sin x)^2 + \sin(x^2)$.

Solution

(a) We start by computing the derivative of the power of 14 :

$$y' = 14(x^2+1)^{13}(x^2+1)' = 14(x^2+1)^{13}\,2x = 28x(x^2+1)^{13}$$

(b) Writing $y = (\sin x + 1)^{1/2}$, we get

$$y' = \tfrac{1}{2}(\sin x+1)^{-1/2}(\sin x+1)' = \tfrac{1}{2}(\sin x+1)^{-1/2}\ \cos x = \tfrac{1}{2}\cos x(\sin x+1)^{-1/2}.$$

(c) $y = (e^x+2)^{-1}$; thus,

$$y' = (-1)(e^x+2)^{-2}(e^x+2)' = -e^x(e^x+2)^{-2} = -\frac{e^x}{(e^x+2)^2}.$$

(d) We start by computing the derivative of sin:

$$y' = \cos(x^2+1)\,(x^2+1)' = 2x\cos(x^2+1).$$

(e) $y' = -\sin(\sec x)\,(\sec x)' = -\sin(\sec x)\sec x\tan x$.

(f) We have to be careful about the order:

$$y' = 2(\sin x)^1(\sin x)' + \cos(x^2)\,(x^2)' = 2\sin x\cos x + 2x\cos(x^2) = \sin 2x + 2x\cos(x^2).$$

In simplifying, we used the formula $2\sin x\cos x = \sin 2x$.

Example 10. Compute the derivatives of the following functions.

(a) $y = e^{4x+2}$ (b) $y = \ln(\sin x + 2)$ (c) $y = 2^{3x}$

(d) $y = \ln(x^2+3x+e^x)$ (e) $y = e^{\sin x} + \sin(e^x)$ (f) $y = \sec\sqrt{x^2+x}$.

Solution

(a) $y' = e^{4x+2}(4x+2)' = 4e^{4x+2}$.

(b) $$y' = \frac{1}{\sin x + 2}(\sin x + 2)' = \frac{\cos x}{\sin x + 2}.$$

(c) $$y' = 2^{3x} \ln 2\,(3x)' = 3 \cdot 2^{3x} \ln 2.$$

(d) As in (b),

$$y' = \frac{1}{x^2 + 3x + e^x}(x^2 + 3x + e^x)' = \frac{2x + 3 + e^x}{x^2 + 3x + e^x}.$$

(e) $y' = e^{\sin x}(\sin x)' + \cos(e^x)(e^x)' = \cos x e^{\sin x} + e^x \cos(e^x)$.

(f) Write $y = \sec(x^2 + x)^{1/2}$ and recall that $(\sec x)' = \sec x \tan x$. Then

$$\begin{aligned} y' &= \sec(x^2 + x)^{1/2} \tan(x^2 + x)^{1/2}((x^2 + x)^{1/2})' \\ &= \sec(x^2 + x)^{1/2} \tan(x^2 + x)^{1/2} \tfrac{1}{2}(x^2 + x)^{-1/2}(2x + 1). \end{aligned}$$

Example 5 Find y' for each of the following.

(a) $x^3 y^5 + 3x = 8y^3 + 1$ [Solution]

(b) $x^2 \tan(y) + y^{10} \sec(x) = 2x$ [Solution]

(c) $\mathbf{e}^{2x+3y} = x^2 - \ln(xy^3)$ [Solution]

Solution

(a) $x^3 y^5 + 3x = 8y^3 + 1$

$$3x^2 y^5 + 5x^3 y^4 y' + 3 = 24y^2 y'$$

$$3x^2 y^5 + 3 = 24y^2 y' - 5x^3 y^4 y'$$

$$3x^2 y^5 + 3 = \left(24y^2 - 5x^3 y^4\right) y'$$

$$y' = \frac{3x^2 y^5 + 3}{24y^2 - 5x^3 y^4}$$

(b) $x^2 \tan(y) + y^{10} \sec(x) = 2x$

$$2x\tan(y)+x^2\sec^2(y)y'+10y^9y'\sec(x)+y^{10}\sec(x)\tan(x)=2$$

$$\left(x^2\sec^2(y)+10y^9\sec(x)\right)y'=2-y^{10}\sec(x)\tan(x)-2x\tan(y)$$

$$y'=\frac{2-y^{10}\sec(x)\tan(x)-2x\tan(y)}{x^2\sec^2(y)+10y^9\sec(x)}$$

(c) $\mathbf{e}^{2x+3y}=x^2-\ln\left(xy^3\right)$

Solution:-

$$\mathbf{e}^{2x+3y}(2+3y')=2x-\frac{y^3+3xy^2y'}{xy^3}$$

$$2\mathbf{e}^{2x+3y}+3y'\mathbf{e}^{2x+3y}=2x-\frac{y^3}{xy^3}-\frac{3xy^2y'}{xy^3}$$

$$2\mathbf{e}^{2x+3y}+3y'\mathbf{e}^{2x+3y}=2x-\frac{1}{x}-\frac{3y'}{y}$$

$$\left(3\mathbf{e}^{2x+3y}+3y^{-1}\right)y'=2x-x^{-1}-2\mathbf{e}^{2x+3y}$$

$$y'=\frac{2x-x^{-1}-2\mathbf{e}^{2x+3y}}{3\mathbf{e}^{2x+3y}+3y^{-1}}$$

Example 2 Differentiate $y=x^x$

Solution

$$\ln y=\ln x^x$$
$$\ln y=x\ln x$$

Differentiate both sides using implicit differentiation.

$$\frac{y'}{y}=\ln x+x\left(\frac{1}{x}\right)=\ln x+1$$

As with the first example multiply by y and substitute back in for y.

$$\begin{aligned} y' &= y(1+\ln x) \\ &= x^x(1+\ln x) \end{aligned}$$

Example 3 Differentiate $y=(1-3x)^{\cos(x)}$

$$\ln y = \ln\left[(1-3x)^{\cos(x)}\right] = \cos(x)\ln(1-3x)$$

Next, do some implicit differentiation.

$$\frac{y'}{y} = -\sin(x)\ln(1-3x)+\cos(x)\frac{-3}{1-3x} = -\sin(x)\ln(1-3x)-\cos(x)\frac{3}{1-3x}$$

Finally, solve for y' and substitute back in for y.

$$\begin{aligned} y' &= -y\left(\sin(x)\ln(1-3x)+\cos(x)\frac{3}{1-3x}\right) \\ &= -(1-3x)^{\cos(x)}\left(\sin(x)\ln(1-3x)+\cos(x)\frac{3}{1-3x}\right) \end{aligned}$$

Example

Suppose we want to differentiate $y = \ln\left(\frac{1-3x}{1+2x}\right)$.

$$y = \ln(1-3x) - \ln(1+2x) \quad \frac{dy}{dx} = \frac{-3}{1-3x} - \frac{2}{1+2x}$$

$$\frac{dy}{dx} = \frac{-3(1+2x)-2(1-3x)}{(1-3x)(1+2x)} = \frac{-3-6x-2+6x}{(1-3x)(1+2x)} = -\frac{5}{(1-3x)(1+2x)}$$

Example

Suppose we want to differentiate $y = x^{\sin x}$.

$$\ln y = \ln x^{\sin x} \qquad \ln y = \sin x \, \ln x$$

$$\ln y = \sin x \, \ln x \qquad \frac{d}{dx}(\ln y) = \frac{d}{dy}(\ln y) \times \frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} = \sin x \frac{1}{x} + \ln x \, \cos x = \frac{\sin x + x \ln x \cos x}{x}$$

$$\begin{aligned} \frac{dy}{dx} &= y\left(\frac{\sin x + x \ln x \cos x}{x}\right) \\ &= x^{\sin x}\left(\frac{\sin x + x \ln x \cos x}{x}\right) \end{aligned}$$

Example:-

1. $y = x^{1/x}$.
2. $y = (\sin x)^x$.

Solution:

1. $\ln y = \dfrac{\ln x}{x}$

   $\dfrac{y'}{y} = \dfrac{\frac{1}{x}x - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$

   $y' = y\dfrac{1 - \ln x}{x^2} = \boxed{x^{1/x}\dfrac{1 - \ln x}{x^2}}$

2. $\ln y = x \ln(\sin x)$

   $\dfrac{y'}{y} = \ln(\sin x) + x \cot x$

   $y' = y\{\ln(\sin x) + x \cot x\} = \boxed{(\sin x)^x \{\ln(\sin x) + x \cot x\}}$

Example:-

$$y = x^{\ln(\sqrt{x})}$$

$$\Rightarrow \ln(y) = \ln(\sqrt{x})\ln(x)$$

$$\ln(y) = \frac{1}{2}(\ln(x))^2$$

$$\frac{y'}{y} = 2 \times \frac{1}{2}(\ln(x)) \times \frac{1}{x}$$

$$y' = \frac{1}{x} y \ln(x)$$

$$y' = \{x^{\ln(\sqrt{x})-1}\}\ln(x)$$

Problem 1: Compute the derivative of $y = \sin(x)^{\cos(x)}$.

$$\begin{aligned}
& \ln(y) && = \cos(x) \cdot \ln(\sin(x)) \\
\Rightarrow \quad & \frac{d}{dx}[\ln(y)] && = \frac{d}{dx}[\cos(x) \cdot \ln(\sin(x))] \\
\Rightarrow \quad & \frac{1}{y} \cdot \frac{dy}{dx} && = -\sin(x) \cdot \ln(\sin(x)) + \cos(x) \cdot \frac{1}{\sin x} \cdot \cos(x)
\end{aligned}$$

$$\frac{dy}{dx} = y \cdot \left(-\sin(x) \cdot \ln(\sin(x)) + \cos(x) \cdot \frac{1}{\sin x} \cdot \cos(x) \right).$$

$$\frac{dy}{dx} = \left(\sin(x)^{\cos(x)}\right) \cdot \left(-\sin(x) \cdot \ln(\sin(x)) + \cos(x) \cdot \frac{1}{\sin x} \cdot \cos(x) \right).$$

Problem:- compute the derivativesof the following function:-

(a) $y = \ln\sqrt{x} + \sqrt{\ln x}$ (b) $y = \dfrac{1 - \ln x}{1 + \ln x}$

(d) $y = e^x + x^e + e^e$ (e) $y = \sec^2(\ln x)$ (c) $f(x) = 2^{e^x}$