Linear Equation

In order to solve a linear first order differential equation we MUST start with the differential equation in the form shown below. If the differential equation is not in this form then the process we're going to use will not work.

$$\frac{dy}{dt} + p(t)\,y = g(t) \qquad (1)$$

Now, we are going to assume that there is some magical function somewhere out there in the world, $\mu(t)$, called an **integrating factor**.

where,

$$\mu(t) = \mathbf{e}^{\int p(t)\,dt} \qquad (10)$$

Solution Process
The solution process for a first order linear differential equation is as follows.

1. Put the differential equation in the correct initial form, (1).
2. Find the integrating factor, $\mu(t)$, using (10).
3. Multiply everything in the differential equation by $\mu(t)$ and verify that the left side becomes the product rule $\left(\mu(t)\,y(t)\right)'$ and write it as such.
4. Integrate both sides, make sure you properly deal with the constant of integration.
5. Solve for the solution *y(t)*.

Example 1 Find the solution to the following differential equation.

$$\frac{dv}{dt} = 9.8 - 0.196v$$

Solution

First, we need to get the differential equation in the correct form.

$$\frac{dv}{dt} + 0.196v = 9.8$$

From this we can see that *p(t)*=0.196 and so $\mu(t)$ is then.

$$\mu(t) = \mathbf{e}^{\int 0.196 dt} = \mathbf{e}^{0.196t}$$

Note that officially there should be a constant of integration in the exponent from the integration.

Now multiply all the terms in the differential equation by the integrating factor and do some simplification.

$$\mathbf{e}^{0.196t}\frac{dv}{dt} + 0.196\mathbf{e}^{0.196t}v = 9.8\mathbf{e}^{0.196t}$$

$$\left(\mathbf{e}^{0.196t}v\right)' = 9.8\mathbf{e}^{0.196t}$$

Integrate both sides and don't forget the constants of integration that will arise from both integrals.

$$\int \left(\mathbf{e}^{0.196t}v\right)' dt = \int 9.8\mathbf{e}^{0.196t}\, dt$$

$$\mathbf{e}^{0.196t}v + k = 50\mathbf{e}^{0.196t} + c$$

Okay. It's time to play with constants again. We can subtract *k* from both sides to get.

$$\mathbf{e}^{0.196t}v = 50\mathbf{e}^{0.196t} + c - k$$

Both *c* and *k* are unknown constants and so the difference is also an unknown constant. We will therefore write the difference as *c*. So, we now have

$$\mathbf{e}^{0.196t}v = 50\mathbf{e}^{0.196t} + c$$

From this point on we will only put one constant of integration down when we integrate both sides knowing that if we had written down one for each integral, as we should, the two would just end up getting absorbed into each other.

The final step in the solution process is then to divide both sides by $\mathbf{e}^{0.196t}$ or to multiply both sides by $\mathbf{e}^{-0.196t}$. Either will work, but we usually prefer the multiplication route. Doing this gives the general solution to the differential equation.

$$v(t) = 50 + c\mathbf{e}^{-0.196t}$$

Example 2 Solve the following IVP.

$$\frac{dv}{dt} = 9.8 - 0.196v \qquad v(0) = 48$$

Solution

To find the solution to an IVP we must first find the general solution to the differential equation and then use the initial condition to identify the exact solution that we are after. So, since this is the same differential equation as we looked at in Example 1, we already have its general solution.

$$v = 50 + ce^{-0.196t}$$

Now, to find the solution we are after we need to identify the value of *c* that will give us the solution we are after. To do this we simply plug in the initial condition which will give us an equation we can solve for *c*. So, let's do this

$$48 = v(0) = 50 + c \quad \Rightarrow \quad c = -2$$

So, the actual solution to the IVP is.

$$v = 50 - 2e^{-0.196t}$$

Example 3 Solve the following IVP.

$$\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1 \qquad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}, \qquad 0 \le x < \frac{\pi}{2}$$

Solution :

Rewrite the differential equation to get the coefficient of the derivative a one.

$$y' + \frac{\sin(x)}{\cos(x)}y = 2\cos^2(x)\sin(x) - \frac{1}{\cos(x)}$$

$$y' + \tan(x)y = 2\cos^2(x)\sin(x) - \sec(x)$$

$$\sec(x)y' + \sec(x)\tan(x)y = 2\sec(x)\cos^2(x)\sin(x) - \sec^2(x)$$

$$(\sec(x)y)' = 2\cos(x)\sin(x) - \sec^2(x)$$

Integrate both sides.

$$\int (\sec(x)y(x))'\, dx = \int 2\cos(x)\sin(x) - \sec^2(x)\, dx$$

$$\sec(x)y(x) = \int \sin(2x) - \sec^2(x)\, dx$$

$$\sec(x)y(x) = -\frac{1}{2}\cos(2x) - \tan(x) + c$$

Note the use of the trig formula $\sin(2\theta) = 2\sin\theta\cos\theta$ that made the integral easier. Next, solve for the solution.

$$\begin{aligned} y(x) &= -\frac{1}{2}\cos(x)\cos(2x) - \cos(x)\tan(x) + c\cos(x) \\ &= -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + c\cos(x) \end{aligned}$$

Finally, apply the initial condition to find the value of *c*.

$$\begin{aligned} 3\sqrt{2} = y\left(\frac{\pi}{4}\right) &= -\frac{1}{2}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right) + c\cos\left(\frac{\pi}{4}\right) \\ 3\sqrt{2} &= -\frac{\sqrt{2}}{2} + c\frac{\sqrt{2}}{2} \\ c &= 7 \end{aligned}$$

The solution is then.

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)$$

Example 4 Find the solution to the following IVP.

$$t\,y' + 2y = t^2 - t + 1 \qquad y(1) = \frac{1}{2}$$

Solution

First, divide through by the *t* to get the differential equation into the correct form.

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

Now let's get the integrating factor, $\mu(t)$.

$$\mu(t) = \mathbf{e}^{\int \frac{2}{t}dt} = \mathbf{e}^{2\ln|t|}$$

Now, we need to simplify $\mu(t)$. However, we can't use (11) yet as that requires a coefficient of one in front of the logarithm. So, recall that

$$\ln x^r = r\ln x$$

and rewrite the integrating factor in a form that will allow us to simplify it.

$$\mu(t) = \mathbf{e}^{2\ln|t|} = \mathbf{e}^{\ln|t|^2} = |t|^2 = t^2$$

We were able to drop the absolute value bars here because we were squaring the *t*, but often they can't be dropped so be careful with them and don't drop them unless you know that you can. Often the absolute value bars must remain.

Now, multiply the rewritten differential equation (remember we can't use the original differential equation here...) by the integrating factor.

$$\left(t^2 y\right)' = t^3 - t^2 + t$$

Integrate both sides and solve for the solution.

$$\begin{aligned} t^2 y &= \int t^3 - t^2 + t\,dt \\ &= \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c \\ y(t) &= \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{c}{t^2} \end{aligned}$$

Finally, apply the initial condition to get the value of *c*.

$$\frac{1}{2} = y(1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c \qquad \Rightarrow \qquad c = \frac{1}{12}$$

The solution is then,

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$$

Example 5 Find the solution to the following IVP.

$$t\,y' - 2y = t^5 \sin(2t) - t^3 + 4t^4 \qquad y(\pi) = \frac{3}{2}\pi^4$$

Solution

First, divide through by ***t*** to get the differential equation in the correct form.

$$y' - \frac{2}{t}y = t^4 \sin(2t) - t^2 + 4t^3$$

Now that we have done this we can find the integrating factor, $\mu(t)$.

$$\mu(t) = \mathbf{e}^{\int -\frac{2}{t}dt} = \mathbf{e}^{-2\ln|t|}$$

Do not forget that the "-" is part of *p(t)*. Forgetting this minus sign can take a problem that is very easy to do and turn it into a very difficult, if not impossible problem so be careful!

Now, we just need to simplify this as we did in the previous example.

$$\mu(t) = \mathbf{e}^{-2\ln|t|} = \mathbf{e}^{\ln|t|^{-2}} = |t|^{-2} = t^{-2}$$

Again, we can drop the absolute value bars since we are squaring the term.

$$t^{-2}y(t) = \int t^2\sin(2t)\,dt + \int -1 + 4t\,dt$$

$$t^{-2}y(t) = -\frac{1}{2}t^2\cos(2t) + \frac{1}{2}t\sin(2t) + \frac{1}{4}\cos(2t) - t + 2t^2 + c$$

$$y(t) = -\frac{1}{2}t^4\cos(2t) + \frac{1}{2}t^3\sin(2t) + \frac{1}{4}t^2\cos(2t) - t^3 + 2t^4 + ct^2$$

Apply the initial condition to find the value of c.

$$\frac{3}{2}\pi^4 = y(\pi) = -\frac{1}{2}\pi^4 + \frac{1}{4}\pi^2 - \pi^3 + 2\pi^4 + c\pi^2 = \frac{3}{2}\pi^4 - \pi^3 + \frac{1}{4}\pi^2 + c\pi^2$$

$$\pi^3 - \frac{1}{4}\pi^2 = c\pi^2$$

$$c = \pi - \frac{1}{4}$$

The solution is then

$$y(t) = -\frac{1}{2}t^4\cos(2t) + \frac{1}{2}t^3\sin(2t) + \frac{1}{4}t^2\cos(2t) - t^3 + 2t^4 + \left(\pi - \frac{1}{4}\right)t^2$$

Example 6 Find the solution to the following IVP and determine all possible behaviors of the solution as $t \to \infty$. If this behavior depends on the value of y_0 give this dependence.

$$2y' - y = 4\sin(3t) \qquad y(0) = y_0$$

Solution

First, divide through by a 2 to get the differential equation in the correct form.

$$y' - \frac{1}{2}y = 2\sin(3t)$$

Now find $\mu(t)$.

$$\mu(t) = \mathbf{e}^{\int -\frac{1}{2}dt} = \mathbf{e}^{-\frac{t}{2}}$$

Multiply $\mu(t)$ through the differential equation and rewrite the left side as a product rule.

$$\left(\mathbf{e}^{-\frac{t}{2}}y\right)' = 2\mathbf{e}^{-\frac{t}{2}}\sin(3t)$$

Integrate both sides (the right side requires integration by parts – you can do that right?) and solve for the solution.

$$\mathbf{e}^{-\frac{t}{2}}y = \int 2\mathbf{e}^{-\frac{t}{2}}\sin(3t)\,dt + c$$

$$\mathbf{e}^{-\frac{t}{2}}y = -\frac{24}{37}\mathbf{e}^{-\frac{t}{2}}\cos(3t) - \frac{4}{37}\mathbf{e}^{-\frac{t}{2}}\sin(3t) + c$$

$$y(t) = -\frac{24}{37}\cos(3t) - \frac{4}{37}\sin(3t) + c\mathbf{e}^{\frac{t}{2}}$$

Apply the initial condition to find the value of c and note that it will contain y_0 as we don't have a value for that.

$$y_0 = y(0) = -\frac{24}{37} + c \quad \Rightarrow \quad c = y_0 + \frac{24}{37}$$

So, the solution is

$$y(t) = -\frac{24}{37}\cos(3t) - \frac{4}{37}\sin(3t) + \left(y_0 + \frac{24}{37}\right)\mathbf{e}^{\frac{t}{2}}$$