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Engineering Mathematics IV

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Ordinary Differential Equations (ODE) and Applications

Differential Equation

The first definition that we should cover should be that of **differential equation**. A differential equation is any equation which contains derivatives, either ordinary derivatives or partial derivatives.

There is one differential equation that everybody probably knows, that is Newton's Second Law of Motion. If an object of mass m is moving with acceleration a and being acted on with force F then Newton's Second Law tells us.

$$F = ma \tag{1}$$

To see that this is in fact a differential equation we need to rewrite it a little. First, remember that we can rewrite the acceleration, a, in one of two ways.

$$a = \frac{dv}{dt} \qquad \text{OR} \qquad a = \frac{d^2u}{dt^2} \tag{2}$$

Where v is the velocity of the object and u is the position function of the object at any time t. We should also remember at this point that the force, F may also be a function of time, velocity, and/or position.

So, with all these things in mind Newton's Second Law can now be written as a differential equation in terms of either the velocity, v, or the position, u, of the object as follows.

$$m\frac{dv}{dt} = F(t, v) \tag{3}$$

$$m\frac{d^2u}{dt^2} = F\left(t, u, \frac{du}{dt}\right) \tag{4}$$

So, here is our first differential equation. We will see both forms of this in later chapters.

Here are a few more examples of differential equations.

$$ay'' + by' + cy = g(t) \tag{5}$$

$$\sin(y)\frac{d^2y}{dx^2} = (1-y)\frac{dy}{dx} + y^2\mathbf{e}^{-5y} \tag{6}$$

$$y^{(4)} + 10y''' - 4y' + 2y = \cos(t) \tag{7}$$

$$\alpha^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{8}$$

$$a^2 u_{xx} = u_{tt} \tag{9}$$

$$\frac{\partial^3 u}{\partial x^2 \partial t} = 1 + \frac{\partial u}{\partial y} \tag{10}$$

Order
The **order** of a differential equation is the largest derivative present in the differential equation. In the differential equations listed above (3) is a first order differential equation, (4), (5), (6), (8), and (9) are second order differential equations, (10) is a third order differential equation and (7) is a fourth order differential equation.

Note that the order does not depend on whether or not you've got ordinary or partial derivatives in the differential equation.

We will be looking almost exclusively at first and second order differential equations in these notes. As you will see most of the solution techniques for second order differential equations can be easily (and naturally) extended to higher order differential equations and we'll discuss that idea later on.

Ordinary and Partial Differential Equations
A differential equation is called an **ordinary differential equation**, abbreviated by **ode,** if it has ordinary derivatives in it. Likewise, a differential equation is called a **partial differential equation**, abbreviated by **pde,** if it has partial derivatives in it. In the differential equations above (3) - (7) are ode's and (8) - (10) are pde's.

The vast majority of these notes will deal with ode's. The only exception to this will be the last chapter in which we'll take a brief look at a common and basic solution technique for solving pde's.

Linear Differential Equations
A **linear differential equation** is any differential equation that can be written in the following form.

$$a_n(t)y^{(n)}(t)+a_{n-1}(t)y^{(n-1)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t)=g(t) \qquad \textbf{(11)}$$

The important thing to note about linear differential equations is that there are no products of the function, $y(t)$, and its derivatives and neither the function or its derivatives occur to any power other than the first power. Also note that neither the function or its derivatives are "inside" another function, for example, $\sqrt{y'}$ or $\mathbf{e}^y$.

The coefficients $a_0(t), \ldots ,a_n(t)$ and $g(t)$ can be zero or non-zero functions, constant or non-constant functions, linear or non-linear functions. Only the function, $y(t)$, and its derivatives are used in determining if a differential equation is linear.

If a differential equation cannot be written in the form, (11) then it is called a **non-linear** differential equation.

In (5) - (7) above only (6) is non-linear, the other two are linear differential equations. We can't classify (3) and (4) since we do not know what form the function *F* has. These could be either linear or non-linear depending on *F*.

Solution

A **solution** to a differential equation on an interval $\alpha < t < \beta$ is any function $y(t)$ which satisfies the differential equation in question on the interval $\alpha < t < \beta$. It is important to note that solutions are often accompanied by intervals and these intervals can impart some important information about the solution. Consider the following example.

Example 1 Show that $y(x) = x^{-\frac{3}{2}}$ is a solution to $4x^2 y'' + 12xy' + 3y = 0$ for $x > 0$.

Solution

We'll need the first and second derivative to do this.

$$y'(x) = -\frac{3}{2}x^{-\frac{5}{2}} \qquad\qquad y''(x) = \frac{15}{4}x^{-\frac{7}{2}}$$

Plug these as well as the function into the differential equation.

$$4x^2\left(\frac{15}{4}x^{-\frac{7}{2}}\right) + 12x\left(-\frac{3}{2}x^{-\frac{5}{2}}\right) + 3\left(x^{-\frac{3}{2}}\right) = 0$$

$$15x^{-\frac{3}{2}} - 18x^{-\frac{3}{2}} + 3x^{-\frac{3}{2}} = 0$$

$$0 = 0$$

So, $y(x) = x^{-\frac{3}{2}}$ does satisfy the differential equation and hence is a solution. Why then did we include the condition that $x > 0$? We did not use this condition anywhere in the work showing that the function would satisfy the differential equation.

To see why recall that

$$y(x) = x^{-\frac{3}{2}} = \frac{1}{\sqrt{x^3}}$$

In this form it is clear that we'll need to avoid $x = 0$ at the least as this would give division by zero.

Also, there is a general rule of thumb that we're going to run with in this class. This rule of thumb is : Start with real numbers, end with real numbers. In other words, if our differential equation only contains real numbers then we don't want solutions that give complex numbers. So, in order to avoid complex numbers we will also need to avoid negative values of *x*.

So, we saw in the last example that even though a function may symbolically satisfy a differential equation, because of certain restrictions brought about by the solution we cannot use all values of the independent variable and hence, must make a restriction on the independent variable. This will be the case with many solutions to differential equations.

In the last example, note that there are in fact many more possible solutions to the differential equation given. For instance, all of the following are also solutions

$$y(x) = x^{-\frac{1}{2}}$$
$$y(x) = -9x^{-\frac{3}{2}}$$
$$y(x) = 7x^{-\frac{1}{2}}$$
$$y(x) = -9x^{-\frac{3}{2}} + 7x^{-\frac{1}{2}}$$

We'll leave the details to you to check that these are in fact solutions. Given these examples can you come up with any other solutions to the differential equation? There are in fact an infinite number of solutions to this differential equation.

So, given that there are an infinite number of solutions to the differential equation in the last example (provided you believe us when we say that anyway....) we can ask a natural question. Which is the solution that we want or does it matter which solution we use? This question leads us to the next definition in this section.

Initial Condition(s)

Initial Condition(s) are a condition, or set of conditions, on the solution that will allow us to determine which solution that we are after. Initial conditions (often abbreviated i.c.'s when we're feeling lazy...) are of the form,

$$y(t_0) = y_0 \qquad \text{and/or} \qquad y^{(k)}(t_0) = y_k$$

So, in other words, initial conditions are values of the solution and/or its derivative(s) at specific points. As we will see eventually, solutions to "nice enough" differential equations are unique and hence only one solution will meet the given initial conditions.

The number of initial conditions that are required for a given differential equation will depend upon the order of the differential equation as we will see.

Example 2 $y(x) = x^{-\frac{3}{2}}$ is a solution to $4x^2y'' + 12xy' + 3y = 0$, $y(4) = \frac{1}{8}$, and $y'(4) = -\frac{3}{64}$.

Solution

As we saw in previous example the function is a solution and we can then note that

$$y(4) = 4^{-\frac{3}{2}} = \frac{1}{\left(\sqrt{4}\right)^3} = \frac{1}{8}$$

$$y'(4) = -\frac{3}{2}4^{-\frac{5}{2}} = -\frac{3}{2}\frac{1}{\left(\sqrt{4}\right)^5} = -\frac{3}{64}$$

and so this solution also meets the initial conditions of $y(4)=\frac{1}{8}$ and $y'(4)=-\frac{3}{64}$. In fact, $y(x)=x^{-\frac{3}{2}}$ is the only solution to this differential equation that satisfies these two initial conditions.

Initial Value Problem

An **Initial Value Problem** (or **IVP**) is a differential equation along with an appropriate number of initial conditions.

Example 3 The following is an IVP.

$$4x^2y''+12xy'+3y=0 \qquad y(4)=\frac{1}{8},\ \ y'(4)=-\frac{3}{64}$$

Example 4 Here's another IVP.

$$2t\,y'+4y=3 \qquad y(1)=-4$$

As we noted earlier the number of initial conditions required will depend on the order of the differential equation.

Interval of Validity

The **interval of validity** for an IVP with initial condition(s)

$$y(t_0)=y_0 \qquad \text{and/or} \qquad y^{(k)}(t_0)=y_k$$

is the largest possible interval on which the solution is valid and contains t_0. These are easy to define, but can be difficult to find, so we're going to put off saying anything more about these until we get into actually solving differential equations and need the interval of validity.

General Solution

The **general solution** to a differential equation is the most general form that the solution can take and doesn't take any initial conditions into account.

Example 5 $y(t)=\frac{3}{4}+\frac{c}{t^2}$ is the general solution to

$$2t\,y'+4y=3$$

We'll leave it to you to check that this function is in fact a solution to the given differential equation. In fact, all solutions to this differential equation will be in this form. This is one of the first differential equations that you will learn how to solve and you will be able to verify this shortly for yourself.

Actual Solution

The **actual solution** to a differential equation is the specific solution that not only satisfies the differential equation, but also satisfies the given initial condition(s).

Example 6 What is the actual solution to the following IVP?

$$2t\,y' + 4y = 3 \qquad y(1) = -4$$

Solution

This is actually easier to do than it might at first appear. From the previous example we already know (well that is provided you believe our solution to this example...) that all solutions to the differential equation are of the form.

$$y(t) = \frac{3}{4} + \frac{c}{t^2}$$

All that we need to do is determine the value of *c* that will give us the solution that we're after. To find this all we need do is use our initial condition as follows.

$$-4 = y(1) = \frac{3}{4} + \frac{c}{1^2} \qquad \Rightarrow \qquad c = -4 - \frac{3}{4} = -\frac{19}{4}$$

So, the actual solution to the IVP is.

$$y(t) = \frac{3}{4} - \frac{19}{4t^2}$$

From this last example we can see that once we have the general solution to a differential equation finding the actual solution is nothing more than applying the initial condition(s) and solving for the constant(s) that are in the general solution.

Implicit/Explicit Solution

In this case it's easier to define an explicit solution, then tell you what an implicit solution isn't, and then give you an example to show you the difference. So, that's what we'll do.

An **explicit solution** is any solution that is given in the form $y = y(t)$. In other words, the only place that *y* actually shows up is once on the left side and only raised to the first power. An **implicit solution** is any solution that isn't in explicit form. Note that it is possible to have either general implicit/explicit solutions and actual implicit/explicit solutions.

Example 7 $y^2 = t^2 - 3$ is the actual implicit solution to $y' = \frac{t}{y}, \quad y(2) = -1$

At this point we will ask that you trust us that this is in fact a solution to the differential equation. You will learn how to get this solution in a later section. The point of this example is that since there is a y^2 on the left side instead of a single $y(t)$ this is not an explicit solution!

Example 8 Find an actual explicit solution to $y' = \frac{t}{y}, \quad y(2) = -1$.

Solution

We already know from the previous example that an implicit solution to this IVP is $y^2 = t^2 - 3$. To find the explicit solution all we need to do is solve for $y(t)$.

$$y(t) = \pm\sqrt{t^2 - 3}$$

Now, we've got a problem here. There are two functions here and we only want one and in fact only one will be correct! We can determine the correct function by reapplying the initial condition. Only one of them will satisfy the initial condition.

In this case we can see that the "-" solution will be the correct one. The actual explicit solution is then

$$y(t) = -\sqrt{t^2 - 3}$$

In this case we were able to find an explicit solution to the differential equation. It should be noted however that it will not always be possible to find an explicit solution.

Also, note that in this case we were only able to get the explicit actual solution because we had the initial condition to help us determine which of the two functions would be the correct solution.

We've now gotten most of the basic definitions out of the way and so we can move onto other topics.